Homework 4 Solutions

5.58

(a)

The solid sphere has a constant charge density $\rho = 3Q/4\pi R^3$. We will divide the sphere into a bunch of rings as in problem 5.56 of homework 3, only now we have a continuum of rings of varying radius for each value of the polar angle θ. The charge of each volume element is $\rho r^2 \sin\theta dr d\theta d\phi$, so the charge of each ring is $dQ = 2\pi\rho r^2 \sin\theta dr d\theta$. The current in each ring is thus $dI = dQ\omega/2\pi = \rho\omega r^2 \sin\theta dr d\theta$. The radius of each ring is $r\sin\theta$, so the corresponding dipole moment is $d\vec{m} = \hat{z}\pi\rho\omega r^4 \sin^3\theta dr d\theta$. To find the total dipole moment of the sphere, we need to integrate this over all values of θ and r:

$$\vec{m} = \int_0^R \int_0^\pi d\vec{m} = \frac{QR^2\omega}{5}\hat{z}. \tag{1}$$

A simple check of this result is to recall from homework 3 that the gyromagnetic ratio of a rotating charged ring of charge Q and mass M is $Q/2M$. We also found the same ratio for a rotating hollow charged sphere. In fact, any object which can be described as a collection of rings will have this same ratio, including the solid sphere we are considering here. A well known result from classical mechanics is that the moment of inertia of a solid sphere is $2MR^2/5$. Multiplying this by ω (to get the angular momentum) and then by the ratio $Q/2M$, we see that we get back (1).

(b)

Note that we could immediately apply equation 5.89 from the text using the result from part (a). However, to get a feel for how to tackle more difficult problems, we'll work a little harder for the answer. We'll start with another result from problem 5.57:

$$\langle\vec{B}\rangle = \frac{3}{4\pi R^3}\int_V \vec{B} = \frac{3}{4\pi R^3}\int_V \nabla\times\vec{A} = -\frac{3}{4\pi R^3}\oint_S \vec{A}\times d\vec{a}. \tag{2}$$

V is the solid sphere, and S is its surface. Next, we substitute in for $\vec{A}$ using

$$\vec{A} = \frac{\mu_0}{4\pi}\int d^3r' \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}. \tag{3}$$

The current density is given by

$$\vec{J}(\vec{r}') = \rho\omega r' \sin\theta' \hat{\phi}'. \tag{4}$$

This should be compared with the quantity dI of part (a): dI is just the area element $r'd\theta'dr'$ times $\vec{J}$. Putting it all together, we get

$$\langle\vec{B}\rangle = -\frac{3\mu_0\rho\omega}{16\pi^2 R^3}\int d^3r' r' \sin\theta' \hat{\phi}' \times \oint_S \frac{d\vec{a}}{|\vec{r} - \vec{r}'|}. \tag{5}$$

From problem 5.57 in the book, the surface integral above is $4\pi\vec{r}'/3$. We then get

$$\langle\vec{B}\rangle = -\frac{\mu_0\rho\omega}{4\pi R^3}\int dr' d\theta' d\phi' r'^4 \sin^2\theta' \hat{\phi}' \times \hat{r}' = -\frac{\mu_0\rho\omega R^2}{20\pi}\int d\theta' d\phi' \sin^2\theta' \hat{\theta}'. \tag{6}$$

In cartesian coordinates, $\hat{\theta}'$ is

$$\hat{\theta}' = (\cos\theta' \cos\phi', \cos\theta' \sin\phi', -\sin\theta'). \tag{7}$$

The first two components will integrate to zero when we perform the ϕ' integral. The integral then reduces to

$$\langle\vec{B}\rangle = \frac{\mu_0\rho\omega R^2}{10}\hat{z}\int d\theta' \sin^3\theta' = \frac{2\mu_0\rho\omega R^2}{15}\hat{z} = \frac{\mu_0 Q\omega}{10\pi R}\hat{z}. \tag{8}$$

This agrees with what we would get from equation 5.89.

(c)

The limit $r \gg R$ is the dipole limit. The vector potential for a dipole moment has already been derived in equation 5.85:

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}. \tag{9}$$

Plugging in the result from part (a) gives

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0\omega Q R^2 \sin\theta}{20\pi r^2}\hat{\phi}. \tag{10}$$

(d)

Now we are going to find $\vec{A}$ *exactly.* We will use the hint, which means using the answer for a rotating spherical *shell* of radius R', given in example 5.11 in the book:

$$\vec{A}(\vec{r}) = \frac{\sigma\omega\mu_0 R'^4}{3r^2}\sin\theta\hat{\phi}. \tag{11}$$

