Black Holes (Ph 161)

An introduction to General Relativity.

Lecture II

Ph 161 Black Holes

Homework Assignment 2

Due Tuesday, February 7, 2006

This should be your own work; do not copy problem solutions.

(1.) Write a few paragraphs discussing the Equivalence Principle and its relationship to geometry. Specifically, describe the experiments which test the relative acceleration of different bodies in a gravitational field (Eötvös experiments) and discuss what it is they measure, how they measure it, and their current precision. Then explain the *geometric* significance of the idea that "everything falls (accelerates)at the same rate in a gravitational field," and the analogy to the geometry on a 2-dimensional curved surface.

(2.) Give an argument as to why there can be no *global* inertial, Minkowski coordinate systems in the presence of a gravitational field.

(3.) With a coordinate transformation from locally inertial (locally Minkowski) coordinates $\{\xi^\alpha\}$ to a laboratory coordinate system $\{x^\mu\}$ (i.e., one at rest on the surface of the earth) derive the geodesic equations. (This is indeed exactly what we did in class!)

(4.) Hartle Chapter 7: problem 7.

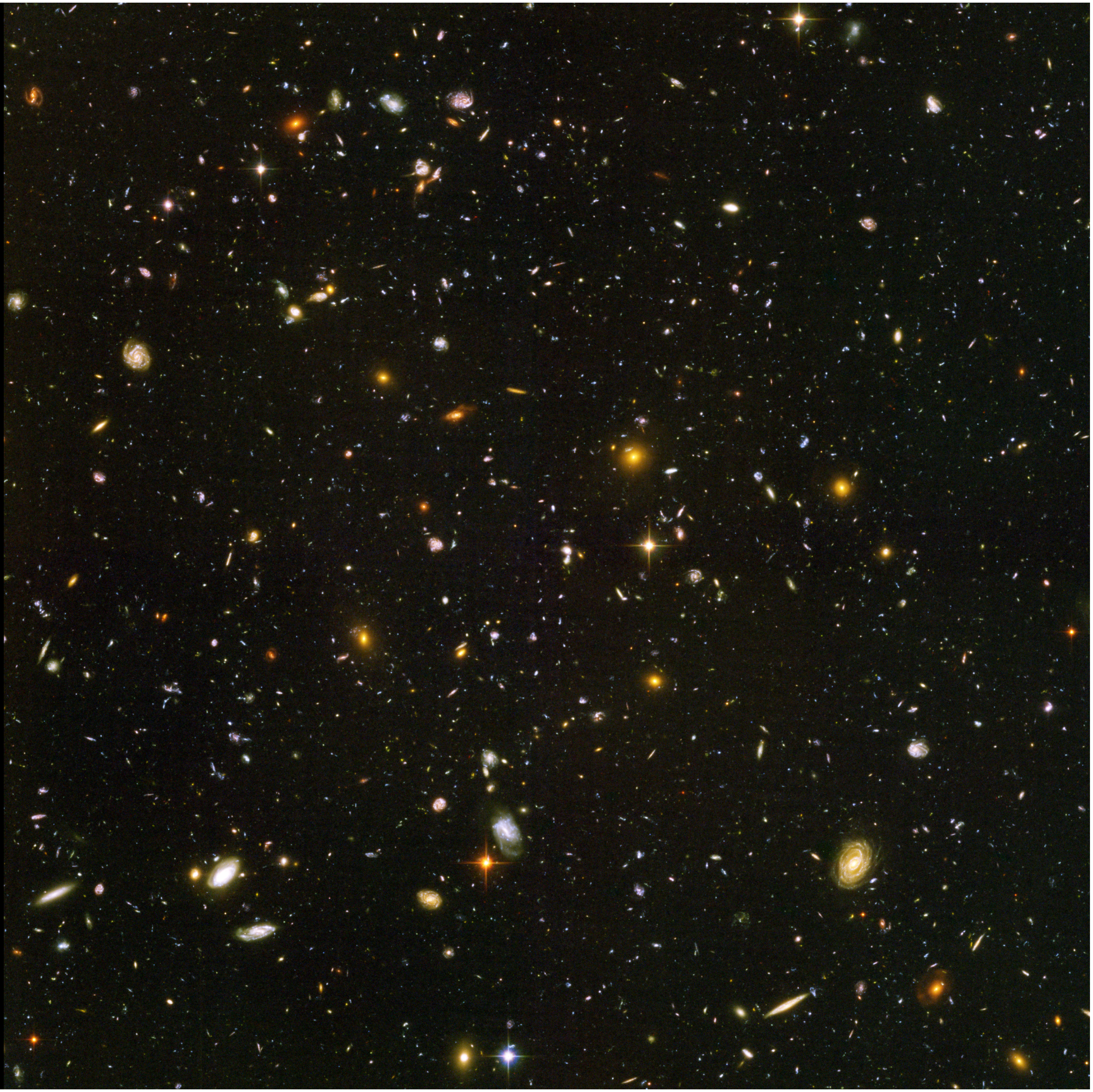
Read Chapters 6 and 7 in Hartle's book.

A quick and dirty tour of all of the whole universe

- the large scale structure/evolution of spacetime!

