Strategies for Biped Gymnastics

Zexiang Li
Leg Lab, MIT. and
Dept. of Math.
University of California at Berkeley

1 Introduction

Recently, there has been a great deal of interest in nonholonomic systems. For example, R. Brockett ([1]) studied the theory and control for a class of motors designed by Pansonic Company ([2]). Relying on the principle of ***holonomy*** (See ([3]), this class of motors could excel, in terms of mass-to-torque ratio, the traditional D.C. motors by several orders of magnitude. T. Kane and M. Scher ([4]) looked at the falling cats problem. They explained how falling cats can manage to land on their feet even released from complete rest while upside-down; C. Frohlich ([5]) examined how a diver or a gymnast can do rotational maneuvers in midair without violating angular momentum conservation; M. Berry ([6]) studied the phase shifting problem of a bead moving in a slowly rotating hoop. He established a general principle, known as the holonomy principle, underling all the previous problems. J. Marsden, R. Montgomery and R. Ratiu ([7]) presented a unified framework for systematically studying these problems.

In robotics research, an interesting area is legged locomotion vehicles. Several working systems have been reported in ([8], [9]). In particular, Raibert's biped machine is capable of performing forward somersault. Forward somersault, or flip, is a gymnastic maneuver in which the performer runs forward, springs off the ground with booth feet, rotates the body forward through 360^o degrees, and lands in a balanced posture on one or both feet. Human gymnasts can do a forward flip as an isolated maneuver or as part of a floor routine in which the flip is proceeded and followed by other maneuvers. The best gymnasts can do double and even triple flips. The average teenager can learn to do a forward flip in a few weeks with proper coaching and practice.

In this report we study the dynamics of a planar cat (see Figure 1) and then propose several possible strategies that a cat can use to perform forward somersault. The notation used here follows closely that of [10].

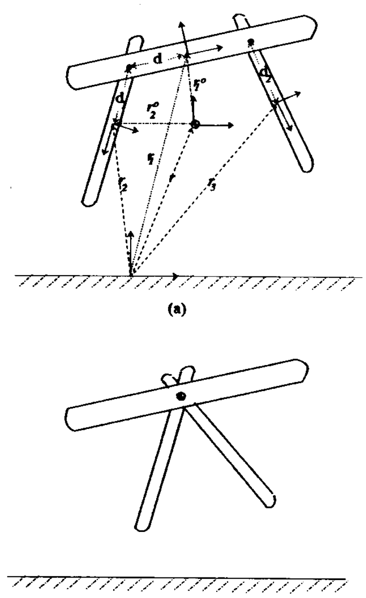

Figure 1: (a) A planar cat, (b) a symmetrical cat.

2 Dynamics

Consider a free-fall configuration of the planar cat shown in Figure 1a. A special case of this, which we shall call the symmetrical cat, is shown in Figure 1b. Let C_r be the inertia reference frame, C_o the frame which is fixed to the mass center of the biped and has the same orientation as the inertia frame. Let $C_i, i = 1,2,3$, be the frame fixed to the mass center of body i. A configuration of C_i relative to C_r is described by an element $(r_i, R(\theta_i))$ of the Euclidean group $SE(2)$ of $\Re^2$. Note that $r_i \in \Re^2$ describes the position of, and

$$R(\theta_i) = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix} \in SO(2)$$

describes the orientation of, C_i. Let r_i^o be the position vector of C_i relative to C_o. The following procedure derives the total kinetic energy of the system. First, the kinetic energy of body i is given by the following integral

$$K_i = \frac{1}{2}\int_{\mathcal{B}_i} \rho(X_i)\|\frac{d}{dt}(R(\theta_i)X_i + r_i)\|^2 dX_i \quad (1)$$

where $\rho(X_i)$ is the mass density function, and $\mathcal{B}_i$ is the set of $\Re^2$ occupied by body i.

