Fluid Mechanics

Fluids are described by fields such as $\rho(\vec{x},t)$, $\vec{v}(\vec{x},t)$, $p(\vec{x},t)$, $T(\vec{x},t)$, etc. The mass current is $\vec{j} = \rho\vec{v}$, hence our first equation is that of continuity:

$$\partial_t \rho + \vec{\nabla}\cdot\vec{j} = 0$$

i.e. $$\partial_t \rho + \vec{\nabla}\cdot(\rho\vec{v}) = \partial_t\rho + \rho\,\vec{\nabla}\cdot\vec{v} + \vec{v}\cdot\vec{\nabla}\rho = 0$$

Now consider a small volume of fluid. The force on this volume is

Ω

$$\vec{F} = -\oint_{\partial\Omega} p\,\hat{n}\,d\Sigma = -\int_{\Omega}\vec{\nabla}p\,dV \; ; \; \partial\Omega = \text{boundary of } \Omega$$

Thus, per unit volume, we have $\vec{f} = -\vec{\nabla}p$. Newton's second law now gives

$$\rho\frac{d\vec{v}}{dt} = -\vec{\nabla}p + \vec{f}_{ext} \quad ; \quad \vec{f}_{ext} = \text{external force density}$$

But $\frac{d\vec{v}}{dt}$ is the rate of change of $\vec{v}$ of a fluid particle as it moves:

$$\frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + \frac{\partial\vec{v}}{\partial x_i}\frac{\partial x_i}{\partial t} = \frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v}$$

Thus, Euler's eqn:

$$\boxed{\rho\frac{\partial\vec{v}}{\partial t} + \rho\vec{v}\cdot\vec{\nabla}\vec{v} = -\vec{\nabla}p + \vec{f}_{ext}}$$

For a fluid in a gravitational field, $\vec{f}_{ext} = \rho\vec{g}$. Thus, for hydrostatic equilibrium, we have

$$\vec{\nabla}p = \vec{f}_{ext}$$

Thus far, we've not accounted for dissipative processes due to internal friction (i.e. viscosity) of the fluid, and heat exchange (thermal conductivity). Fluids where these effects are negligible are called <u>ideal</u>.

From thermodynamics, the enthalpy is

$$dH = TdS + Vdp$$

and with $h = H/M$ (enthalpy per unit mass),

$$dh = Ts + \frac{1}{\rho}dp \qquad ; \qquad s = S/M$$

An ideal fluid involves no heat exchange, hence the entropy of fluid particles remains constant as they move. Thus,

$$dh = Tds + \frac{1}{\rho}dp = \frac{1}{\rho}dp$$

and so $\rho^{-1}\vec{\nabla}p = \vec{\nabla}h$, and we have

$$\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v} = -\vec{\nabla}h$$ no external field

Note

$$\begin{aligned}\epsilon_{ijk}\, v_j\, \epsilon_{klm}\, \partial_l v_m &= \epsilon_{kij}\epsilon_{klm}\, v_j\, \partial_l v_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, v_j\, \partial_l v_m \\ &= v_j\,\partial_i v_j - v_j\,\partial_j v_i\end{aligned}$$

$$\Rightarrow \quad \vec{v}\times(\vec{\nabla}\times\vec{v}) = -\vec{v}\cdot\vec{\nabla}\vec{v} + \frac{1}{2}\vec{\nabla}\vec{v}^{\,2}$$

so

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{2}\vec{\nabla}\vec{v}^{\,2} - \vec{v}\times\vec{\omega} = -\vec{\nabla}h$$

$$\vec{\omega} = \vec{\nabla}\times\vec{v} = \text{vorticity}$$

Take the curl:

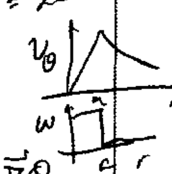
$$\frac{\partial \vec{\omega}}{\partial t} + \vec{\nabla}\times(\vec{\omega}\times\vec{v}) = \frac{\partial}{\partial t}\vec{\nabla}\times\vec{v} - \vec{\nabla}\times(\vec{v}\times\vec{\nabla}\times\vec{v}) = 0$$

Note

$$\oint_{\partial\Sigma} \vec{v}\cdot d\vec{l} = \int_{\Sigma} \hat{n}\cdot\vec{\omega}\, dA$$

Rankine vortex

(iii) $v_\theta = \begin{cases}\Omega r, & r<a \\ \dfrac{\Omega a^2}{r}, & r>a\end{cases}$ $\qquad \omega = 2\Omega\,\Theta(a-r)$



(i) $\vec{v} = \vec{\Omega}\times\vec{r} \Rightarrow \vec{\omega} = 2\vec{\Omega}$; (ii) $\vec{v} = K\vec{\nabla}\theta \Rightarrow \vec{\omega} = K\vec{\nabla}\times\vec{\nabla}\theta$
$= 2\pi K\hat{z}\,\delta(\vec{r}\,)$

Bernoulli's Eqn

In an external field,

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{2}\vec{\nabla} v^2 - \vec{v}\times\vec{\omega} = -\vec{\nabla} h + \vec{f}/\rho$$
$$= -\vec{\nabla}(h+\chi) \quad ; \quad \vec{f}/\rho = -\vec{\nabla}\chi$$

$$\frac{\partial \vec{v}}{\partial t} + \vec{\omega}\times\vec{v} = -\vec{\nabla}\left(h+\chi+\frac{1}{2}v^2\right)$$