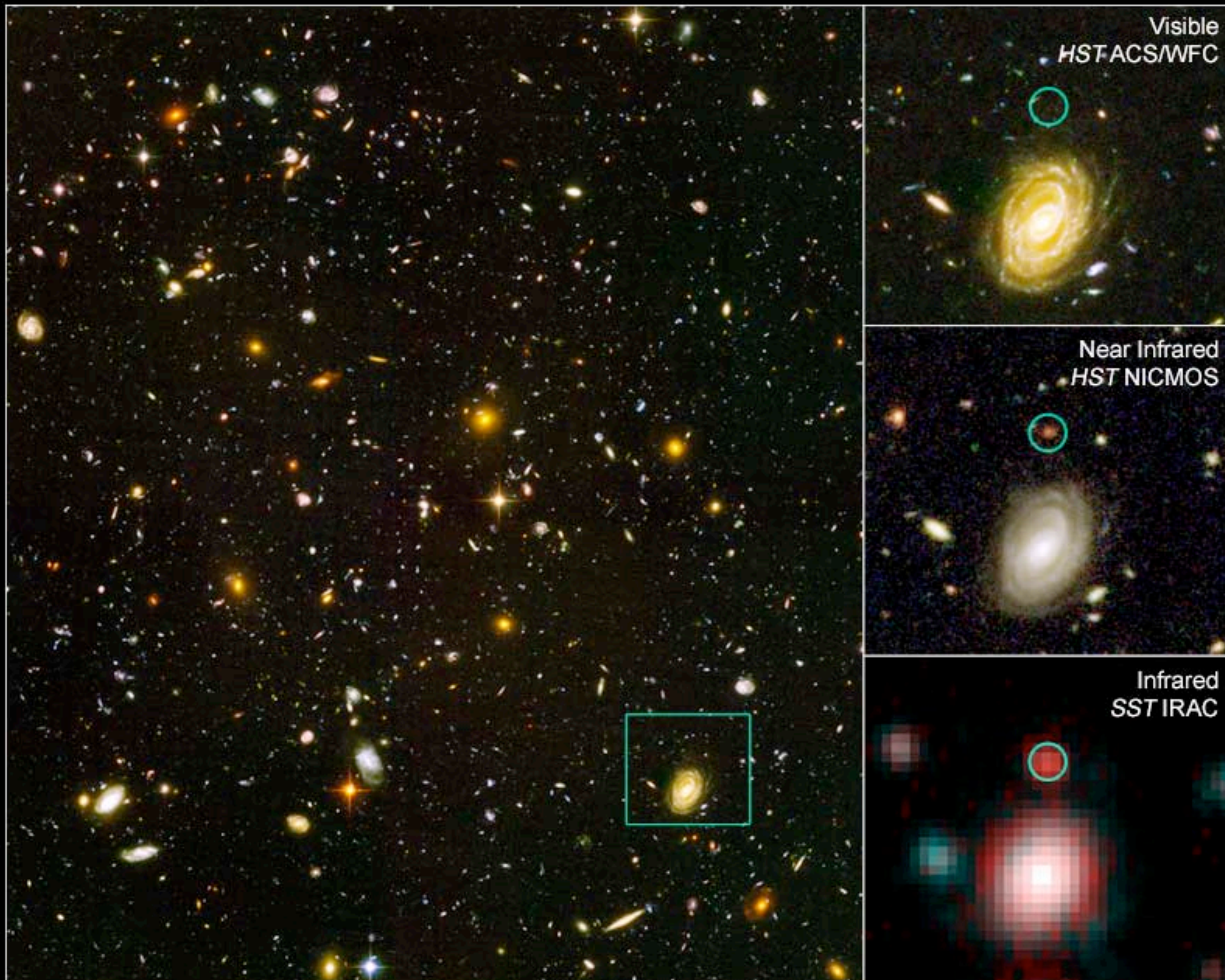
Hubble (HST)
Ultra Deep Field

Some of the first
galaxies to form.



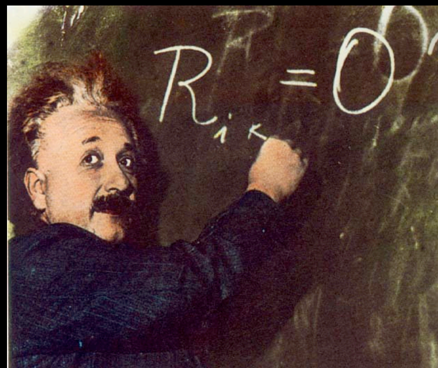
Distant Galaxy in the Hubble Ultra Deep Field

HST ACS NICMOS ▪ *SST* IRAC



NASA, ESA, and B. Mobasher (STScI/ESA)

STScI-PRC05-28



Albert Einstein

George Gamow

A. Friedmann

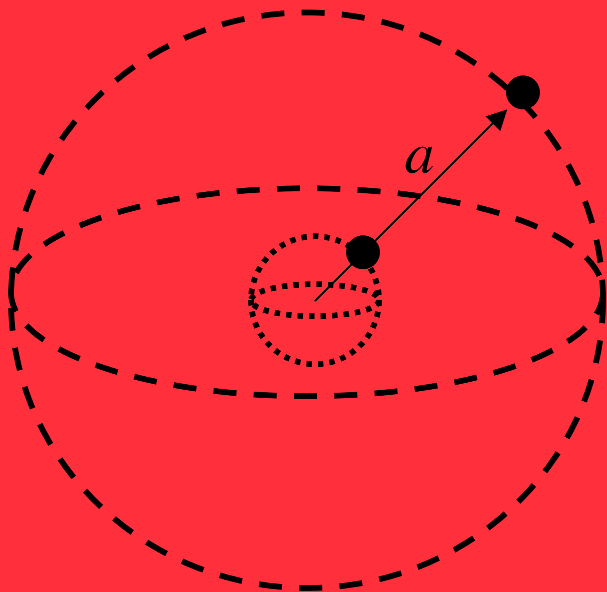
George LeMaitre

Homogeneity and isotropy of the universe:
implies that *total energy* inside a co-moving spherical surface is constant with time.

total energy **= (kinetic energy of expansion) + (gravitational potential energy)**

mass-energy density = ρ

test mass = *m*



$$\approx \tfrac{1}{2} m \dot{a}^2$$

$$\approx -\frac{G\left[\tfrac{4}{3}\pi a^3 \rho\right] m}{a}$$

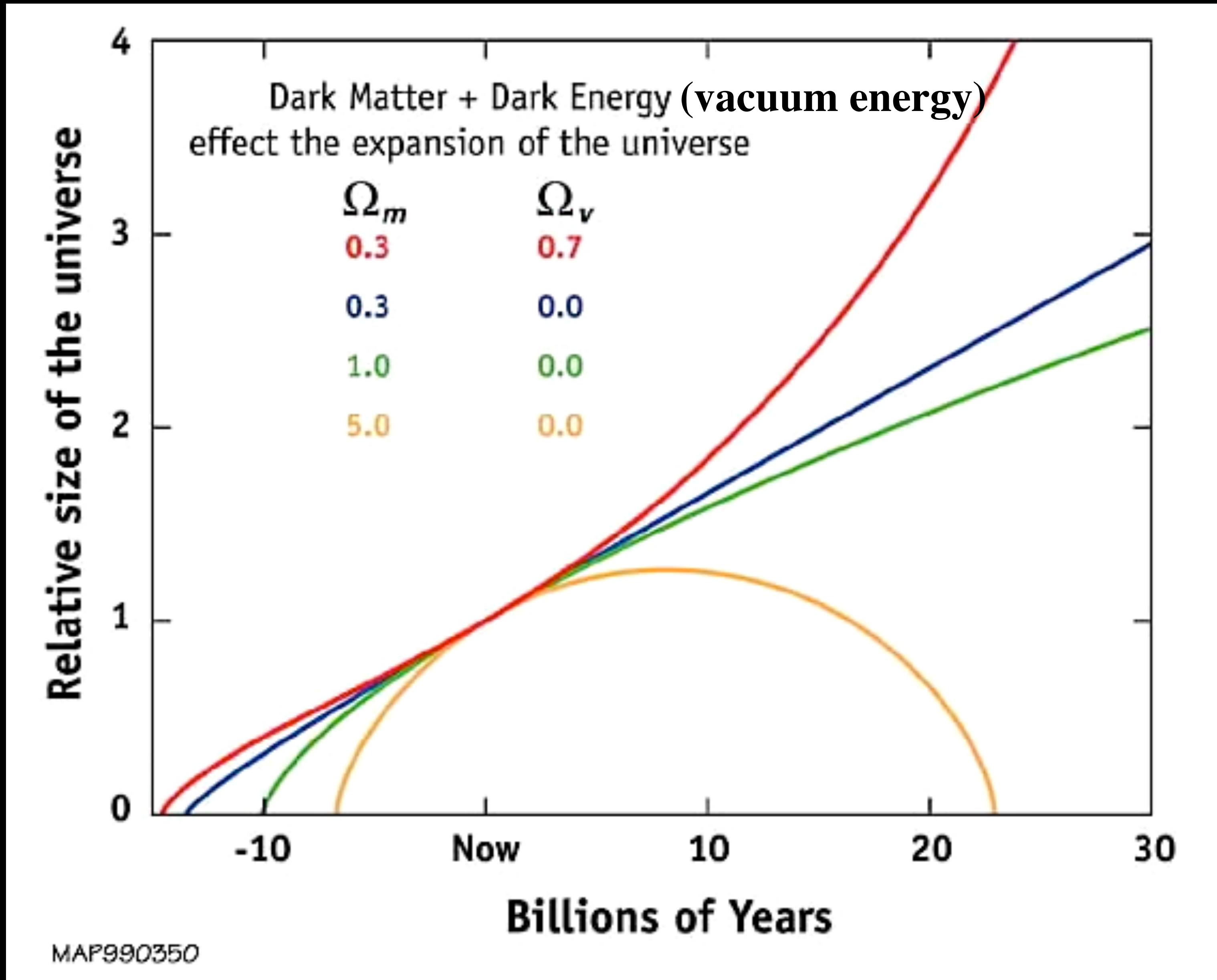
$$\dot{a}^2 + k = \frac{8}{3}\pi G \rho\, a^2$$