Expanding (1) gives

$$K_i = \frac{1}{2}\int_{\mathcal{B}_i} \rho(X_i)\left(\|\dot{R}(\theta_i)X_i\|^2 + 2\langle\dot{R}(\theta_i)X_i, \dot{r}_i\rangle + \|\dot{r}_i\|^2\right) dX_i \quad (2)$$

Since C_i is positioned at the mass center of body i, the second term in (2) vanishes, i.e.,

$$\int_{\mathcal{B}_i} \rho(X_i)\langle\dot{R}(\theta_i)X_i, \dot{r}_i\rangle dX_i = \langle\dot{R}(\theta_i)\int_{\mathcal{B}_i} \rho(X_i)X_i dX_i, \dot{r}_i\rangle = 0.$$

Write $X_i = (x_i, y_i)^T$, and let

$$I_i = \int_{\mathcal{B}_i} \rho(X_i)(x_i^2 + y_i^2)dx_i dy_i$$

be the moment of inertia about the $Z-$ axis, and

$$m_i = \int_{\mathcal{B}_i} \rho(X_i)dx_i dy_i$$

the mass of body i. Then, the kinetic energy integral (2) can be further simplified to

$$K_i = \frac{1}{2}I_i w_i^2 + \frac{1}{2}m_i\|\dot{r}_i\|^2. \quad (3)$$

where $w_i = \dot{\theta}_i$ is the angular velocity of body i.

Summing (3) over i yields the total kinetic energy of the system

$$K = \frac{1}{2}\sum_{i=1}^{i=3} I_i w_i^2 + \frac{1}{2}\sum_{i=1}^{i=3} m_i\|\dot{r}_i\|^2. \quad (4)$$

In order to decouple rotational motion from translational motion, we need to express the kinetic energy (4) as the sum of a translational component of the mass center and a rotational component about the mass center of the system. For this, Figure 1 reveals the following kinematic relations.

$$r_i = r + r_i^o, \; i = 1, 2, 3. \quad (5)$$

Substituting (5) into the second component of (4) yields

$$\frac{1}{2}\sum_{i=1}^{i=3} m_i\|\dot{r}_i\|^2 = \frac{1}{2}m\|\dot{r}\|^2 + \frac{1}{2}\sum_{i=1}^{i=3} m_i\|\dot{r}_i^o\|^2 \quad (6)$$

where we have used the fact that

$$\langle\sum_{i=1}^{i=3} m_i\dot{r}_i^o, \dot{r}\rangle = 0$$

as C_b is positioned at the mass center of the system, and

$$m = \sum_{i=1}^{i=3} m_i$$

is the total mass.

Furthermore, the following kinematic relations are straightforward from Figure 1.

$$r_2 = r_1 + R(\theta_1)\begin{bmatrix} -d \\ 0 \end{bmatrix} + R(\theta_2)\begin{bmatrix} d_1 \\ 0 \end{bmatrix} \quad (7)$$

and

$$r_3 = r_1 + R(\theta_1)\begin{bmatrix} d \\ 0 \end{bmatrix} + R(\theta_3)\begin{bmatrix} d_2 \\ 0 \end{bmatrix}. \quad (8)$$

Substituting (7) and (8) into the following equation

$$r_1^o = r_1 - r = r_1 - \sum_{i=1}^{i=3}\epsilon_i r_i, \;\; \epsilon_i = m_i/m,$$

yields

$$\begin{aligned} r_1^o &= (1-\epsilon_1)r_1 - \epsilon_2 r_2 - \epsilon_3 r_3 \quad (9)\\ &= (\epsilon_2 - \epsilon_3)R(\theta_1)\begin{bmatrix} d \\ 0 \end{bmatrix} - \epsilon_2 R(\theta_2)\begin{bmatrix} d_1 \\ 0 \end{bmatrix} - \epsilon_3 R(\theta_3)\begin{bmatrix} d_2 \\ 0 \end{bmatrix} \end{aligned}$$