In steady state, $\frac{\partial \vec{v}}{\partial t} = 0$, so

$$\vec{v}\cdot\vec{\nabla}\left(h+\chi+\frac{1}{2}v^2\right) = 0$$

which says that the function $\varphi = h+\chi+\frac{1}{2}v^2$ is constant along streamlines. Streamlines give the paths of fluid particles in a steady flow:

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z} \quad \Rightarrow \quad d\vec{x} \parallel \vec{v}$$

For steady, irrotational flows, we have $\vec{\nabla}\varphi = 0$, and $\varphi = \text{const}$ everywhere. For incompressible irrotational flow, we also have $\rho = \text{constant}$, which means

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\rho\cdot\vec{v} + \rho\vec{\nabla}\cdot\vec{v} = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{v} = 0$$

Now ρ is no longer an unknown f^{n}, and Bernoulli's eqn takes the simpler form

$$\varphi = \frac{1}{2}v^2 + \frac{P}{\rho} + \chi = \text{const.} \qquad \text{(Bernoulli)}$$

i.e.

$$\frac{\partial \vec{v}}{\partial t} + \vec{\omega}\times\vec{v} = -\vec{\nabla}\left(\frac{1}{2}v^2 + \frac{P}{\rho} + \chi\right)$$
$$= 0 \quad \text{(steady state, irrotational)}$$

Energy Flux

Let u be the specific internal energy of the fluid, i.e. internal energy per unit mass. Thermodynamically, $u = u(\rho, T)$. Now let ε be the total internal energy of the moving fluid:

$$\varepsilon = u + \frac{1}{2} v^2$$

We then have

$$\frac{\partial(\rho\varepsilon)}{\partial t} = u \frac{\partial\rho}{\partial t} + \rho\frac{\partial u}{\partial t} + \rho\vec{v}\cdot\frac{\partial\vec{v}}{\partial t} + \frac{1}{2}\vec{v}^2\frac{\partial\rho}{\partial t}$$

Next, invoke Euler

$$\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v} = -\frac{1}{\rho}\vec{\nabla}p + \frac{1}{\rho}\vec{f}_{ext} = -\frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}\chi$$

and thermodynamics

$$du = T\,ds - p\,d\left(\frac{V}{M}\right) = T\,ds + \frac{p}{\rho^2}d\rho$$
$$dh = T\,ds + \frac{1}{\rho}dp \qquad (h = u + p/\rho)$$

and continuity

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$

Thus,

$$\frac{\partial}{\partial t}\left(\rho u + \frac{1}{2}\rho v^2\right) = \rho\frac{\partial u}{\partial t} + u\frac{\partial\rho}{\partial t} + \frac{1}{2}v^2\frac{\partial\rho}{\partial t} + \rho\vec{v}\cdot\frac{\partial\vec{v}}{\partial t}$$
$$= \rho T\frac{\partial s}{\partial t} + \frac{p}{\rho}\frac{\partial\rho}{\partial t} + u\frac{\partial u}{\partial t} + \frac{1}{2}v^2\frac{\partial\rho}{\partial t} - \rho\vec{v}\cdot\left(\frac{\vec{\nabla}p}{\rho} + \vec{\nabla}\frac{1}{2}v^2 + \vec{\nabla}\chi\right)$$
$$= \rho T\frac{\partial s}{\partial t} - \left(u + \frac{1}{2}v^2 + \frac{p}{\rho}\right)\vec{\nabla}\cdot(\rho\vec{v}) - \rho\vec{v}\cdot\left(\vec{\nabla}h + \vec{\nabla}\frac{1}{2}v^2 + \vec{\nabla}\chi - T\vec{\nabla}s\right)$$
$$= \rho T\left(\frac{\partial s}{\partial t} + \vec{v}\cdot\vec{\nabla}s\right) - \vec{\nabla}\cdot\left[\rho\vec{v}\left(h + \frac{1}{2}v^2\right)\right] + \vec{v}\cdot\vec{f}_{ext}$$

Now let us add another condition: adiabaticity. This says

$$\frac{ds}{dt} = \frac{\partial s}{\partial t} + \vec{v} \cdot \vec{\nabla} s = 0 \qquad \left(\text{note } s = \frac{S}{M} = \frac{1}{m}\frac{S}{N}\right)$$

We then arrive at

$$\underbrace{\frac{\partial}{\partial t}(\rho \varepsilon)}_{\text{energy density}} + \vec{\nabla} \cdot \underbrace{\left(\rho \vec{v}\left(h + \tfrac{1}{2}v^2\right)\right)}_{\text{energy flux}} = \underbrace{\vec{v} \cdot \vec{f}_{ext}}_{\text{source term}}$$

So the fluid carries "on its back" an energy density $h + \frac{1}{2}v^2$.

We may write the total energy flux past a surface Σ as

$$\Phi_\varepsilon(\Sigma) = -\int_\Sigma dA\, \hat{n} \cdot \rho \vec{v}\left(h + \tfrac{1}{2}v^2\right)$$

$$= -\int_\Sigma dA\, \underbrace{\hat{n} \cdot \rho \vec{v}\left(u + \tfrac{1}{2}v^2\right)}_{\substack{\text{energy transport} \\ \text{(kinetic + potential)}}} - \int_\Sigma \underbrace{dA\, \hat{n} \cdot p\vec{v}}_{\text{pressure forces}}$$

<u>Momentum Flux</u>

We have

$$\frac{\partial}{\partial t}(\rho v_i) = \rho \frac{\partial v_i}{\partial t} + v_i \frac{\partial \rho}{\partial t}$$

$$= \rho\left\{ -v_k \frac{\partial v_i}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} - \frac{\partial \chi}{\partial x_i}\right\} - v_i \frac{\partial}{\partial x_k}(\rho v_k)$$

$$= -\frac{\partial}{\partial x_k}\left(\delta_{ik} p + \rho v_i v_k\right) - \rho \frac{\partial \chi}{\partial x_i}$$