total energy **> 0 expand forever k = -1**

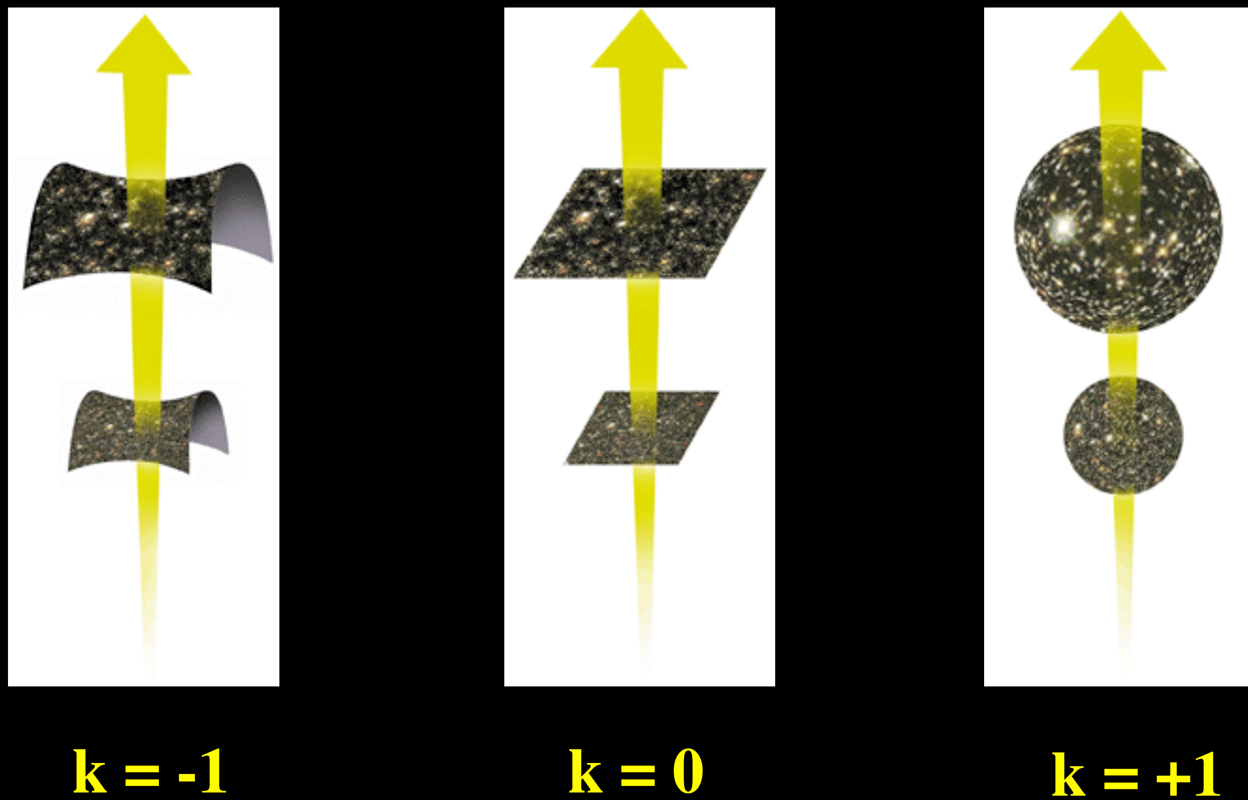
total energy **= 0 for ρ = ρ_{crit} k = 0**

total energy **< 0 re-collapse k = +1**

$$\Omega = \rho/\rho_{\mathrm{crit}} = \Omega_\gamma + \Omega_\nu + \underbrace{\Omega_{\mathrm{baryon}} + \Omega_{\mathrm{dark\ matter}}}_{\approx 0.3} + \Omega_{\mathrm{vacuum}} \approx 1 \quad \mathbf{(k=0)}$$



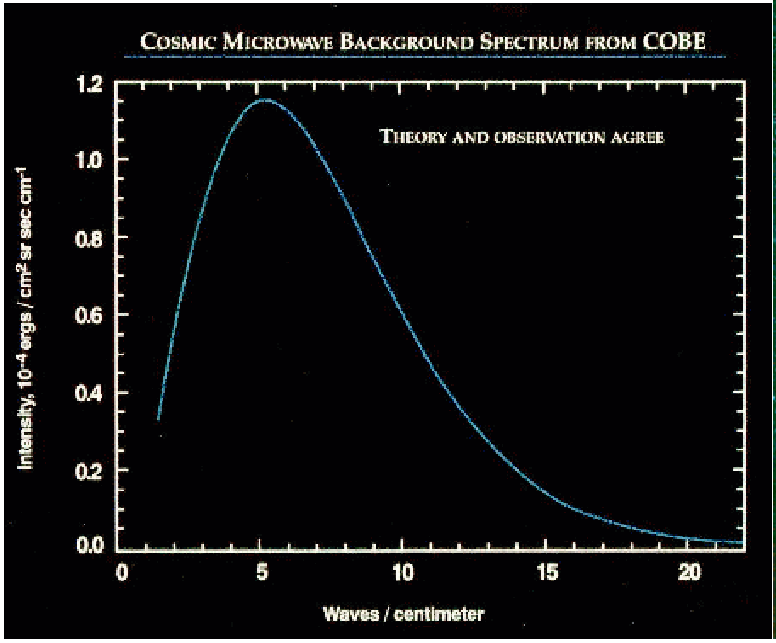
Dark Matter + Dark Energy (vacuum energy)
effect the expansion of the universe
Ω_m Ω_v
0.3 0.7
0.3 0.0
1.0 0.0
5.0 0.0
Relative size of the universe
0
1
2
3
4
-10
Now
10
20
30
Billions of Years
MAP990350