We can then write (see Figure 1)

$$\begin{aligned} r_2^o &= r_1^o + R(\theta_1)\begin{bmatrix} -d \\ 0 \end{bmatrix} + R(\theta_2)\begin{bmatrix} d_1 \\ 0 \end{bmatrix} \\ &= (\epsilon_2 - \epsilon_3 - 1)R(\theta_1)\begin{bmatrix} d \\ 0 \end{bmatrix} \\ &+ (1-\epsilon_2)R(\theta_2)\begin{bmatrix} d_1 \\ 0 \end{bmatrix} - \epsilon_3 R(\theta_3)\begin{bmatrix} d_2 \\ 0 \end{bmatrix} \quad (10) \end{aligned}$$

and

$$\begin{aligned} r_3^o &= r_1^o + R(\theta_1)\begin{bmatrix} d \\ 0 \end{bmatrix} + R(\theta_3)\begin{bmatrix} d_2 \\ 0 \end{bmatrix} \\ &= (1 + \epsilon_2 - \epsilon_3)R(\theta_1)\begin{bmatrix} d \\ 0 \end{bmatrix} - \epsilon_2 R(\theta_2)\begin{bmatrix} d_1 \\ 0 \end{bmatrix} \\ &+ (1-\epsilon_3)R(\theta_3)\begin{bmatrix} d_2 \\ 0 \end{bmatrix} \quad (11) \end{aligned}$$

Thus, differentiating r_i^o with respect to time t, yields

$$\begin{aligned} \|\dot{r}_1^o\|^2 &= (\epsilon_2 - \epsilon_3)^2 d^2 w_1^2 + \epsilon_2^2 d_1^2 w_2^2 + \epsilon_3^2 d_2^2 w_3^2 \\ &- 2\epsilon_2(\epsilon_2 - \epsilon_3)dd_1\cos\theta_{21}w_1 w_2 \quad (12)\\ &- 2\epsilon_3(\epsilon_2 - \epsilon_3)dd_2\cos\theta_{31}w_1 w_3 + 2\epsilon_2\epsilon_3 d_1 d_2\cos\theta_{32}w_2 w_3 \end{aligned}$$

where $\theta_{ij} = \theta_i - \theta_j$ denotes the relative orientation of body i with respect to body j, and

$$\begin{aligned} \|\dot{r}_2^o\|^2 &= (1 + \epsilon_3 - \epsilon_2)^2 d^2 w_1^2 + (1-\epsilon_2)^2 d_1^2 w_2^2 + \epsilon_3^2 d_2 w_3^2 \\ &+ 2(\epsilon_2 - \epsilon_3 - 1)(1-\epsilon_2)dd_1\cos\theta_{21}w_1 w_2 \\ &- 2\epsilon_3(\epsilon_2 - \epsilon_3 - 1)dd_2\cos\theta_{31}w_3 w_1 \\ &- 2\epsilon_3(1-\epsilon_2)d_1 d_2\cos\theta_{32}w_2 w_3; \quad (13) \end{aligned}$$

$$\begin{aligned} \|\dot{r}_3^o\|^2 &= (1 + \epsilon_2 - \epsilon_3)^2 d^2 w_1^2 + \epsilon_2^2 d_1^2 w_2^2 + (1-\epsilon_3)^2 d_2^2 w_3^2 \\ &- 2(1 + \epsilon_2 - \epsilon_3)\epsilon_2 dd_1\cos\theta_{21}w_1 w_2 \\ &+ 2(1 + \epsilon_2 - \epsilon_3)(1-\epsilon_3)dd_2\cos\theta_{31}w_1 w_3 \\ &- 2\epsilon_2(1 - \epsilon_3 d_1 d_2\cos\theta_{32}w_2 w_3. \quad (14) \end{aligned}$$

Combining (12), (13), (14), and (6) with (4), and rearranging the results, we can write the total kinetic energy in the form

$$K = \frac{1}{2}w^T J w + \frac{1}{2}m\|\dot{r}\|^2 \quad (15)$$

where $w = (w_1, w_2, w_3)^T$ is the angular velocity vector of the system, and J is the symmetric moment of inertia matrix, given by

$$J = \begin{bmatrix} \bar{I}_1 & \lambda_{12} & \lambda_{13} \\ \lambda_{12} & \bar{I}_2 & \lambda_{23} \\ \lambda_{13} & \lambda_{23} & \bar{I}_3 \end{bmatrix}$$

where

$$\tilde{I}_1 = I_1 + \{(\epsilon_2-\epsilon_3)^2 + (1+\epsilon_3-\epsilon_2)^2 + (1+\epsilon_2-\epsilon_3)^2\}d^2$$
$$\tilde{I}_2 = I_2 + \{\epsilon_2^2 + (1-\epsilon_2)^2 + \epsilon_2^2\}d_1^2$$
$$\tilde{I}_3 = I_3 + \{2\epsilon_3^2 + (1-\epsilon_3)^2\}d_2^2$$