Let

$$\Pi_{ik} \equiv p\,\delta_{ik} + \rho v_i v_k = \Pi_{ki}$$

Then

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_k}\Pi_{ik} = f_i^{ext}$$

The quantity Π_{ik} is the momentum flux tensor. Thus,

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\varepsilon \\ \rho v_i\end{pmatrix} + \frac{\partial}{\partial x_k}\begin{pmatrix}\rho\left(\frac{1}{2}v^2+h\right)v_k \\ p\delta_{ik}+\rho v_i v_k\end{pmatrix} = \begin{pmatrix} v_k f_k \\ f_i\end{pmatrix}$$

Circulation

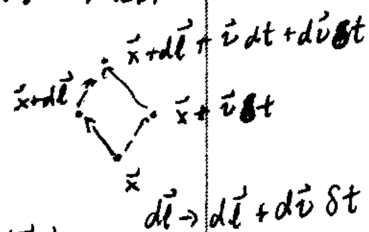
Consider the integral

$$\Gamma(C) = \oint_C \vec{v}\cdot d\vec{l}$$

called the circulation of the fluid along the closed path γ. There are two ways to compute $\Gamma(C)$. One is to consider C as a fixed path in space. Then

$$\Gamma = \int_{\mathrm{int}(C)} \vec{\nabla}\times\vec{v}\cdot\hat{n}\, dA$$

from which we obtain that $\Gamma(C)=0$ for irrotational flow, independent of C. Another definition is to consider $C=C(t)$ as a path that moves with the fluid. Then



$$\Gamma(C(t+\delta t)) = \oint_{C(t+dt)} \vec{v}(\vec{x},t)\cdot d\vec{l}$$

$$= \oint_{C(t)} v(\vec{x}+\delta\vec{x}, t+\delta t)\cdot(d\vec{l}+\delta t\, d\vec{v})$$

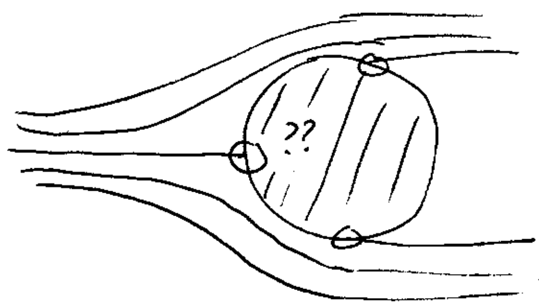
$$= \Gamma(C(t)) + \oint_{C(t)} \delta t\,\frac{d\vec{v}}{dt}\cdot d\vec{l} + \oint_{C(t)} d\left(\tfrac{1}{2}v^2\right)\delta t$$

$$\Rightarrow \frac{\delta\Gamma}{\delta t} = \frac{d\Gamma}{dt} = \oint_C \frac{d\vec{v}}{dt}\cdot d\vec{l} = -\oint_C \vec{\nabla}(h+\chi)\cdot d\vec{l} = 0$$

Hence circulation *in the comoving frame* is preserved. This is known as *Lord Kelvin's Theorem*. (We have assumed the flow is adiabatic, *nota bene*.)

Potential Flow

If $\vec{\omega}=0$ everywhere, the flow is *irrotational*. Now due to conservation of circulation, if $\vec{\omega}=0$ at any point on a streamline, it must vanish everywhere along the streamline. Now consider a steady flow at infinity. $\vec{\omega}=0$ on all streamlines, so $\vec{\omega}=0$ everywhere in space! This is not valid, however, for streamlines which terminate at a point on a solid surface, since it is then impossible to draw a closed contour in the fluid around



that point. The flow may be singular at that point. Physically, there is a unique sol^{n} to such problems. When viscous effects are included, one finds a *boundary layer* is formed. The boundary layer selects one from among an infinite number of discontinuous $sol^{\underline{ns}}$.

For potential flow, then, we may write

$$\vec{v} = \vec{\nabla}\phi$$

so $\vec{\nabla}\times\vec{v} = \vec{\nabla}\times\vec{\nabla}\phi = 0$. Euler's eqn now gives

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{2}\vec{\nabla} v^2 - \vec{v}\times\vec{\omega} = -\vec{\nabla}(h+\chi)$$

$$\Rightarrow \quad \vec{\nabla}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}(\vec{\nabla}\phi)^2 + h + \chi\right) = 0$$

Thus,

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(\vec{\nabla}\phi)^2 + h + \chi = 0$$

Note the rhs could in principle be an arbitrary $f^{\underline{n}}$ of t, but we can always redefine

$$\phi \to \tilde{\phi} = \phi + \int^t dt'\, g(t')$$

to eliminate this $f^{\underline{n}}$. In steady flow, $\frac{\partial \phi}{\partial t} = 0$, so

$$\frac{1}{2}(\vec{\nabla}\phi)^2 + h + \chi = \text{const.}$$

<u>Incompressible Fluids</u>

For incompressible potential flow,

$$\vec{\nabla}\cdot\vec{v} = \nabla^2\phi = 0$$

which is just Laplace's eqn! Boundary conditions on surfaces: $\vec{v}\cdot\hat{n}\Big|_{\text{surface}} = 0$. We also have

$$\frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{p}{\rho} + \chi = g(t) \quad \text{arbitrary}$$

Note that if $\chi = 0$, then $\vec{v}^2$ is maximum when p is minimum, i.e. $p = 0$. Conversely, $p = p_{max}$ occurs for $\vec{v} = 0$, called a <u>stagnation point</u>.

