Lecture 11

Isothermal Sphere

Consider an ideal gas which is self-gravitating and isothermal

$$p = \frac{\rho k_B T}{m} \quad \text{ideal gas}$$

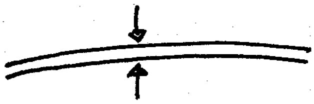
k_B Boltzmann constant
p pressure
T temperature
m mass per particle
ρ density

Equation of hydrostatic support:

(1) $$\frac{dp}{dr} = \frac{k_B T}{m}\frac{d\rho}{dr} = -\rho \frac{G M(r)}{r^2}$$

$M(r)$ total mass within r

proof:



dr dp net pressure on thin spherical shell

$$-\underbrace{\rho\, 4\pi r^2 dr}_{\text{mass of shell}} \frac{G \cdot M(r)}{r^2} \quad \text{gravitational force on spherical shell}$$

balancing force from pressure : $4\pi r^2 \cdot dp$

$$4\pi r^2 dp = -4\pi r^2 \rho \frac{G \cdot M(r)}{r^2} dr$$

(1) $$\frac{dp}{dr} = -\rho \frac{G \cdot M(r)}{r^2} \quad \checkmark$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \cdot \rho$$

multiply (1) by $\frac{r^2 m}{\rho k_B T}$ and take $\frac{d}{dr}$ on both sides:

$$r^2 \frac{1}{\rho} \frac{d\rho}{dr} = - \frac{m G M(r)}{k_B T}$$

$$\frac{d}{dr} \ln \rho = \frac{1}{\rho} \frac{d\rho}{dr}$$

$$(2) \quad \frac{d}{dr} \left(r^2 \frac{d \ln \rho}{dr} \right) = - \frac{G \cdot m}{k_B T} 4\pi r^2 \cdot \rho$$

Independently, introduce now a simple DF f for a steady-state spherical distribution:

$$(3) \quad f(\mathcal{E}) = \frac{\rho_1}{(2\pi \sigma^2)^{3/2}} e^{\frac{\mathcal{E}}{\sigma^2}} = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} \exp\left(\frac{\psi - \frac{1}{2} v^2}{\sigma^2} \right)$$

$$\mathcal{E} = -E + \phi_0 = \psi - \frac{1}{2} v^2$$

$$\psi = -\phi + \phi_0$$

$\mathcal{E} > 0$ for bound particles

vanishes at infinity

$$\rho = 4\pi G \int f \, d^3V$$

$$\rho = \rho_1 \, e^{\frac{\psi}{\sigma^2}}$$

(4) $$\frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{d\psi}{dr}\right) = -4\pi G \cdot \rho$$ Poisson Equation

(5) $$\frac{d}{dr}\left(r^2 \frac{d \ln \rho}{dr}\right) = -\frac{4\pi G}{\sigma^2} r^2 \rho$$ using $\rho = \rho_1 e^{\frac{\psi}{\sigma^2}}$

choosing $\sigma^2 = \frac{k_B T}{m}$ Eq. 5 becomes identical to Eq. 2 of isothermal self-gravitating ideal gas

Physical explanation: The distribution of velocities at each point in the stellar-dynamical isothermal system (sphere) is the Maxwell distribution

$$F(v) = N e^{-\frac{1}{2}\frac{v^2}{\sigma^2}}.$$

Kinetic theory, however, tells us that this is also the equilibrium Maxwell-Boltzmann distribution which would emerge if the stars were allowed to bounce elastically off each other like the molecules of a gas. Therefore, if the DF of a system is given by Eq. 3 it is a matter of indifference whether the particles of the system collide with one another, or not.

The mean-square speed of stars at a point in the isothermal sphere is

$$\overline{v^2} = \frac{\int_0^\infty \exp\left(\frac{\psi - \frac{1}{2}v^2}{\sigma^2}\right) v^4 \, dv}{\int_0^\infty \exp\left(\frac{\psi - \frac{1}{2}v^2}{\sigma^2}\right) \cdot v^2 dv} = 2\sigma^2 \frac{\int_0^\infty x^4 e^{-x^2} dx}{\int_0^\infty x^2 e^{-x^2} dx} = 3\sigma^2$$

$\overline{v^2} = 3\sigma^2$ independent of position

The dispersion in any one component of velocity, for example, $\left(\overline{v_r^2}\right)^{\frac{1}{2}}$ is equal to σ

It is easy to find one solution of Eq. 5:

$\rho = C \cdot r^{-b}$ Ansatz

$$\frac{d}{dr} \ln\rho = -b \frac{1}{r}$$

$$\frac{d}{dr}\left(r^2 \frac{d}{dr} \ln\rho\right) = -b \quad \longrightarrow \quad -\frac{4\pi G}{\sigma^2} C \cdot r^{2-b}$$

$$b = 2$$

$$\rho(r) = \frac{\sigma^2}{2\pi G \cdot r^2} \qquad C = \frac{2\sigma^2}{4\pi G}$$

singular isothermal sphere

We want solutions which are well behaved at origin.