and

$$\lambda_{12} = \{\epsilon_2(\epsilon_2-\epsilon_3) + (\epsilon_2-\epsilon_3-1)(1-\epsilon_2) - (1+\epsilon_2-\epsilon_3)\epsilon_2\}\, dd_1\cos\theta_{21}$$
$$\lambda_{13} = \{-\epsilon_3(\epsilon_2-\epsilon_3) - \epsilon_3(\epsilon_2-\epsilon_3-1) + (1+\epsilon_2-\epsilon_3)(1-\epsilon_3)\}\, dd_2\cos\theta_{31}$$
$$\lambda_{23} = \{\epsilon_2\epsilon_3 - \epsilon_3(1-\epsilon_2) - \epsilon_2(1-\epsilon_3)\}\, d_1d_2\cos\theta_{32}$$

Note that for the symmetrical cat, i.e., $d = 0, d_2 = d_3 \triangleq d_l$, $\epsilon_2 = \epsilon_3 \triangleq \epsilon_l$, entries of the moment of inertia matrix get simplified to

$$\tilde{I}_1 = I_1,\ \tilde{I}_2 = I_2 + [2\epsilon_l^2 + (1-\epsilon_l)^2]d_l^2,\ \tilde{I}_3 = I_3 + [2\epsilon_l^2 + (1-\epsilon_l)^2]d_l^2$$
$$\lambda_{12} = 0,\ \lambda_{13} = 0,\ \lambda_{23} = \epsilon_l(3\epsilon_l - 2)d_l^2\cos\theta_{32}.$$

The potential energy of the cat is

$$V = mgr_z \tag{16}$$

and the Lagrangian of the cat is

$$L = \frac{1}{2}w^TJw + \frac{1}{2}m\|\dot{r}\|^2 - mgr_z. \tag{17}$$

Once the cat is airborne, the only external force acting on is the gravity force. The legs, on the other hand, are acted on by internal torques from the body, namely, torques applied to the motors located at the hinges. Let $\hat{g} = \begin{bmatrix} 0 \\ g \end{bmatrix}$ be the gravitational force vector, and $\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$ be the input torque vector. Then, the Lagrangian equations of motion of the system are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = P^T\tau,\ \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix},\ P^T = \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{18}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = -m\hat{g}. \tag{19}$$

Using (17) in the above equations, we have

$$J\ddot{\theta} + N(\theta,\dot{\theta}) = P^T\tau, \tag{20}$$
$$\ddot{r} = \hat{g}. \tag{21}$$

where

$$N(\theta,\dot{\theta}) = \{\frac{d}{dt}J\}w - \begin{bmatrix} w_1\frac{\partial}{\partial\theta_1}(w_2\lambda_{12} + w_3\lambda_{13}) \\ w_2\frac{\partial}{\partial\theta_2}(w_1\lambda_{12} + w_3\lambda_{23}) \\ w_3\frac{\partial}{\partial\theta_3}(w_1\lambda_{13} + w_2\lambda_{23}) \end{bmatrix}.$$

$N(\theta,\dot{\theta})$ is the component that contains the centrifugal and Coriolis force terms.

Remark 2.1 The moment of inertia matrix, J, depends on $\theta_{ij}, i > j$, only, and $N(\theta,\dot{\theta})$ is a function of $\theta_{ij}, i > j$, and w.

It is important to observe that translational motion is decoupled from rotational motion. Integrating the translational component, yields

$$r_x(t) = v_x(t_0)(t-t_0) + r_x(t_0);$$
$$r_y(t) = v_y(t_0)(t-t_0) + r_y(t_0) - \frac{1}{2}g(t-t_0)^2.$$

where t_0 is time when the cat takes off, $(r_x(t_0), r_y(t_0))$ is the position of the mass center at take-off, and $(v_x(t_0), v_y(t_0))$ is the take-off velocity. The flight time, $T = (t_1 - t_0)$, can be obtained from the above equation by setting $r_y(t_1) = r_y(t_0)$, i.e.,

$$\boxed{T = \frac{2v_y(t_0)}{g}.}$$

In order to land safely, the cat has to complete the maneuver before T. We discuss how this can be done in the next section.