For two-dimensional flow, we have

$$v_x = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x}, \quad v_y = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} \qquad \text{(Cauchy-Riemann!)}$$

$$\vec{\nabla} \cdot \vec{v} = 0$$

Note then if

$$W = \phi + i\psi \qquad \begin{array}{l} z = x+iy \\ \bar{z} = x-iy \end{array} \qquad \begin{array}{l} \frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y} \\ \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y} \end{array}$$

then

$$2\frac{\partial W}{\partial \bar{z}} = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y} + i\left(+\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}\right) = 0$$

$$\Rightarrow \quad W = W(z) \quad \underline{\text{analytic}}$$

Note

$$\frac{\partial W}{\partial z} = \frac{1}{2}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right) + \frac{i}{2}\left(-\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}\right)$$

$$= \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y} = v_x - i v_y$$

i.e.,

$$\frac{\partial W}{\partial z} = \frac{dW}{dz} = \bar{v}$$

the complex (conjugate) velocity. W = "complex potential".

<u>Example</u>

Moving sphere of radius R, velocity $\vec{V}$ in a fluid.

Must have $\vec{v}(\infty) = 0 \Rightarrow$

$$\phi = \vec{A} \cdot \vec{\nabla}\left(\frac{1}{r}\right) \qquad \begin{array}{l} r \text{ relative to center of sphere} \\ \text{solves Laplace's eqns} \end{array}$$

At surface, we must have $(\vec{v} - \vec{V}) \cdot \hat{n}\Big|_{r=R} = 0$. This gives

$$\vec{A} = \frac{1}{2}R^3\vec{V}$$

Thus,

$$\phi = -\frac{R^3}{2r^2}\,\vec{V}\cdot\hat{n} \quad , \quad \vec{v} = \frac{R^3}{2r^3}\left\{3\hat{n}(\vec{v}\cdot\hat{n}) - \vec{v}\right\}$$

Pressure :

$$P = P_0 - \frac{1}{2}\rho\vec{v}^2 - \rho\frac{\partial\phi}{\partial t}$$

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial\vec{V}}\cdot\frac{d\vec{V}}{dt} - \vec{V}\cdot\vec{\nabla}\phi$$

$$\Rightarrow \quad P = P_0 + \frac{1}{8}\rho u^2(9\cos^2\theta - 5) + \frac{1}{2}\rho R\,\hat{n}\cdot\frac{d\vec{V}}{dt}$$

where $\hat{V}\cdot\hat{n} = \cos\theta$.

Classical Aerofoil Theory

Consider a 2D flow. Then $W = W(z)$ with

$$\frac{dW}{dz} = v_x - i v_y$$

$$v_x = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad v_y = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

$$\vec{v}\cdot\vec{\nabla}\psi = \frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y} = 0$$

$\Rightarrow$ ψ constant along streamlines

and

$$\frac{\partial^2 W}{\partial\bar{z}\partial z} = 0 = \frac{\partial}{\partial\bar{z}}(v_x - i v_y)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(v_x - i v_y)$$

$$= \frac{1}{2}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x}\right)$$

$$= \frac{1}{2}\vec{\nabla}\cdot\vec{v} - \frac{i}{2}\vec{\nabla}\wedge\vec{v} = 0$$

i.e. $\vec{v}$ is incompressible and irrotational.

- Uniform flow at angle α

$$\left.\begin{array}{l} v_x = v_0\cos\alpha \\ v_y = v_0\sin\alpha \end{array}\right\} \Rightarrow \quad v_x - i v_y = v_0 e^{-i\alpha} = \frac{dW}{dz}$$

$$\therefore \quad W(z) = v_0 z e^{-i\alpha}$$

- Line vortex

$$\vec{v} = \frac{\Gamma}{2\pi}\vec{\nabla}\theta \quad \Leftrightarrow \quad \oint \vec{v}\cdot d\vec{l} = \Gamma$$

$$= \frac{\Gamma}{2\pi}\frac{x\hat{y} - y\hat{x}}{x^2+y^2} \quad \Rightarrow \quad v_x - i v_y = \frac{\Gamma}{2\pi r^2}(-y - ix)$$

$$= \frac{\Gamma}{2\pi i}\frac{\bar{z}}{r^2} = \frac{\Gamma}{2\pi i z}$$

$$\therefore \quad W(z) = \frac{\Gamma}{2\pi i}\ln z$$

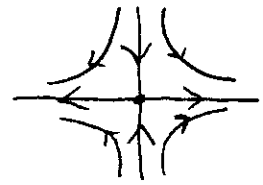
Shifting the origin of the line vortex to z_0:

$$W(z) = \frac{\Gamma}{2\pi i}\ln(z - z_0)$$

- Stagnation point: $W(z) = \frac{1}{2}\alpha z^2$

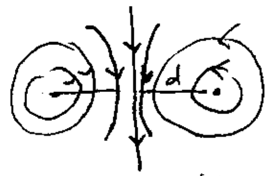
$$v_x - i v_y = \alpha z = \alpha x + i \alpha y$$

$$v_x = \alpha x \ , \quad v_y = -\alpha y$$



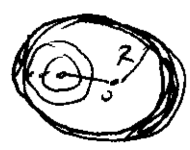
- Vortex near a wall:

$$W(z) = \frac{\Gamma}{2\pi i} \ln\left(\frac{z-d}{z+d}\right)$$



We want ψ to be constant on the boundary, so $\hat{n}\cdot\vec{\nabla}\psi = 0$, or $\hat{n}\times\nabla\phi = 0$ which is analogous to the boundary condition at the surface of a conductor!
Here, we solve using the method of images.