$$\tilde{\rho} = \frac{\rho}{\rho_0}, \quad \tilde{r} = \frac{r}{r_0} \quad \text{rescaled variables}$$

$$r_0 = \sqrt{\frac{9\sigma^2}{4\pi G \cdot \rho_0}}$$

King radius where density falls to $0.5013\,\rho_0$

ρ_0 central density

Eq. 5 becomes

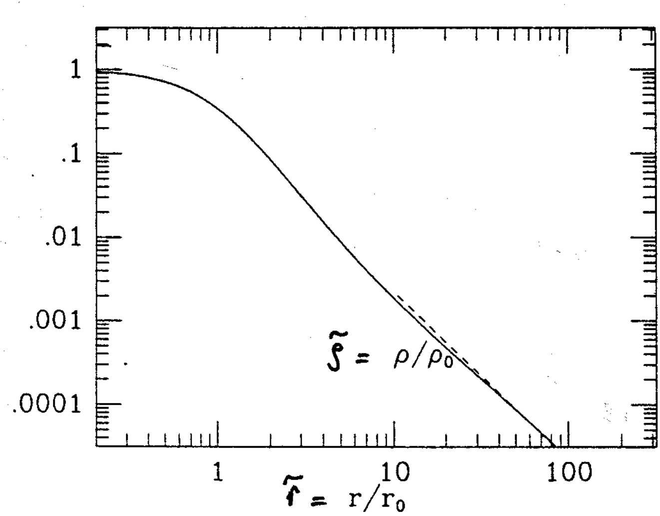
$$\frac{d}{d\tilde{r}}\left(\tilde{r}^2 \frac{d\ln\tilde{\rho}}{d\tilde{r}}\right) = -9\,\tilde{r}^2\tilde{\rho} \tag{6}$$

or

$$\frac{d}{d\tilde{r}}\left[\tilde{r}^2 \frac{d(\psi/\sigma^2)}{d\tilde{r}}\right] = -9\,\tilde{r}^2 \exp\left[\frac{\psi(r)-\psi(0)}{\sigma^2}\right] \tag{7}$$

numerical integration with $\tilde{\rho}(0) = 1$ and $\frac{d\tilde{\rho}}{d\tilde{r}} = 0$

boundary conditions:



dashed line is the density profile of singular isothermal sphere

Total mass is infinite

The circular speed at r is given by

$$v_c^2(r) = \frac{G \cdot M(r)}{r} \qquad (8)$$

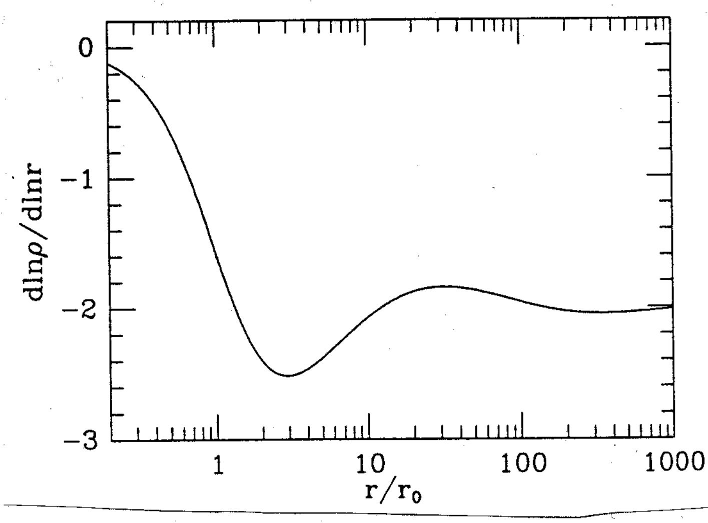
From Eq. (1)

$$\sigma^2 \frac{d\rho}{dr} = -\rho \frac{GM}{r^2} = -\rho v_c^2 \frac{1}{r}$$

$$-\sigma^2 \, d\ln\rho = v_c^2 \, d\ln r$$

$$v_c^2 = -\sigma^2 \frac{d\ln\rho}{d\ln r}$$

numerically :



← −2 asymptotically

for large $\frac{r}{r_0}$ $\quad v_c = \sqrt{2}\,\sigma$

We would like to modify now isothermal sphere in a minimal fashion to make the total mass finite

King Model

$$f_K(\mathcal{E}) = \begin{cases} \rho_1 (2\pi\sigma^2)^{-\frac{3}{2}} \left(e^{\frac{\mathcal{E}}{\sigma^2}} - 1 \right) & \mathcal{E} > 0 \\ 0 & \mathcal{E} \leq 0 \end{cases} \tag{9}$$

$$\rho_K(\psi) = \frac{4\pi\rho_1}{(2\pi\sigma^2)^{3/2}} \int_0^{\sqrt{2\psi}} \left[\exp\left(\frac{\psi - \frac{1}{2}v^2}{\sigma^2} \right) - 1 \right] v^2 dv$$

$$= \rho_1 \left[e^{\frac{\psi}{\sigma^2}} \cdot \mathrm{erf}\left(\frac{\sqrt{\psi}}{\sigma} \right) - \sqrt{\frac{4\psi}{\pi\sigma^2}} \left(1 + \frac{2\psi}{3\sigma^2} \right) \right] \tag{10}$$