Note that for symmetric cat, the equations of motion (the rotational component) becomes

$$J\ddot{\theta} + N(\theta,\dot{\theta}) = P^T\tau, \tag{22}$$

where

$$N(\theta,\dot{\theta}) = \begin{bmatrix} dd_1\sin\theta_{21}\,\dot{\theta}_2^2 - dd_2\sin\theta_{31}\,\dot{\theta}_3^2 \\ -dd_1\sin\theta_{21}\,\dot{\theta}_1^2 - \epsilon(3\epsilon_l - 2)d_1d_2\sin\theta_{32}\,\dot{\theta}_3^2 \\ dd_2\sin\theta_{31}\dot{\theta}_1^2 + \epsilon_l(3\epsilon_l - 2)d_1d_2\sin\theta_{32}\dot{\theta}_2^2 \end{bmatrix}.$$

3 Maneuvering Strategies

In this section, we discuss possible strategies which a cat can use to perform forward or backward somersault.

Since rotational motion and translational motion are decoupled from each other, the configuration space in consideration will be the orientation space, $Q = S^1 \times S^1 \times S^1$, of the system.

The tangent bundle of Q, denoted as TQ, consists of pairs $\{(\theta, w), \theta \in Q, w \in T_\theta Q\}$, whereas a configuration $\theta = (\theta_1, \theta_2, \theta_3) \in Q$ gives the orientation of the cat relative to the inertia reference frame.

The Lagrangian defines a function on TQ:

$$L : TQ \longrightarrow \Re, (\theta,\dot{\theta}) \longmapsto \frac{1}{2}\dot{\theta}^TJ(\theta)\dot{\theta}.$$

When the cat is in the air, its angular momentum is conserved. This conservation law arises from the rotational symmetry of the system. In other words, let $G = S^1$ be the rotational group of $\Re^2$, then the configuration space is acted on by G by rotation as follows:

$$G \times Q \longrightarrow Q : (\alpha, (\theta_1,\theta_2,\theta_3)) \longmapsto (\theta_1 + \alpha, \theta_2 + \alpha, \theta_3 + \alpha) \triangleq \theta + \alpha.$$

Since $J(\theta)$ depends on $\theta_{ij}, i < j$, only, the Lagrangian is invariant under this action. The Lie algebra, $\mathcal{G}$, of G can be identified with the real line $\Re$. For $\xi = 1 \in \mathcal{G}$, the infinitesimal generator $\xi_Q(\theta)$ is simply

$$\frac{d}{dt}(\theta_1 + t, \theta_2 + t, \theta_3 + t) = (1,1,1) \triangleq e.$$

The corresponding momentum μ for each $w \in T_\theta Q$ is given by the formula

$$\mu = e^TJ(\theta)w = [1,1,1]J(\theta)w. \tag{23}$$

To summarize, we have a configuration space $Q = S^1 \times S^1 \times S^1$, an action on Q by the rotational group G. To each $\theta \in Q$, the orbit, G_θ, of the action is defined as

$$G_\theta = \{\theta + \alpha, \alpha \in G\}$$

Notation 3.1 Let $\psi = \begin{bmatrix} \psi_2 \\ \psi_3 \end{bmatrix}$ be the set of hinge variables, i.e.,

$$\psi = \begin{bmatrix} \psi_2 \\ \psi_3 \end{bmatrix} = \begin{bmatrix} \theta_2 - \theta_1 \\ \theta_3 - \theta_1 \end{bmatrix}$$

and $M = S^1 \times S^2$ the hinge space.

There exists a natural projection from Q to M, given by

$$P : Q \longrightarrow M, \theta \longmapsto P\theta = \psi.$$

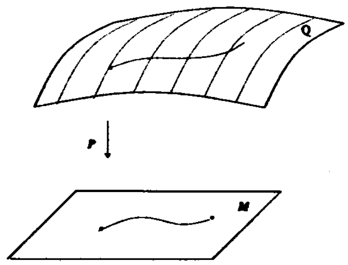

Figure 2: A global perspective of the system

For each $\psi \in M$, the preimage under P is the orbit G_θ, where $\theta = \begin{bmatrix} 0 \\ \psi \end{bmatrix}$. The triplet (Q, G, M) is called a *principal bundle*. A global perspective of the system is given by Figure 2.