- Vortex near a cylindrical wall



$$W(z) = \frac{\Gamma}{2\pi i}\ln(z-d) - \frac{\Gamma}{2\pi i}\ln\left(z - \frac{R^2}{d}\right)$$

- Milne-Thomson circle theorem

Let $W(z) = f(z)$ inside a circle of radius R, where $f(z)$ has singularities for $|z| > R$. Then

$$W(z) = f(z) + \overline{f(R^2/\bar{z})}$$

has the same singularities, and has $|z| = R$ a streamline, since $W(Re^{i\theta}) = f(Re^{i\theta}) + \overline{f(Re^{i\theta})} \in \mathbb{R} \Rightarrow \psi = 0$.

- Application: flow past a fixed cylinder

$$f(z) = v_0 z \ ; \quad f(R^2/\bar{z}) = v_0 R^2/\bar{z}$$

$$W(z) = v_0\left(z + \frac{R^2}{z}\right)$$

We can also add a vortex at $z=0$:

$$W(z) = v_0\left(z + \frac{R^2}{z}\right) + \frac{\Gamma}{2\pi i}\ln z$$

$$\phi = v_0\left(r + \frac{R^2}{r}\right)\cos\theta + \frac{\Gamma\theta}{2\pi}$$

$$\psi = v_0\left(r - \frac{R^2}{r}\right)\sin\theta - \frac{\Gamma}{2\pi}\ln r$$

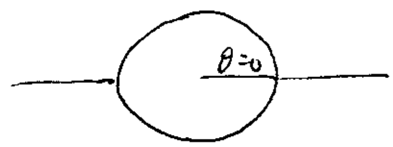
Thus,

$$v_r = \frac{\partial\phi}{\partial r} = v_0\left(1 - \frac{R^2}{r^2}\right)\cos\theta = \frac{\partial\phi}{\partial r}$$

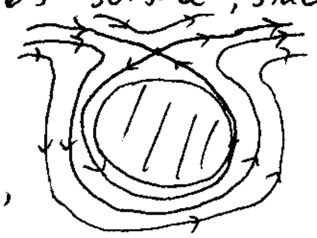
$$v_\theta = -v_0\left(1 + \frac{R^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r} = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

For $r=R$, $v_r = 0$ and

$$v_\theta = -2v_0\sin\theta + \frac{\Gamma}{2\pi R}$$



Thus, we have slip at the cylinder's surface, since $v_\theta \neq 0$.



- $\Gamma > 4\pi R v_0$

  stagnation point at $\cos\theta = 0$,

  $\sin\theta = +1 \Rightarrow \theta = \frac{\pi}{2}$,

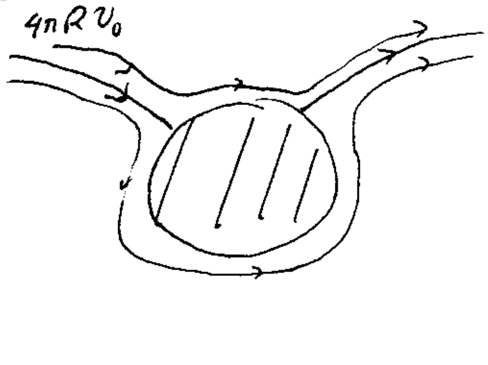
$$\frac{\Gamma}{2\pi v_0} = r + \frac{R^2}{r} \Rightarrow r^2 - \frac{\Gamma r}{2\pi v_0} + R^2 = 0$$

$$r = \frac{\Gamma}{4\pi v_0} \pm \sqrt{\left(\frac{\Gamma}{4\pi v_0}\right)^2 - R^2}$$

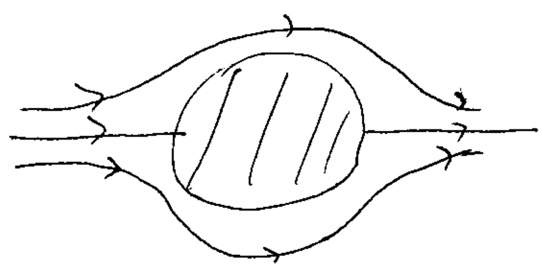
only + root outside as $r_+ r_- = R^2$

- $4\pi R v_0 > \Gamma > 0$: stagnation at $r = R$,

$$\sin\theta^* = \frac{\Gamma}{4\pi R v_0} \Rightarrow \theta = \theta^*, \pi - \theta^*$$



• $\Gamma = 0 \Rightarrow$ stagnation points at $r = R$, $\theta = 0, \pi$



etc.

Now,

$$p + \frac{1}{2}\rho v^2 = \text{constant}$$

$$p(\theta) - p(-\theta) = \frac{1}{2}\rho \vec{v}^2(-\theta) - \frac{1}{2}\rho \vec{v}^2(\theta)$$

$$\vec{v}^2 = v_r^2 + v_\theta^2 = v_0^2\left(1 - \frac{R^2}{r^2}\right)^2 \cos^2\theta + v_0^2\left(1 + \frac{R^2}{r^2}\right)^2 \sin^2\theta + \frac{\Gamma^2}{4\pi^2 r^2} - \frac{\Gamma v_0}{\pi r}\left(1 + \frac{R^2}{r^2}\right)\sin\theta$$

$$= -2 v_0^2 \frac{R^2}{r^2}\cos 2\theta - \frac{\Gamma v_0}{\pi r}\left(1 + \frac{R^2}{r^2}\right)\sin\theta + v_0^2\left(1 + \frac{R^4}{r^4}\right) + \frac{\Gamma^2}{4\pi^2 r^2}$$