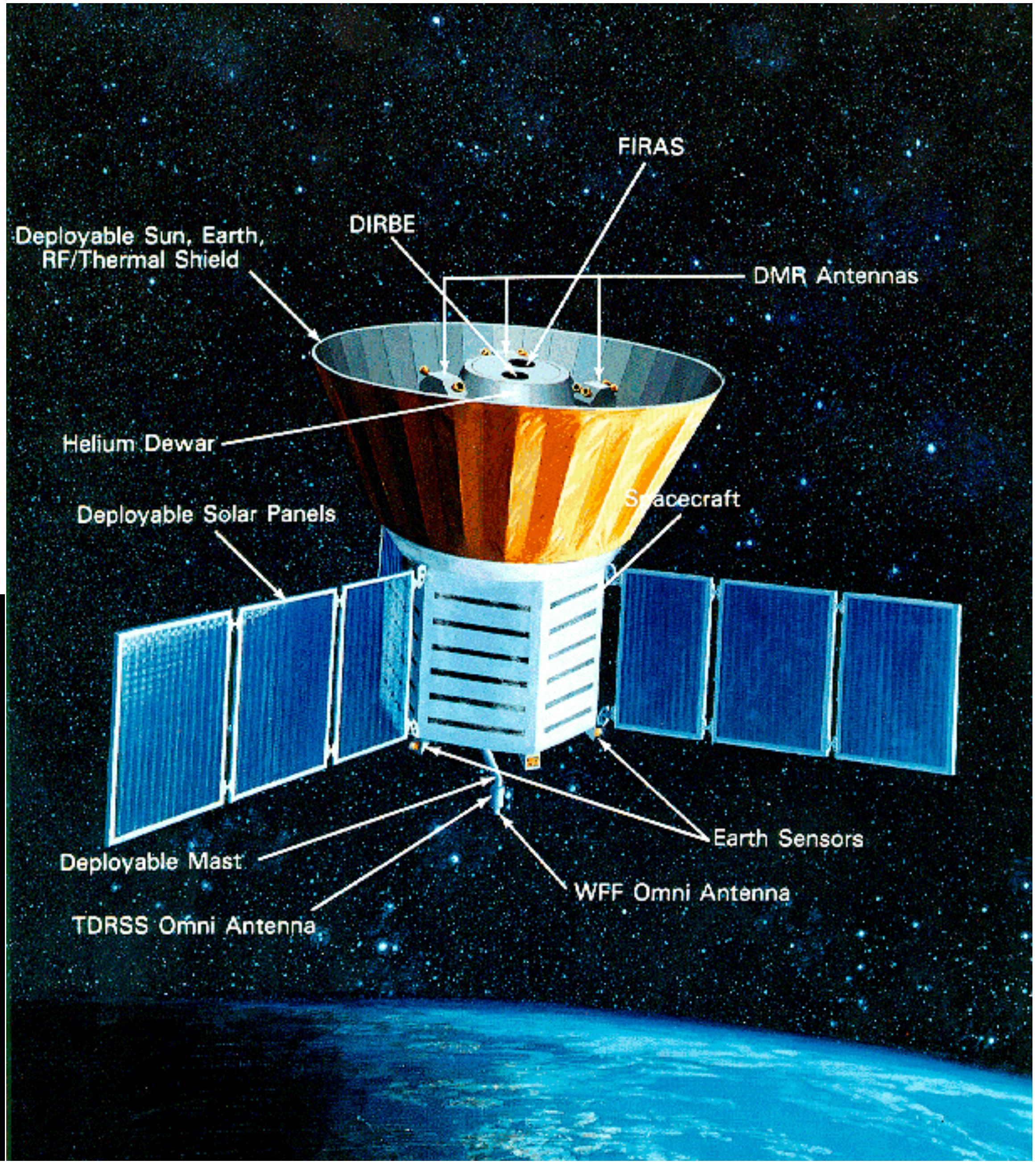

k = -1
k = 0
k = +1

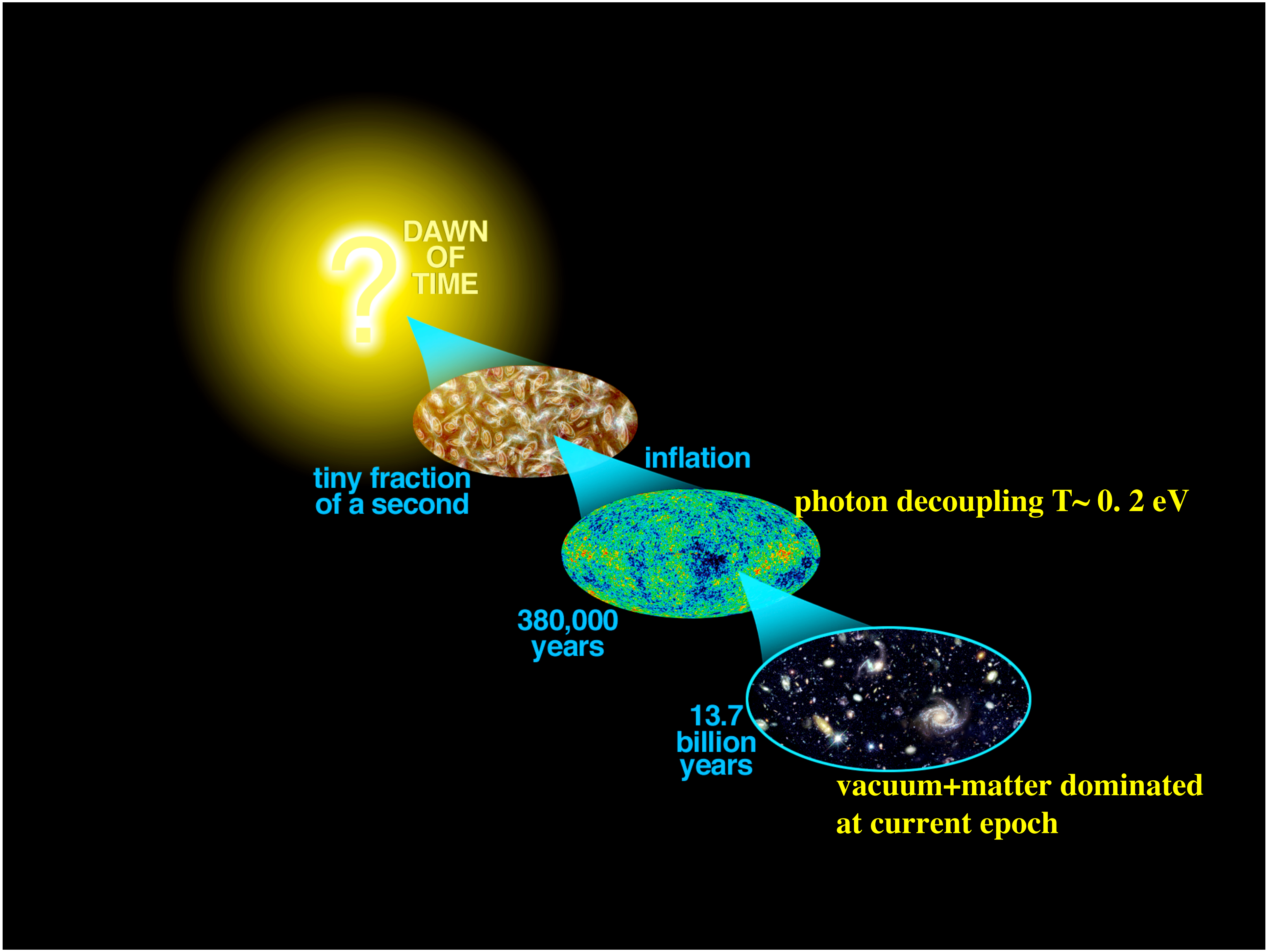
The COBE satellite - the microwave background radiation