We should be able to get $\vec{A}$ for the solid sphere case by adding up this solution for all values of the radius R', from 0 to R. Before we do this, we should write the surface charge density σ in terms of our volume charge density ρ: $\sigma = \rho dR'$. The sum over shells becomes an integral which is easy to perform:

$$\vec{A}(\vec{r}) = \frac{\rho\omega\mu_0}{3r^2}\sin\theta\hat{\phi}\int_0^R dR' R'^4 = \frac{\rho\omega\mu_0 R^5}{15r^2}\sin\theta\hat{\phi} = \frac{Q\omega\mu_0 R^2}{20\pi r^2}\sin\theta\hat{\phi}. \tag{12}$$

This is exactly the same as the approximate value we got in part (c)! When you think about it though, it is not at all surprising that we got the correct answer in part (c). Now that we know the correct answer, it is clear from the form of (18) that if we keep only the leading order term in powers of R^2/r^2, we get back the exact answer since it contains only one such power, ie the full answer *is* the leading order term in this expansion.

(e)

We can again make use of the results from example 5.11 to compute the magnetic field inside the rotating sphere. Let's recall these results:

$$\vec{A}^{shell} = \left\{ \begin{array}{ll} \frac{\sigma\omega\mu_0 R'}{3} r\sin\theta\hat{\phi}, & r \le R' \\ \frac{\sigma\omega\mu_0 R'^4}{3}\frac{\sin\theta}{r^2}\hat{\phi}, & r \ge R' \end{array} \right\}. \tag{13}$$

As in part (d), we can integrate the shell solutions to obtain the solution for the solid sphere. If we want to compute $\vec{A}^{solidsphere}(r,\theta,\phi)$ inside the sphere, then we need to integrate $\vec{A}^{shell}_{inside}$ for $R' > r$, integrate $\vec{A}^{shell}_{outside}$ for $R' < r$, and add the two results:

$$\vec{A}^{solidsphere}_{inside}(r,\theta,\phi) = \int_r^R dR'\vec{A}^{shell}_{inside}(R',r,\theta,\phi) + \int_0^r dR'\vec{A}^{shell}_{outside}(R',r,\theta,\phi). \tag{14}$$

Plugging in from (19), replacing $\sigma \to \rho dR'$, and doing the integrals gives

$$\vec{A} = \frac{\rho\omega\mu_0}{3}\left[\frac{1}{2}r(R^2 - r^2) + \frac{1}{5}r^3\right]\sin\theta\hat{\phi}. \tag{15}$$

Next, we take the curl of this expression to find the magnetic field inside the sphere:

$$\vec{B} = \nabla\times\vec{A} = \frac{\rho\omega\mu_0}{3}\left\{\hat{r}\left[R^2 - \frac{3}{5}r^2\right]\cos\theta - \hat{\theta}\left[R^2 - \frac{6}{5}r^2\right]\sin\theta\right\}. \tag{16}$$

Finally, we'd like to take the average of this result and check that it agrees with the answer from part (b):

$$\langle \vec{B} \rangle = \frac{3}{4\pi R^3} \int_V \vec{B}. \tag{17}$$

We'll start with the angular pieces:

$$\begin{aligned} \langle \hat{r} \cos\theta \rangle &= \frac{1}{4\pi} \int_0^\pi d\theta \int_0^{2\pi} d\phi \sin\theta \hat{r} \cos\theta = \frac{1}{3}\hat{z}, \\ \langle \hat{\theta} \sin\theta \rangle &= \frac{1}{4\pi} \int_0^\pi d\theta \int_0^{2\pi} d\phi \sin\theta \hat{\theta} \sin\theta = -\frac{2}{3}\hat{z}. \end{aligned} \tag{18}$$

Plugging these results into (22) and (23) and collecting terms then leads to

$$\langle \vec{B} \rangle = \frac{\rho\omega\mu_0}{R^3}\hat{z} \int_0^R dr r^2 (R^2 - r^2) = \frac{2\rho\omega\mu_0 R^2}{15}\hat{z} = \frac{\mu_0 Q \omega}{10\pi R}\hat{z}. \tag{19}$$

This agrees with equation (8) above.

5.60

(a)

We write the vector potential as

$$\vec{A} = \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|}. \tag{20}$$

As in equation 5.77, we can expand the denominator of the integrand:

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\hat{r}\cdot\hat{r}'}} = \frac{1}{r}\sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\hat{r}\cdot\hat{r}'). \tag{21}$$

Plugging this into (26) gives the multipole expansion:

$$\vec{A} = \frac{\mu_0}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int d^3r' r'^n P_n(\hat{r}\cdot\hat{r}') \vec{J}(\vec{r}'). \tag{22}$$

(b)

The monopole potential is given by the first term in the above expansion:

$$\vec{A}(\vec{r})_{mon} = \frac{\mu_0}{4\pi}\frac{1}{r} \int d^3r' \vec{J}(\vec{r}'). \tag{23}$$

Obviously, if this is to vanish, we need to show that the integral of $\vec{J}$ vanishes. We can replace this integral with an integral over the curl of the magnetic field:

$$\int d^3r' \vec{J} = \frac{1}{\mu_0} \int d^3r' \nabla \times \vec{B} = \frac{1}{\mu_0} \oint d\vec{a} \times \vec{B}. \tag{24}$$

In the last step, we used the identity from problem 1.60b of the book. In this case the volume integral is over all space, and the surface integral is over the "surface at infinity". To make this well defined, we can imagine integrating over a large ball of radius R, performing the surface integral as a function of R, and then taking $R \to \infty$. However, we must have the $\vec{B}$ field vanishing at infinity since the current sources are localized—if they were not localized, the multipole expansion would not be sensible. Therefore, $\vec{B}$ vanishes at infinity, and the surface integral vanishes.