$$\Rightarrow \quad p(\theta) - p(-\theta)\Big|_{r=R} = -\frac{4\Gamma v_0}{\pi R}\sin\theta \cdot \frac{1}{2}\rho$$

$$\frac{F}{L} = R\int_0^\pi d\theta \, \sin\theta \, \Delta p(\theta) = -\Gamma \rho v_0 = \text{lift force per unit length of cylinder}$$

$$\Gamma < 0 \Rightarrow \frac{F}{L} > 0 \Rightarrow \text{positive lift.}$$

If instead we took $v = v_0 e^{-i\alpha}$, i.e. $\vec{v} = v_0 \cos\alpha\, \hat{x} + v_0 \sin\alpha\, \hat{y}$, then

$$W(z) = v_0\left(z e^{-i\alpha} + \frac{a^2}{z} e^{i\alpha}\right) + \frac{\Gamma}{2\pi i}\ln z$$

$$\frac{F_y}{L} = -\Gamma \rho v_0 \cos\alpha$$

Aerofoil Theory and Electrostatics in $d=2$ $\quad (\vec{A}\wedge\vec{B} \equiv \hat{z}\cdot\vec{A}\times\vec{B} = A_xB_y - A_yB_x)$

In fluid flow, we have, assuming incompressibility,

$$\vec{\nabla}\cdot\vec{v} = 0 \qquad\qquad \vec{\nabla}\wedge\vec{e} = 0$$

$$\vec{\nabla}\wedge\vec{v} = \omega \qquad \longleftrightarrow \qquad \vec{\nabla}\cdot\vec{e} = \omega \equiv 4\pi\rho$$

$$\vec{e} = \vec{v}\times\hat{z} = (v_y, -v_x)$$

$$\vec{v} = \hat{z}\times\vec{e} = (-e_y, e_x)$$

Away from surfaces, we have $\omega = 0$, i.e. irrotational flow. Thus

$$v_x = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \qquad\longleftrightarrow\qquad e_x = -\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial y} = -\frac{\partial\varphi}{\partial x} = -\frac{\partial\chi}{\partial y}$$

$$v_y = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \qquad\qquad e_y = -\frac{\partial\psi}{\partial y} = -\frac{\partial\phi}{\partial x} = -\frac{\partial\varphi}{\partial y} = +\frac{\partial\chi}{\partial x}$$

streamline $\psi = $ constant $\qquad$ equipotential $\varphi = $ constant

$$\vec{v} = \vec{\nabla}\phi = \vec{\nabla}\psi\times\hat{z} \qquad\qquad \vec{e} = -\vec{\nabla}\psi = \vec{\nabla}\phi\times\hat{z}$$

$$= -\vec{\nabla}\varphi = -\vec{\nabla}\chi\times\hat{z}$$

Complex potential :

$$W = \phi + i\psi \qquad\qquad \Xi = \psi - i\phi = -iW = \varphi + i\chi$$

$$\frac{\partial W}{\partial \bar{z}} = 0 \qquad \longleftrightarrow \qquad \frac{\partial \Xi}{\partial \bar{z}} = 0$$

$$e = -iv$$
$$\bar{e} = +i\bar{v}$$

$$\bar{v} = v_x - iv_y = W'(z) \qquad\qquad \bar{e} = e_x - ie_y = -\Xi'(z)$$

Boundary conditions :


Σ

$$\oint_{\partial\Sigma} \vec{v}\cdot\hat{n}\,dl = \int_\Sigma \vec{\nabla}\cdot\vec{v}\,dA = 0 \qquad\qquad \oint_{\partial\Sigma}\vec{e}\cdot d\vec{l} = \int_\Sigma \vec{\nabla}\wedge\vec{e}\,dA = 0$$

$$\Rightarrow \quad \hat{n}\cdot\vec{v}\big|_{\partial\Sigma} = 0 \qquad\qquad \Rightarrow \quad \hat{n}\wedge\vec{e}\big|_{\partial\Sigma} = 0$$


Σ

$$\oint_{\partial\Sigma}\vec{v}\cdot d\vec{l} = \int_\Sigma \vec{\nabla}\wedge\vec{v}\,dA = \int_\Sigma \omega\,dA \qquad\qquad \oint_{\partial\Sigma}\vec{e}\cdot\hat{n}\,dl = \int_\Sigma \vec{\nabla}\cdot\vec{e}\,dA = \int_\Sigma dA\,4\pi\rho$$

$$\Rightarrow \quad \hat{n}\wedge\vec{v}\big|_{\partial\Sigma} = \omega_{2D} \qquad\qquad \Rightarrow \quad \hat{n}\cdot\vec{e}\big|_{\partial\Sigma} = 4\pi\rho_{2D} = \omega_{2D}$$

line vortex | line charge

$$\vec{\nabla}\wedge\vec{v} = \Gamma\,\delta(\vec{r}-\vec{r}_0) \qquad \vec{\nabla}\cdot\vec{e} = \lambda\,\delta(\vec{r}-\vec{r}_0)$$

$$\oint \vec{v}\cdot d\vec{l} = \Gamma \quad \longleftrightarrow \quad \oint \vec{e}\cdot\hat{n}\,dl = \lambda$$

$$W = \frac{\Gamma}{2\pi i}\ln(z-z_0) \qquad \Xi = -\frac{\lambda}{2\pi}\ln(z-z_0)$$

$$\phi = \frac{\Gamma}{2\pi}\arg(z-z_0) \qquad \varphi = -\frac{\lambda}{2\pi}\ln|z-z_0|$$