Blackbody radiation

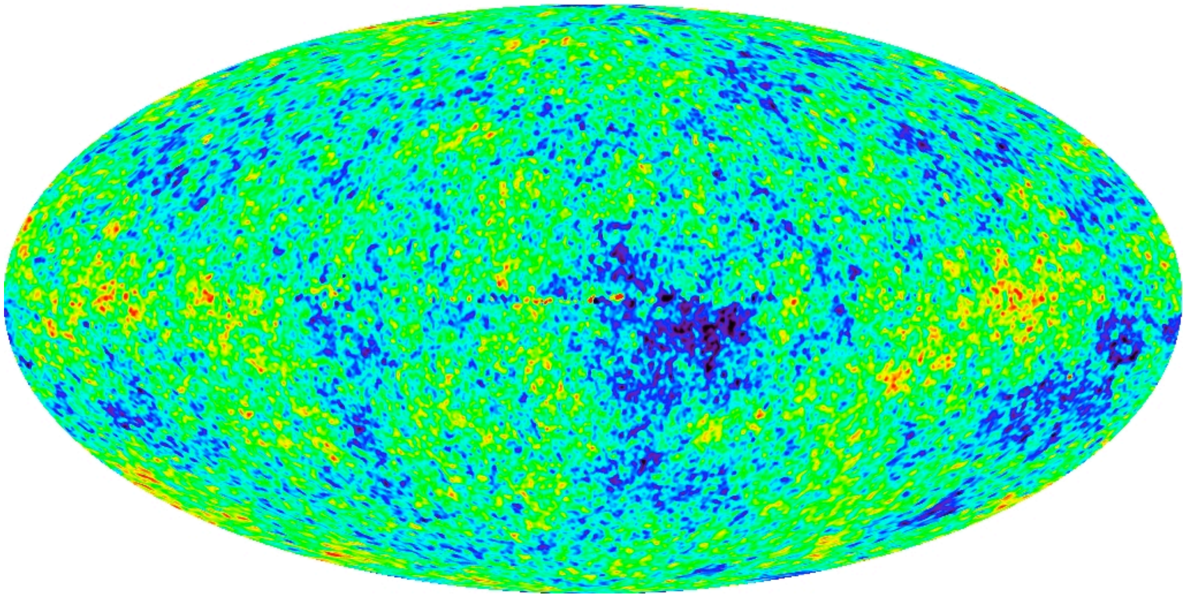
$$T \approx 2.725\,\mathrm{K}$$



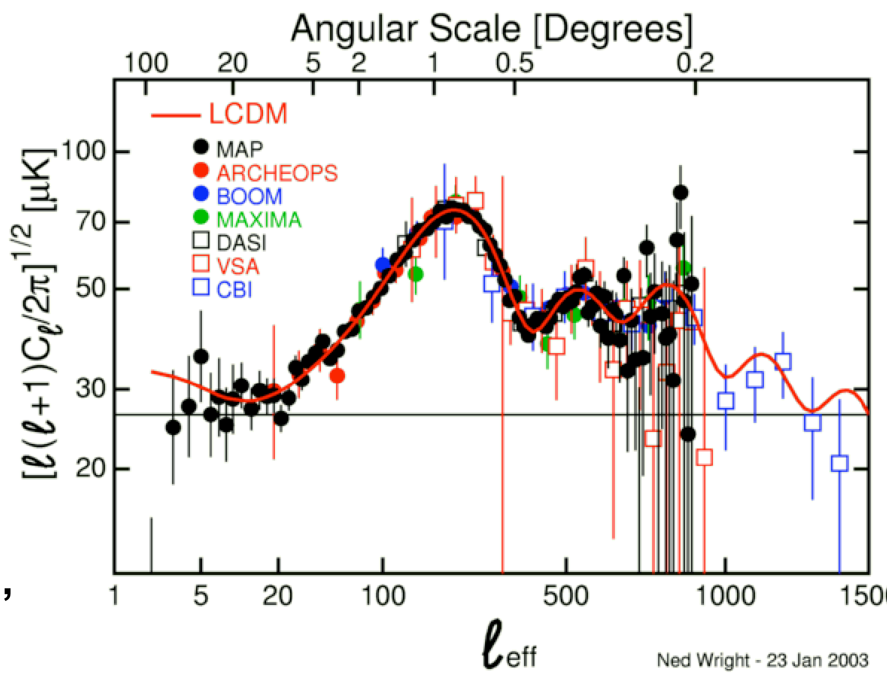



DAWN OF TIME
?
tiny fraction of a second
inflation
photon decoupling T~ 0. 2 eV
380,000 years
13.7 billion years
vacuum+matter dominated
at current epoch

WMAP cosmic microwave background satellite

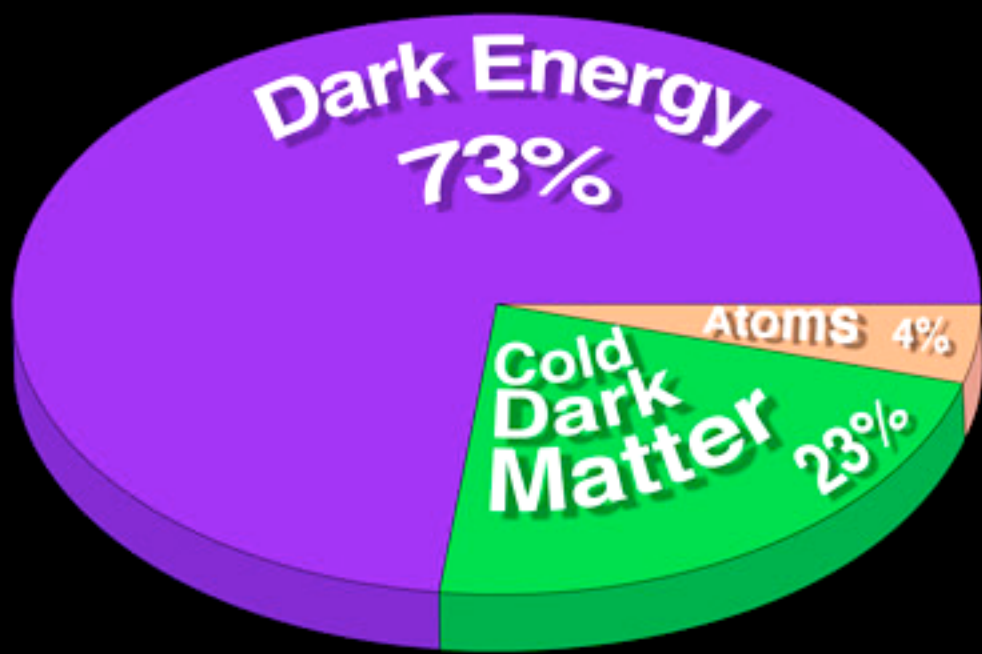


Fluctuations in CMB temperature give
Insight into the composition, size, and age
of the universe and the large scale character
of spacetime.