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$

$\mathrm{erf}(0) = 0$

$\mathrm{erf}(\infty) = 1$

$\mathrm{erf}(-z) = -\mathrm{erf}(z)$

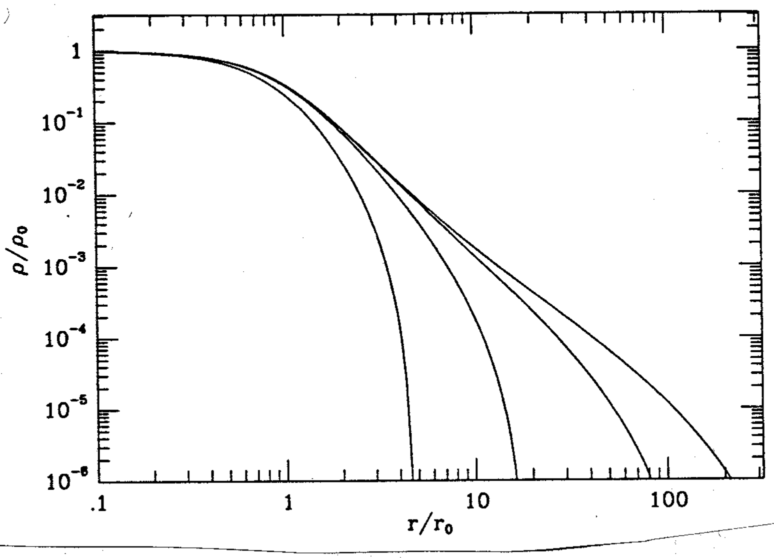
$$\lim_{x \to \infty} \left(1 - \mathrm{erf}(x) \right) = \frac{e^{-x^2}}{\sqrt{\pi}\, x}$$

Poisson equation:

$$\frac{d}{dr}\left(r^2 \frac{d\psi}{dr} \right) = -4\pi G \cdot \rho_1 \cdot r^2 \left[e^{\frac{\psi}{\sigma^2}} \cdot \mathrm{erf}\left(\frac{\sqrt{\psi}}{\sigma} \right) - \sqrt{\frac{4\psi}{\pi\sigma^2}} \left(1 + \frac{2\psi}{3\sigma^2} \right) \right] \tag{11}$$

$$\left. \begin{array}{l} \psi(0) \\ \dfrac{d\psi}{dr} = 0 \end{array} \right\} \text{boundary conditions}$$

King models form a single sequence as a function of $\psi(0)/\sigma^2$. In the limit $\psi(0)/\sigma^2 \to \infty$, the sequence goes over into the isothermal sphere.



$$\frac{\psi(0)}{\sigma^2} = 12,\ 9,\ 6,\ 3$$

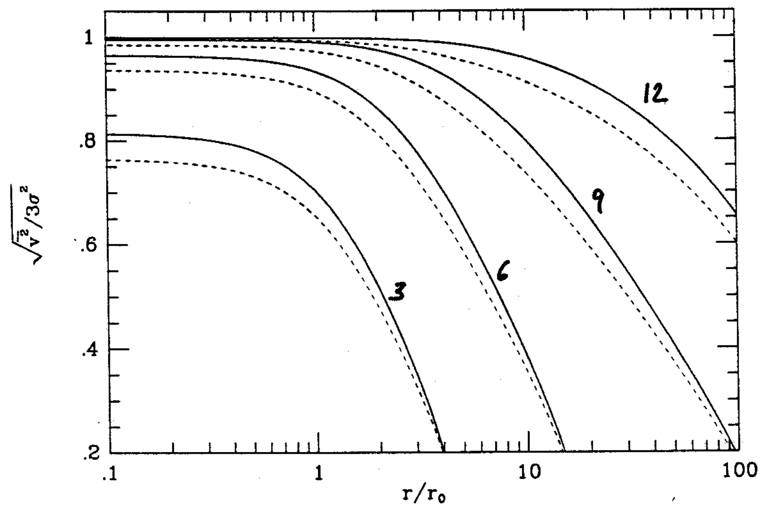
sequence

The parameter σ is not exactly the actual velocity dispersion $(\overline{v^2})^{\frac{1}{2}}$ of the stars.

$$\overline{v}^2 = 3\,\overline{v_r^2}$$

$$\overline{v^2}(r) = \frac{J_2}{J_0} \qquad J_n = \int_0^{\sqrt{2\psi}} \left[\exp\left(\frac{\psi - \frac{1}{2}v^2}{\sigma^2}\right) - 1\right] v^{n+2}\, dv$$

numerical velocity profile of King sequence:



$$\frac{\psi(0)}{\sigma^2} = 12,\ 9,\ 6,\ 3$$