<u>Conformal Mapping</u>

Suppose $Z_1 = f(z)$ is analytic, with analytic inverse $z = F(Z)$. Then if $W(z)$ is analytic, so is $W(F(Z)) \equiv \tilde{W}(Z)$:

$$\frac{\partial\tilde{W}}{\partial\bar{Z}} = \frac{\partial W}{\partial z}\frac{\partial z}{\partial\bar{Z}} = \frac{\partial W}{\partial z}\frac{\partial F}{\partial\bar{Z}} = 0$$

Furthermore,

$$\bar{V} = V_x - iV_y = \frac{d\tilde{W}}{dZ} = W'(F(Z))\,F'(Z) \qquad \text{(chain rule)}$$

describes a 2-dim^{l} potential flow in the (X,Y) plane, with $Z = X+iY$.

Note that

$$\bar{V} = \bar{v}\cdot\frac{dz}{dZ}$$

hence in order to describe the same flow at ∞, we must have $\frac{dz}{dZ} \xrightarrow[Z\to\infty]{} 1$.

Conformal mappings preserve <u>angles</u> at all points where $f'(z) \neq 0$. To see this, consider a curve $z(t)$ and its image $Z_1(t) = f(z(t))$. At $t = t_0$, $\dot{Z}_1(t_0) = f'(z_0)\,\dot{z}(t_0)$, so

$$\arg\dot{Z}_1(t_0) = \arg f'(z_0) + \arg\dot{z}(t_0)$$

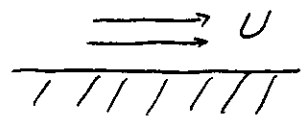
Thus, the angle between the tangents $\dot{Z}_0$ and $\dot{z}_0$ depends <u>only</u> on $f'(z_0)$ and not on the curves themselves.

Examples of Conformal Mappings

- Half-plane to wedge : $f(z) = z^{\alpha/\pi}$ $(0 < \alpha < 2\pi)$

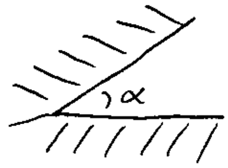
Consider $W(z) = Uz \Rightarrow \bar{V} = V_x - iV_y = U \in \mathbb{R}$,

confined to the region $\operatorname{Im} z > 0$:



Now invoke $f(z) = z^{\alpha/\pi} \Rightarrow F(Z) = Z^{\pi/\alpha}$

$$\tilde{W}(Z) = U Z^{\pi/\alpha}$$



$\alpha < \pi$

$$\bar{V} = \tilde{W}'(Z) = \frac{\pi}{\alpha} U Z^{\frac{\pi}{\alpha}-1}$$

$$= \frac{\pi}{\alpha} U R^{\frac{\pi}{\alpha}-1} e^{i\left(\frac{\pi}{\alpha}-1\right)\Theta}$$

$\alpha > \pi$

Along the $\Theta = 0$ boundary, $\bar{V} = \frac{\pi}{\alpha} U R^{\frac{\pi}{\alpha}-1}$. Along $\Theta = \alpha$, $\bar{V} = -\frac{\pi}{\alpha} U R^{\frac{\pi}{\alpha}-1} e^{-i\alpha} \Rightarrow V = -\frac{\pi}{\alpha} U R^{\frac{\pi}{\alpha}-1} e^{i\alpha}$, which is parallel to the boundary.

$\vec{v}$-field

Note that $\alpha > \pi \Rightarrow$ divergence of $|V|$ at $R = 0$, i.e. at the corner of the wedge.

The electric field lines look like the following:

$\vec{e}$-field

($\rho_{2D} > 0$ shown)

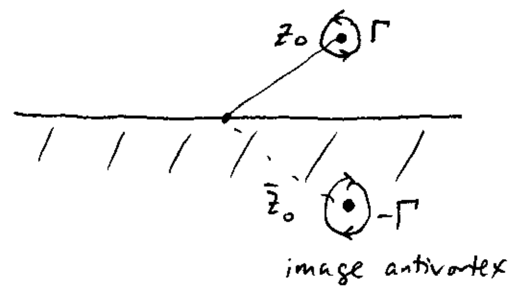
- Interlude : inversions

  - inversion in plane $\operatorname{Im} z = 0$: $g(z) = f(z) + \overline{f(\bar{z})}$

  - inversion in circle $|z| = R$: $g(z) = f(z) + \overline{f\left(\frac{R^2}{\bar{z}}\right)}$

The inversion renders the respective surfaces as streamlines.

- Vortex inside a wedge :



$$W(z) = \frac{\Gamma}{2\pi i} \ln(z - z_0) - \frac{\Gamma}{2\pi i} \ln(z - \bar{z}_0)$$

$$= \frac{\Gamma}{2\pi i} \ln\left(\frac{z - z_0}{z - \bar{z}_0}\right)$$

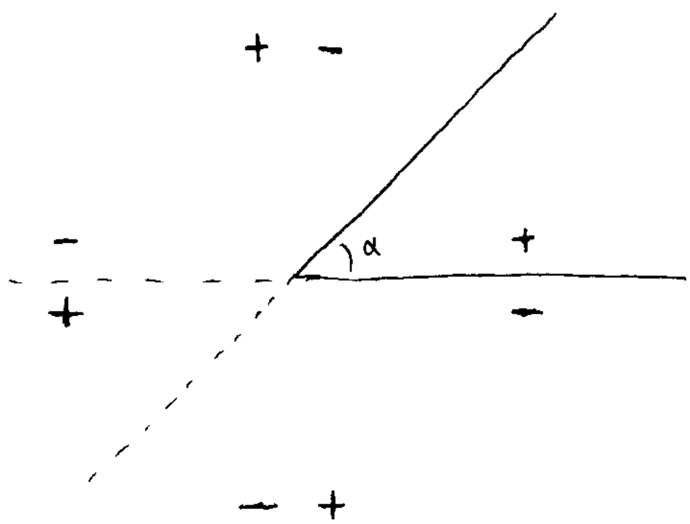
Now map onto a wedge : $z = Z^{\pi/\alpha}$. Let $z_0 \equiv b^{\pi/\alpha} e^{i\frac{\pi\delta}{\alpha}}$, so