Since the number of controls (2 here) is equal to the dimension of M, it is intuitive that a path in M should be fully controllable. This is indeed the case as the following theorem shows.

Theorem 3.1 *Let μ be the initial angular momentum, and $\psi_d(t) \in M, t \in [0, \infty)$, a desired trajectory in the hinge space. Then, there exists a choice of the control inputs $\tau(t) \in \Re^2, t \in [0, \infty)$, such that the true trajectory tracks the desired trajectory, i.e., the trajectory tracking error*

$$e_p(t) = \psi(t) - \psi_d(t)$$

converges to zero asymptotically as $t \to \infty$.

Remark 3.1 (1) The inertia matrix J depends on ψ only, and is nonsingular. (2) The centrifugal and Corolis force term N is a function of ψ and $\dot{\theta}$, i.e., $N = N(\psi, \dot{\theta})$.

Proof. By the above remark, multiplying Equation (20) by $J^{-1}(\psi)$, yields,

$$\ddot{\theta} + J^{-1}(\psi)N(\psi, \dot{\theta}) = J^{-1}(\psi)P^T\tau. \tag{24}$$

We can write

$$\dot{\theta} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \dot{\psi} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \dot{\theta}_1 \triangleq K\dot{\psi} + e\dot{\theta}_1. \tag{25}$$

Using (25) in the conservation of angular momentum Equation (23), we get

$$\mu = e^T J(\psi)K\dot{\psi} + e^T J(\psi)e\dot{\theta}_1. \tag{26}$$

Finally, we have

$$\dot{\theta}_1 = (e^T J(\psi)e)^{-1}(\mu - e^T J(\psi)K\dot{\psi}). \tag{27}$$

Consequently, the angular velocity vector w is a function of ψ, $\dot{\psi}$ and μ. That is,

$$w = K\dot{\psi} + e\{e^T J(\psi)e\}^{-1}\left\{\mu - e^T J(\psi)K\dot{\psi}\right\}, \text{ i.e., } w = w(\psi, \dot{\psi}, \mu). \tag{28}$$

Multiplying (24) by P and using (28) we get the dynamcis equation in the hinge space (or the reduced space).

$$\ddot{\psi} + PJ^{-1}(\psi)\, N(\psi, \dot{\psi}, \mu) = PJ^{-1}(\psi)P^T\tau, \;\; \psi = P\theta. \tag{29}$$

Observe now that P has rank 2 and J is nonsingular. This implies that

$$PJ^{-1}(\psi)P^T \triangleq J_M^{-1}(\psi)$$

is nonsingular.

We claim that the following control law realizes the desired trajectory $\psi_d(t), t \in [0, \infty)$.

$$\tau = J_M(\psi)\left\{\ddot{\psi}_d - K_v\dot{e}_p - K_p e_p\right\} + J_M(\psi)PJ^{-1}(\psi)\, N(\psi, \dot{\psi}, \mu), \tag{30}$$

where $K_v, K_p \in \Re^{2\times 2}$ are properly chosen velocity and position gains.

To see this, substituting (30) into the reduced dynamics equation (29), and after some algebra we have,

$$\ddot{e}_p + K_v\dot{e}_p + K_p e_p = 0. \tag{31}$$

This shows that e_p can be driven to zero by properly choosing K_v and K_p. □

We now discuss maneuvering strategies. Let

$$f(\psi) = [e^T J(\psi)e]^{-1} = [(\bar{I}_1 + \bar{I}_2 + \bar{I}_3) + 2\sum_{i<j}\lambda_{ij}(\psi)]^{-1},$$

$$g_2(\psi) = f(\psi)e^T J(\psi)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = f(\psi)(\bar{I}_2 + \lambda_{12} + \lambda_{23}),$$

$$g_3(\psi) = f(\psi)e^T J(\psi)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = f(\psi)(\bar{I}_3 + \lambda_{13} + \lambda_{23})$$

where we have used the fact that $\theta_{32} = \psi_3 - \psi_2$. Then, Equation (27) can be rewritten as

$$d\theta_1 = f(\psi)\mu dt - g_2(\psi)d\psi_2 - g_3(\psi)d\psi_3. \tag{32}$$

Remark 3.2 In non-linear control terminology, $f(\psi)\mu$ is called the *drifting vector field*, $g_2(\psi), g_3(\psi)$ the *control vector fields*.