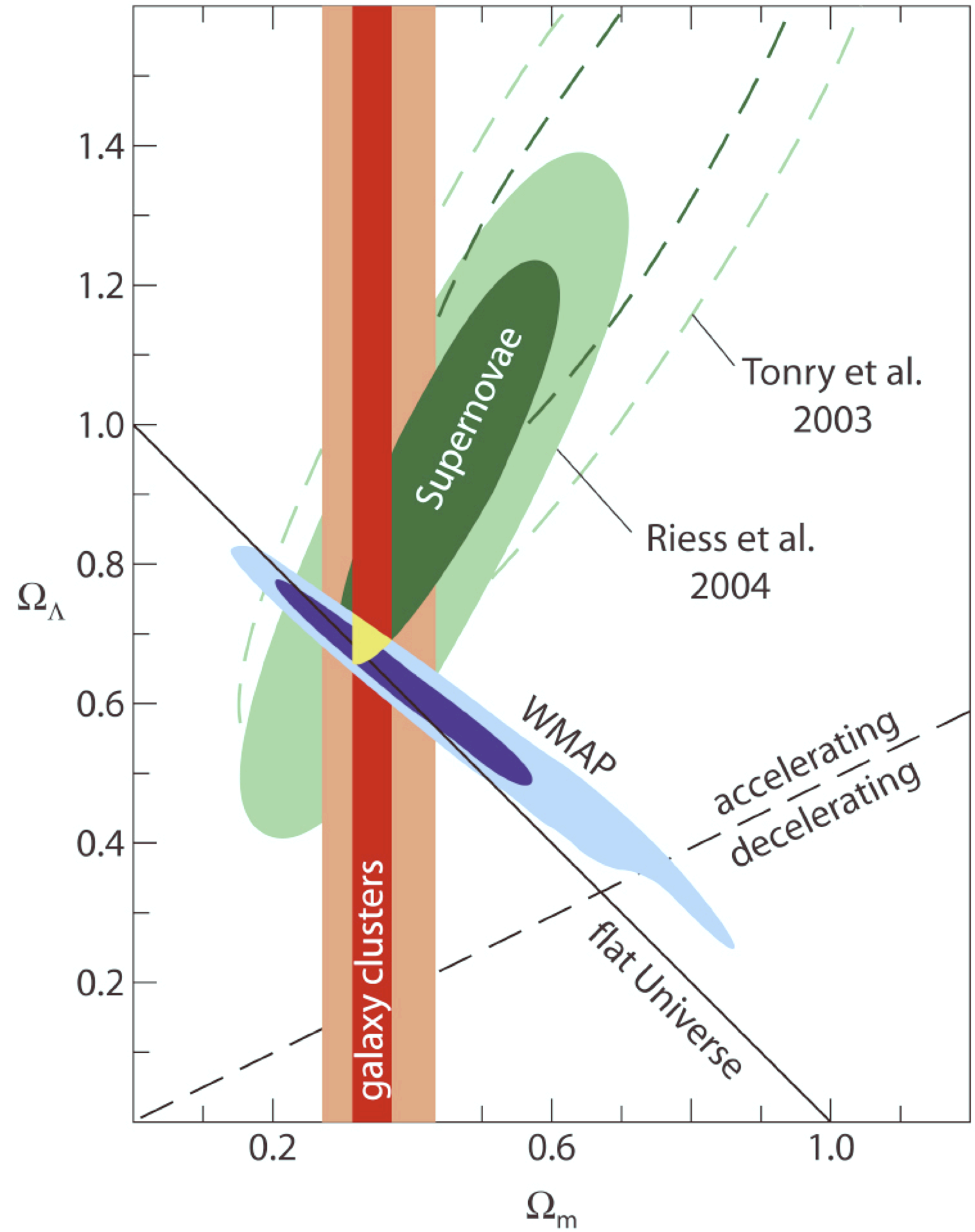


Age = 13.7 Gyr
Spacetime = flat
Composition = 23% unknown nonrelativistic matter, 73% unknown vacuum energy (dark energy), 4% ordinary baryons.

We live in a k = 0, critically closed universe.



observational constraints
on the content of
of nonrelativistic matter
and vacuum energy
(dark energy) in the universe



Constraining the Cosmological Parametres

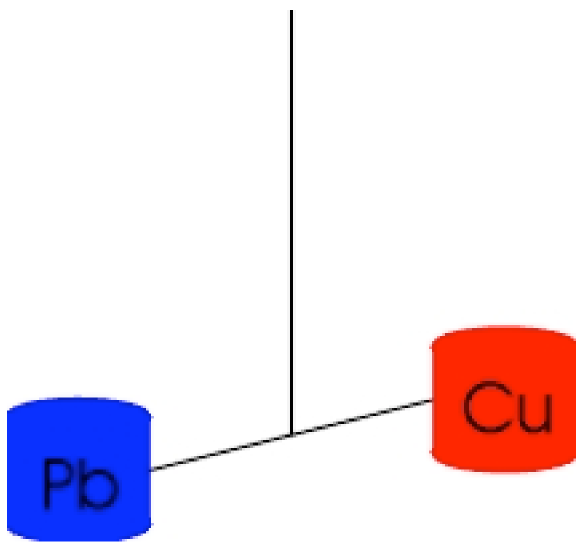
ESO PR Photo 18d/04 (3 June 2004)

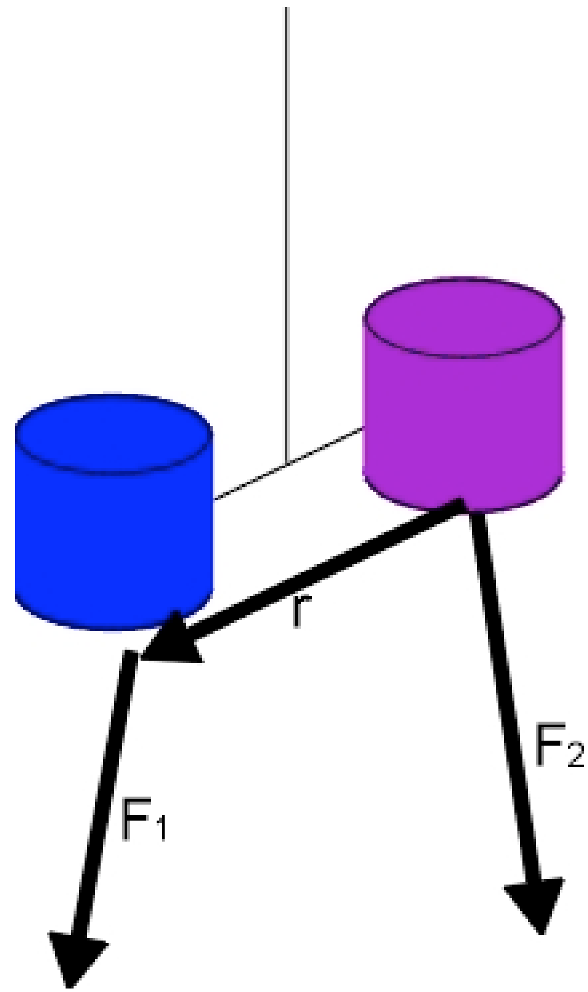
The Equivalence Principle

- Eotvos experiments
- meaning for freely falling bodies
- geometric implications
- geodesics

