7.38

In example 3.8 of the book, we learn that the potential due to a conducting sphere in an otherwise uniform electric field is given by

$$V(r,\theta) = -E_0 \left(r - \frac{R^3}{r^2} \right) \cos\theta. \tag{1}$$

For $a \ll d$, this result still holds for the geometry shown in Fig. 7.48. This is because the deviations in the uniform electric field caused by the sphere will disappear far away from the sphere. So near the upper plate, the field will be uniform and will satisfy the appropriate boundary condition at the plate. You might worry that the solution won't work for the hemisphere, but a quick inspection of (1) reveals that the boundary condition $V = 0$ at $\theta = \pi/2$ is already satisfied by the solution for the whole sphere, so this solution is immediately applicable to the hemisphere as well by restricting to $\theta \leq \pi/2$.

The electric field is given by

$$\vec{E} = -\nabla V = E_0 \left[\hat{r} \left(1 + \frac{2R^3}{r^3} \right) \cos\theta - \frac{1}{r} \hat{\theta} \left(r - \frac{R^3}{r^2} \right) \sin\theta \right]. \tag{2}$$

The current density is given by $\vec{J} = \sigma \vec{E}$. The current into the sphere is found by computing the flux of the current density through the surface of the sphere:

$$\begin{aligned} I &= \int \vec{J} \cdot d\vec{a} = \sigma \int \vec{E} \cdot \hat{r} a^2 d\Omega = 3\sigma a^2 E_0 \int_0^{2\pi} \int_0^{\pi/2} d\phi d\theta \sin\theta \cos\theta \\ &= 3\pi\sigma a^2 E_0 = 3\pi\sigma a^2 V_0/d. \end{aligned} \tag{3}$$

7.42

(a)

The constancy of $\vec{B}$ follows immediately from Maxwell's equation. Since $\vec{E}$ vanishes inside a perfect conductor, so must its curl, and therefore $\partial\vec{B}/\partial t = 0$ inside a perfect conductor.

(b)

This also follows directly from Maxwell's equation. A loop of perfectly conducting wire, like any other perfect conductor, has vanishing electric field inside. Therefore, if we integrate Faraday's law over an area enclosed by the loop, we find that the magnetic flux through that area is constant:

$$\begin{aligned}\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \int d\vec{a} \cdot \nabla \times \vec{E} &= \oint \vec{E} \cdot d\vec{\ell} = 0 \\ = -\int d\vec{a} \cdot \frac{\partial \vec{B}}{\partial t} &= -\frac{\partial \Phi}{\partial t}.\end{aligned} \tag{4}$$

(c)

The fact that the current in a superconductor is relegated to the surface stems immediately from the fourth Maxwell equation:

$$\mu_0 \vec{J} = \nabla \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}. \tag{5}$$

If both the electric and magnetic fields vanish inside a conductor, then the RHS of the above equation vanishes, implying that the current density vanishes inside the conductor, ie any current within the superconductor must be at the surface.

(d)

This is similar to example 3.8 from the text, except here we are talking about a magnetic field instead of an electric field. That is, we know that $\vec{B}$ vanishes inside the sphere, and that as we go to infinity, $\vec{B} \to B_0 \hat{z}$. The problem is to find $\vec{B}$ everywhere outside the sphere. We can then look at how $\vec{B}$ changes across the surface of the sphere to determine the surface current.

Actually, we won't even bother to find $\vec{B}$. We can instead find the vector potential $\vec{A}$ and use that the normal derivative of $\vec{A}$ as we approach the surface from outside is equal to $-\mu_0 \vec{K}$, where $\vec{K}$ is the surface current (see equation 5.76 of the text). Finding $\vec{A}$ turns out to be exactly the same problem as finding the scalar potential in example 3.8. To see this, first let's choose the following asymptotic form of the vector potential:

$$\vec{A} \to -\tfrac{1}{2}\vec{r} \times \vec{B} = \tfrac{1}{2} r B_0 \sin\theta \hat{\phi} \quad \text{as} \quad r \to \infty. \tag{6}$$

This looks very similar to the asymptotic form of the scalar potential in example 3.8, except that we have a sine instead of a cosine. However, this difficulty is easily dispensed with by defining a new spherical coordinate system where the new z axis is orthogonal to the direction of the $\vec{B}$ field. That is, the new polar angle χ is given by $\chi = \pi/2 - \theta$. We then have the following boundary conditions on A_ϕ:

$$A_\phi = 0 \quad \text{at} \quad r = a, \qquad A_\phi \to \tfrac{1}{2} r B_0 \cos\chi \quad \text{as} \quad r \to \infty. \tag{7}$$