An alternative solution is to exploit an identity given in problem 5.7 of the text. This relates the integral over the current density to the time derivative of the electric dipole moment:

$$\int_V \vec{J} = \frac{d\vec{p}}{dt}. \tag{25}$$

For magnetostatics, the electric dipole moment must be constant in time, so we see that the integral of $\vec{J}$ must vanish.

(c)

We want to express the dipole moment of a current distribution (described by $\vec{J}$) as an integral over the volume occupied by that distribution. To do this, we will divide the distribution into infinitesimal current elements and determine the contribution to $\vec{m}$ made by each such element.

We begin with equation 5.84 from the text:

$$\vec{m} = I \int d\vec{a}. \tag{26}$$

Equation 1.107 allows us to rewrite this as

$$\vec{m} = \frac{I}{2} \oint \vec{r} \times d\vec{\ell}. \tag{27}$$

Let's be clear about what these expressions mean. Equation (31) says that the dipole moment of a loop of current I enclosing an area $\int da$ is the product of I with the area vector of this area, $\int d\vec{a}$. Equation (32) says that we may write the area vector as an integral over the loop enclosing its associated area and thus express $\vec{m}$ in terms of this integral. In these expressions, we do not concern ourselves with the cross section of the loop, ie the thickness of the wire. We have already integrated over this thickness when we describe the current with the quantity I. In particular, $I = \int d\vec{a}' \cdot \vec{J}$, where $\int da'$ is the cross-sectional area, and $\vec{J}$ is the usual current density. ($\int da'$ is not to be confused with the area *enclosed* by the loop, $\int da$.) If we shrink the cross-sectional area to infinitesimal size, then $\vec{J}$ is constant over this area and in the $d\vec{\ell}$ direction. From this and (32) above, we see that an infinitesimal element of the loop makes the following contribution to the dipole moment:

$$d\vec{m} = \tfrac{1}{2} J\vec{r} \times d\vec{\ell} da' = \tfrac{1}{2}\vec{r} \times \vec{J} d\ell da' = \tfrac{1}{2}\vec{r} \times \vec{J} d^3 r. \tag{28}$$

To find the total dipole moment, we integrate this to obtain the desired expression.

5.61

We will think of the rotating cylinder as a stack of magnetic dipoles. The dipole moment of each current ring is

$$d\vec{m} = dI \int d\vec{a} = \pi R^2 \hat{z} dI = \pi R^3 \omega\sigma \hat{z} dz. \tag{29}$$

The magnetic field of a dipole is quoted in problem 5.33:

$$\vec{B}_{dip} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right]. \tag{30}$$

To find the total magnetic field at a distance s from the center of the glass rod, we must integrate over this stack of dipoles:

$$\vec{B} = \frac{\mu_0 R^3 \omega\sigma}{4}\int_{-L/2}^{L/2} dz \frac{3(\hat{r}\cdot\hat{z})\hat{r} - \hat{z}}{(s^2+z^2)^{3/2}}. \tag{31}$$

Here, we've used that the magnitude of the vector from the dipole to the measurement point is given by $r = \sqrt{s^2+z^2}$. This comes from $\vec{r} = s\hat{s} - z\hat{z}$. From this, we can also deduce that

$$\hat{r} = \frac{s}{\sqrt{s^2+z^2}}\hat{s} - \frac{z}{\sqrt{s^2+z^2}}\hat{z}. \tag{32}$$

Plugging this into (36), we get

$$\vec{B} = \frac{\mu_0 R^3 \omega\sigma}{4}\int_{-L/2}^{L/2} dz \frac{3\left[\frac{z^2}{s^2+z^2}\hat{z} - \frac{sz}{s^2+z^2}\hat{s}\right] - \hat{z}}{(s^2+z^2)^{3/2}}. \tag{33}$$

The integral proportional to $\hat{s}$ vanishes since it is an odd function of z. We are then left with

$$\vec{B} = \frac{\mu_0 R^3 \omega\sigma}{4}\hat{z}\int_{-L/2}^{L/2} dz \frac{2z^2 - s^2}{(s^2+z^2)^{5/2}} = -\frac{\mu_0 R^3 \omega\sigma L}{4(s^2+\frac{L^2}{4})^{3/2}}\hat{z}. \tag{34}$$