The EÖTVÖS Experiment

torsion balance

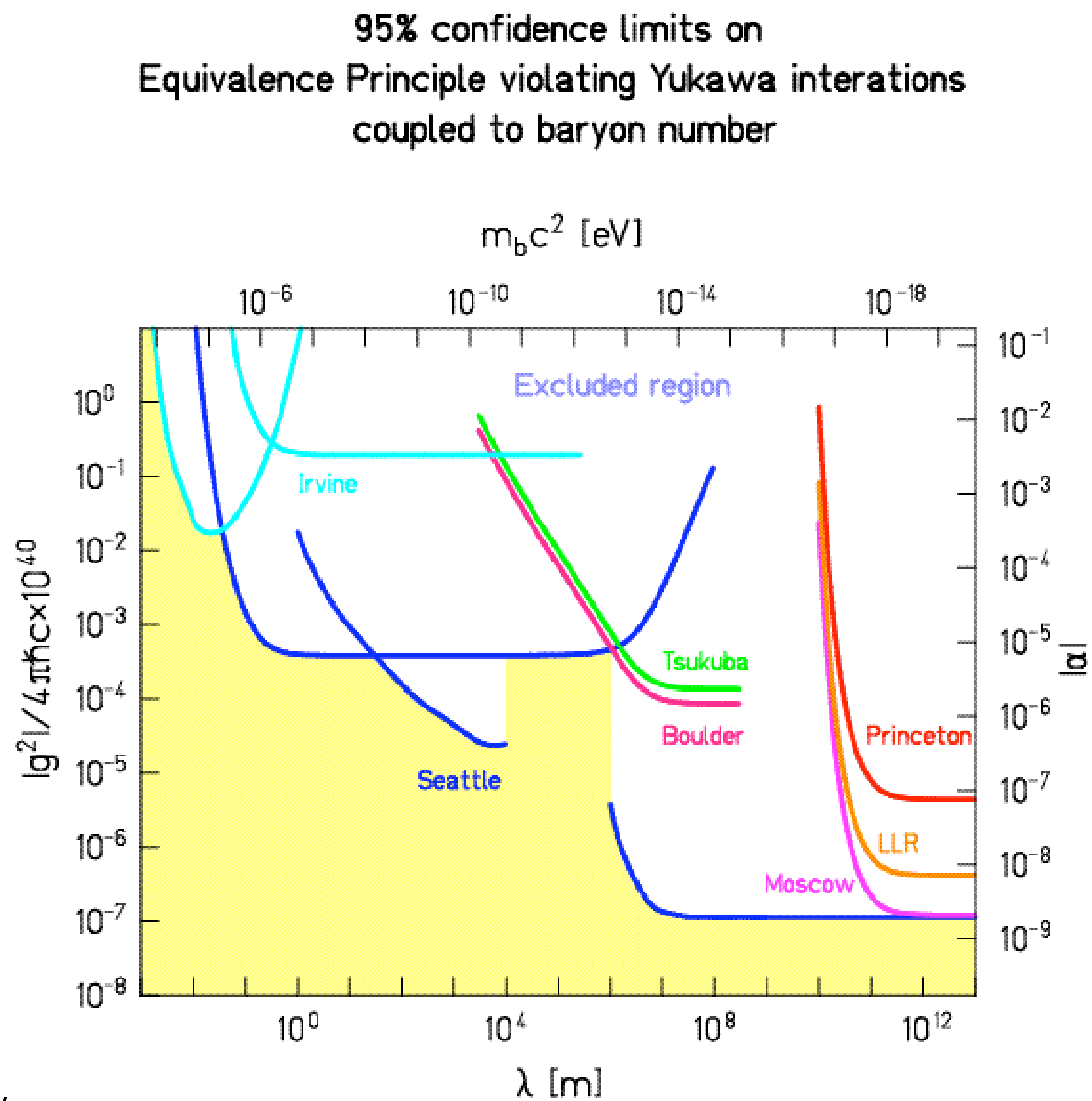




see www.npl.washington.edu/eotwash

Magnitude of torque on fiber: $T = \left[\mathbf{r} \cdot (\mathbf{F}_1 \times \mathbf{F}_2)\right] / |\mathbf{F}_1 + \mathbf{F}_2|$

EotWash lab's results: sensitivity for long range forces is at about 1 part in 10^{13}



www.npl.washington.edu/eotwash/

OK, what does this mean?

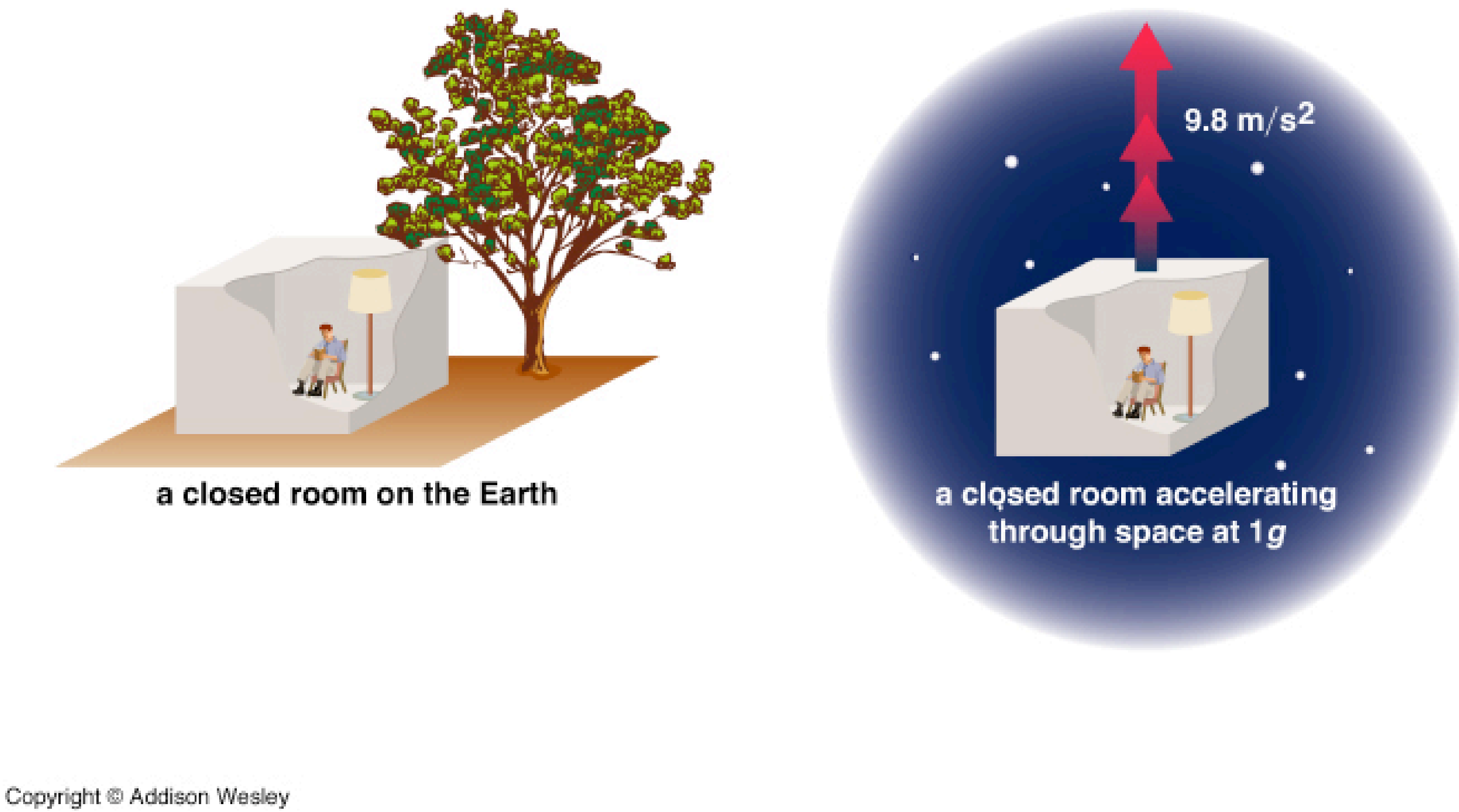
→ **Everything falls at the same rate!**

Apollo 15 astronaut
David R. Scott drops a hammer
and a feather . . . Guess what happens?

www.hq.nasa.gov/ . . ./History/SP-4214/cover.html

One begins to get a creepy feeling that the acceleration produced by "gravity" has nothing to do with what the bodies in question are made out of, but rather is a property of space (spacetime) itself!