A_ϕ must satisfy Laplace's equation, so the problem is now identical to that of example 3.8. The solution is then

$$A_\phi = \tfrac{1}{2} B_0 \left(r - \frac{a^3}{r^2} \right) \sin\theta. \tag{8}$$

The normal derivative of this is just the derivative with respect to r. Evaluated at the surface of the sphere, this is

$$-\mu_0 \vec{K} = \left.\frac{\partial \vec{A}}{\partial r}\right|_{r=a} = \frac{3}{2} B_0 \sin\theta \hat{\phi}. \tag{9}$$

An alternative solution is to consider the magnetic field inside a rotating spherical shell. This is given in equation 5.68 of the text:

$$\vec{B} = \frac{2}{3} \mu_0 \sigma \omega a \hat{z}. \tag{10}$$

We can think of this as the field due to a surface current on the shell. In order to have a vanishing magnetic field inside the shell, we need this magnetic field (the one due to the surface current) to cancel the constant external field, $\vec{B} = B_0 \hat{z}$. In other words, we need

$$\frac{2}{3} \mu_0 \sigma \omega a \hat{z} = -B_0 \hat{z} \quad \Rightarrow \quad \sigma \omega a = -\frac{3}{2} \frac{B_0}{\mu_0}. \tag{11}$$

The surface current is given by $\vec{K} = \sigma \vec{v}$, so we find

$$\vec{K} = \sigma \omega a \sin\theta \hat{\phi} = -\frac{3 B_0}{2\mu_0} \sin\theta \hat{\phi}. \tag{12}$$

This agrees with our previous answer.

7.48

The equation describing cyclotron motion (eqn 5.3) is

$$mv = e B_o s. \tag{13}$$

m is the mass of the electron, v is its tangential velocity, e is its electric charge, B_o is the magnetic field at the orbit, and s is the radius of the circular orbit. If we differentiate both sides of this equation with respect to time, we find

$$m\frac{dv}{dt} = es\frac{\partial B_o}{\partial t} + eB_o\frac{ds}{dt}. \tag{14}$$

Rearranging this a bit, we have

$$\frac{ds}{dt} = \frac{m}{eB_o}\frac{dv}{dt} - \frac{s}{B_o}\frac{\partial B_o}{\partial t}. \tag{15}$$

We would like to write the right hand side of this equation in terms of the average magnetic field over the area enclosed by the orbit, $\langle B\rangle$.

To do this, we need to consider the electric field generated by the changing magnetic field. From Faraday's law, we have

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}. \tag{16}$$

If we integrate (3) over the area enclosed by the orbit, we find

$$\frac{\partial}{\partial t}\int d\vec{a}\cdot\vec{B} = \pi s^2\frac{\partial}{\partial t}\langle B\rangle = -\int d\vec{a}\cdot\nabla\times\vec{E} = -\oint \vec{E}_o\cdot d\vec{\ell} = 2\pi s E_o. \tag{17}$$

$\vec{E}_o$ is the electric field at the orbit. It is tangential to the orbit and accelerates the electron according to

$$m\frac{d\vec{v}}{dt} = \vec{F} = e\vec{E}_o. \tag{18}$$

In (5), we have chosen $d\vec{a}$ to be parallel to the z component of $\vec{B}$ (in fig 7.52, this means the $-z$ direction), so $\vec{E}_o$ is antiparallel to $d\vec{\ell}$, hence the extra minus sign. From (5) and (6), we then see that

$$\frac{m}{e}\frac{dv}{dt} = \tfrac{1}{2}s\frac{\partial}{\partial t}\langle B\rangle. \tag{19}$$

Plugging this into (3), we get

$$\frac{ds}{dt} = \frac{s}{B_o}\frac{\partial}{\partial t}\left(\tfrac{1}{2}\langle B\rangle - B_o\right). \tag{20}$$

We see that choosing the magnetic field such that $\langle B\rangle = 2B_o$ ensures that the orbital radius s is independent of time.

7.50

The changing current in the solenoid induces an emf $\mathcal{E}$ around the square loop given by $\mathcal{E} = -d\Phi/dt$, where Φ is the magnetic flux through the square loop. This is the same as the flux through the cross section of the solenoid since the magnetic field is zero outside the solenoid. The flux through the cross section of the solenoid is given to us as αt, so $\mathcal{E} = -\alpha$.

The induced emf drives a current around the square loop. From figure 7.53 and from Lenz's law, we see that the current travels counterclockwise through the square loop. By assumption, no current travels through either voltmeter. The induced current is given by $I = \mathcal{E}/(R_1 + R_2)$. Since the current travels counterclockwise, the voltage across R_1 is positive, while the voltage across R_2 is negative. The magnitude of the voltage across each resistor is given by $V = IR$, so $V_1 = \alpha R_1/(R_1 + R_2)$ and $V_2 = -\alpha R_2/(R_1 + R_2)$.