Consider now a compact region Ω in M, with boundary $\delta\Omega$. Integrating (32) over $\delta\Omega$, and using Green's Theorem (assuming $\theta_1(0) = 0$), yields,

$$\theta_1(t) = \int f(\psi)\mu dt - \iint_\Omega \left(\frac{\partial g_3(\psi)}{\partial \psi_3} - \frac{\partial g_2(\psi)}{\partial \psi_2}\right) d\psi_2 d\psi_3. \tag{33}$$

The above equation indicates: (1) *the net rotation of the body is given by the superposition of the initial angular momentum term and the area integral in the hinge space*; (2) *Forward somersault (or backward somersault) amounts to having $\theta_1(t)$ undergo -2π (or $+2\pi$) rotation within the time interval* $[0, T]$.

Based on Equation (33), we arrive at the following possible maneuvering strategies for forward somersault:

Strategy A: Using the drifting term only to accomplish forward somersault. At take-off, creates a proper set of initial conditions $v_x(0), v_y(0), \psi(0), \theta_1(0), \mu < 0$, such that

$$|\theta_1(T) + 2\pi| = |f(\psi(0))\mu T + 2\pi| \le \epsilon,$$

where $\epsilon > 0$ is a safety factor. The legs are locked once the cat is airborne. In fact, Hodgins and Raibert([8]) have used this strategy to control the flipping of their biped with moderate degrees of success.

Strategy B: Using internal motion of the legs to accomplish forward somersault. At take-off, make the initial angular momentum μ be zero. Then, when the cat is airborne net body rotation is given by

$$\theta_1(t) = -\iint_\Omega \left(\frac{\partial g_3(\psi)}{\partial \psi_2} - \frac{\partial g_2(\psi)}{\partial \psi_3}\right) d\psi_2 d\psi_3, \;\; t \in [0, T].$$

Choose a closed trajectory $\psi_d(t) \in M, t \in [0, T]$, such that the above area integral gives -2π rotation. Finally, choose the hinge space control according to (30), with $\psi_d(t), t \in [0, T]$, as the desired trajectory. One can use the grid-based searching method proposed in ([11]), which respects hinge-torque constraints, to plan the desired $\psi_d(t)$. Note that a falling cat, which starts from an upside-down configuration with zero angular momentum, uses this strategy to land on her feet.

Strategy C: This combines Strategy A and Strategy B to accomplish forward somersault.

Step A: Creates a proper set of initial conditions at take-off: $v_x(0), v_y(0), \psi(0), \theta_1(0), \mu < 0$. If

$$|f(\psi(0))\mu T + 2\pi| < \epsilon$$

then, lock the legs once the cat is airborne and go to Step C. Else go to Step B.

Step B: (a). Plan a desired trajectory $\psi_d(t) \in M, t \in [0, T]$, so that $\theta_1(T)$ given from (33) is approximately -2π. (b). Choose the the hinge space control law according to (30) to realize the desired rotation. Go to Step C.

Step C: End.

While Strategy A is the simplest of all, it may risk failures due to the difficulties in creating exact initial conditions. On the other hand, Strategy B, which may be of great potentials in space applications, is not practically sound for legged robots. For example, for the biped built in Raibert's laboratory, only 27^o of body rotation can be realized using internal motion only, as the jumping height of the machine is limited. Thus, the best strategy to employ for legged robots is Strategy C. We plan to implement this strategy in Raibert's biped in the near future.

4 Conclusions

In this report, we have studied the dynamics of a planar cat, and have proposed a few strategies that a cat can use to perform forward or backawrd somersault. We plan to test these strategies in the real machines soon.

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