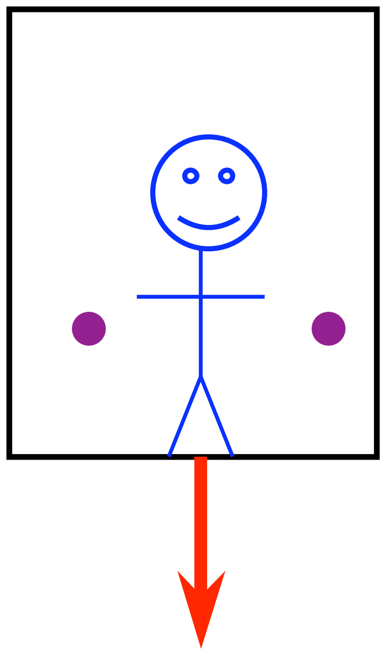
equivalence of inertial and gravitational mass:



cse.ssl.berkeley.edu/bmendez/ay10/2002/notes/pics

elevators in free fall . . . & the E. P.

Someone cuts your elevator's cable and you release the two balls that you have in your hands . . . What happens?



1 g = 9.8 m s^{-2}

If we make the elevator small enough, it looks to us as if THERE IS NO GRAVITY !!!

Cannot tell the difference between an elevator in free fall and the *absence* of gravitation.

Gravitation as Geometry

Statement of the
Equivalence Principle:

In a *sufficiently small* region of space & time we can find a freely falling (locally Minkowski) coordinate system in which the effects of gravitation are absent - the laws of physics are the same as they are in a Minkowski coordinate system with no gravitation.

Statement of the fundamental theorem of differential geometry for 2-D surfaces:

In a *sufficiently small* region on any 2-D surface, the geometry is locally flat and Cartesian. (We can pass a tangent plane through any point on the surface. In a *sufficiently small* region around where this tangent plane touches the surface, the geometry will be flat, like a Cartesian plane.)

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$dl^2 = dx^2 + dy^2$$

Coordinate Transformations

Follows from the chain rule: view coordinates in one system as functions of the coordinates in the other frame.

e.g., consider these four functions: $x^\mu\left(\xi^0, \xi^1, \xi^2, \xi^3\right)$

with $\mu = 0, 1, 2, 3$

chain rule gives:

$$dx^\mu = \frac{\partial x^\mu}{\partial \xi^\alpha} \cdot d\xi^\alpha$$

$$= \frac{\partial x^\mu}{\partial \xi^0} \cdot d\xi^0 + \frac{\partial x^\mu}{\partial \xi^1} \cdot d\xi^1 + \frac{\partial x^\mu}{\partial \xi^2} \cdot d\xi^2 + \frac{\partial x^\mu}{\partial \xi^3} \cdot d\xi^3$$

Coordinate transformation is these 16 functions:

$$dx^\mu = \Lambda^\mu{}_\alpha \, d\xi^\alpha$$

$$\boxed{\Lambda^\mu{}_\alpha = \frac{\partial x^\mu}{\partial \xi^\alpha}}$$

In locally inertial (Minkowski) coordinates $\xi^\alpha(x)$ the particle is unaccelerated, and moving on a straight line

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0$$

$$\frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) = 0 \qquad \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2\xi^\alpha}{\partial x^\mu\,\partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

multiply both sides by $\frac{\partial x^\rho}{\partial\xi^\alpha}$ and sum:

$$\frac{\partial x^\rho}{\partial\xi^\alpha}\cdot\frac{\partial\xi^\alpha}{\partial x^\mu}\cdot\frac{d^2x^\mu}{d\tau^2} + \frac{\partial x^\rho}{\partial\xi^\alpha}\cdot\frac{\partial^2\xi^\alpha}{\partial x^\mu\,\partial x^\nu}\cdot\frac{dx^\mu}{d\tau}\cdot\frac{dx^\nu}{d\tau} = 0$$

$$\underbrace{\left(\frac{\partial x^\rho}{\partial\xi^\alpha}\cdot\frac{\partial\xi^\alpha}{\partial x^\mu}\right)}_{\delta^\rho{}_\mu}\cdot\frac{d^2x^\mu}{d\tau^2} + \underbrace{\left[\frac{\partial x^\rho}{\partial\xi^\alpha}\cdot\frac{\partial^2\xi^\alpha}{\partial x^\mu\,\partial x^\nu}\right]}_{\Gamma^\rho_{\mu\,\nu}}\cdot\frac{dx^\mu}{d\tau}\cdot\frac{dx^\nu}{d\tau} = 0$$

$$\boxed{\frac{d^2x^\rho}{d\tau^2} + \Gamma^\rho_{\mu\,\nu}\cdot\frac{dx^\mu}{d\tau}\cdot\frac{dx^\nu}{d\tau} = 0}$$

Geodesic Equation

$$\frac{d^2 x^\rho}{d\tau^2} + \Gamma^\rho_{\mu\,\nu} \cdot \frac{dx^\mu}{d\tau} \cdot \frac{dx^\nu}{d\tau} = 0$$

So in this coordinate system (x) it looks as if the particle experiences a “gravitational force” and is accelerated.

Note that another way to write this equation is

$$\frac{du^\rho}{d\tau} + \Gamma^\rho_{\mu\,\nu}\, u^\mu\, u^\nu = 0$$

Connection Coefficients (Christoffel symbols)

$$\Gamma^{\rho}_{\mu\,\nu}$$

These are obviously related to how the locally Minkowski coordinates differ from our lab coordinates x

$$\Gamma^{\rho}_{\mu\,\nu} = \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \cdot \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu}\,\partial x^{\nu}}$$

Now, in considering these two coordinate systems, the locally Minkowski coordinates and the “lab” coordinates, we will demand that the spacetime interval is always the same:

$$
\begin{aligned}
ds^2 &= \eta_{\alpha\,\beta}\, d\xi^\alpha\, d\xi^\beta \\
&= \eta_{\alpha\,\beta} \left(\frac{\partial \xi^\alpha}{\partial x^\mu}\, dx^\mu \right) \left(\frac{\partial \xi^\beta}{\partial x^\nu}\, dx^\nu \right) \\
&= \underbrace{\left(\eta_{\alpha\,\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \right)}_{g_{\mu\,\nu}} dx^\mu\, dx^\nu \\
&= g_{\mu\,\nu}\, dx^\mu\, dx^\nu
\end{aligned}
$$

The general metric tensor field is $g_{\mu\,\nu}\,(\mathbf{x})$

It is symmetric: $g_{\mu\,\nu} = g_{\nu\,\mu}$

Therefore, there are 10 independent functions $g_{\mu\,\nu}\,(\mathbf{x})$ at any event (point) in spacetime.

Examples: 2-D, flat Cartesian coordiantes and polar coordinates.