$$\widetilde{W}(Z) = \frac{\Gamma}{2\pi i} \ln\left(\frac{Z^{\pi/\alpha} - b^{\pi/\alpha} e^{i\pi\delta/\alpha}}{Z^{\pi/\alpha} - b^{\pi/\alpha} e^{-i\pi\delta/\alpha}}\right)$$

$$\overline{V}(Z) = \frac{\Gamma}{2\pi i} \frac{1}{Z} \cdot \left\{ \frac{1}{1 - (be^{i\delta}/Z)^{\pi/\alpha}} - \frac{1}{1 - (be^{-i\delta}/Z)^{\pi/\alpha}} \right\}$$

$$= \frac{\Gamma}{2\pi i} \cdot R^{-1} e^{-i\Theta} \left\{ \frac{1}{1 - (be^{i(\delta - \Theta)}/R)^{\pi/\alpha}} - \frac{1}{1 - (be^{i(\Theta - \delta)}/R)^{\pi/\alpha}} \right\}$$

By method of images : need infinite set of image vortices .



$\alpha = \pi/4 \Rightarrow 7$ images

$\frac{\alpha}{\pi}$ irrational $\Rightarrow \infty$ images!

Linear Fractional Transformations

invertible
$$\begin{cases} Z = \dfrac{az+b}{cz+d} & a,b,c,d \in \mathbb{C} \\[2ex] z = -\dfrac{dZ-b}{cZ-a} & \text{if } ad-bc \neq 0 \end{cases}$$

LFTs map circles onto circles. Note that

$$f(z) = \frac{z_1-z_3}{z_2-z_4} \cdot \frac{z-z_2}{z-z_3}$$

satisfies $f(z_1)=1$, $f(z_2)=0$, $f(z_3)=\infty$.

Joukowski Transformation

Consider

$$Z = f(z) = \frac{1}{2}\left(z + \frac{c^2}{z}\right)$$

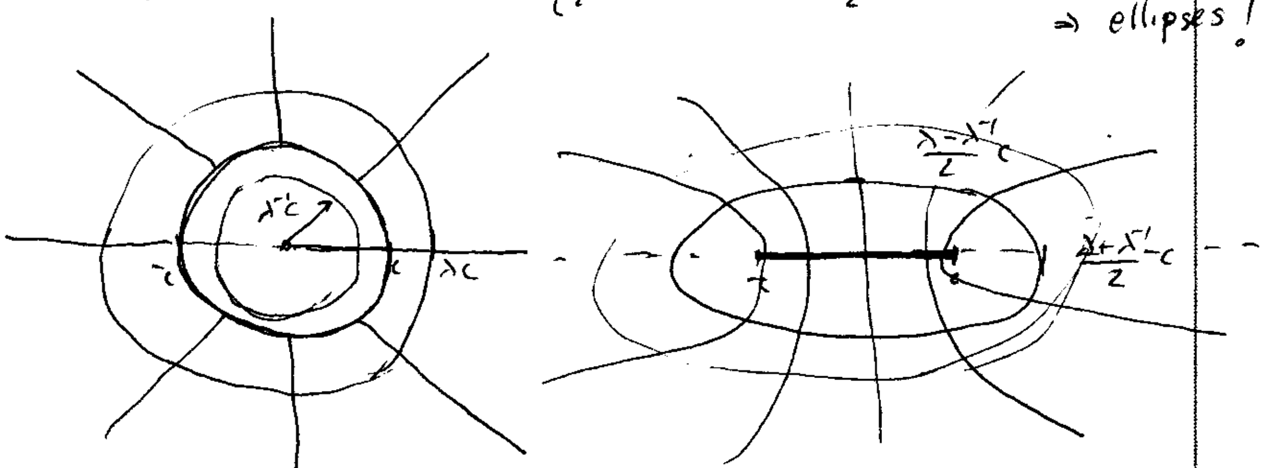
$$z = Z \pm \sqrt{Z^2-c^2} \quad \begin{cases} |z|>|c| \\ |z|<|c| \end{cases}$$

Note (assume $c \in \mathbb{R}$):

$$z = ce^{i\alpha} \;\rightarrow\; Z = \cos\alpha \cdot c$$

$$z = \lambda c e^{i\alpha} \;\rightarrow\; Z = \left\{\tfrac{1}{2}(\lambda+\lambda^{-1})\cos\alpha + \tfrac{i}{2}(\lambda-\lambda^{-1})\sin\alpha\right\}\cdot c$$

$\Rightarrow$ ellipses!



the circles $\lambda c e^{i\alpha}$, $\lambda^{-1} c e^{i\alpha}$ map onto same ellipse

Now consider $v_0 = U e^{i\alpha}$,

$$W(z) = \bar{v}_0 z + v_0 \frac{R^2}{z} + \frac{\Gamma}{2\pi i} \ln z \qquad \text{(flow around cylinder)}$$

Then

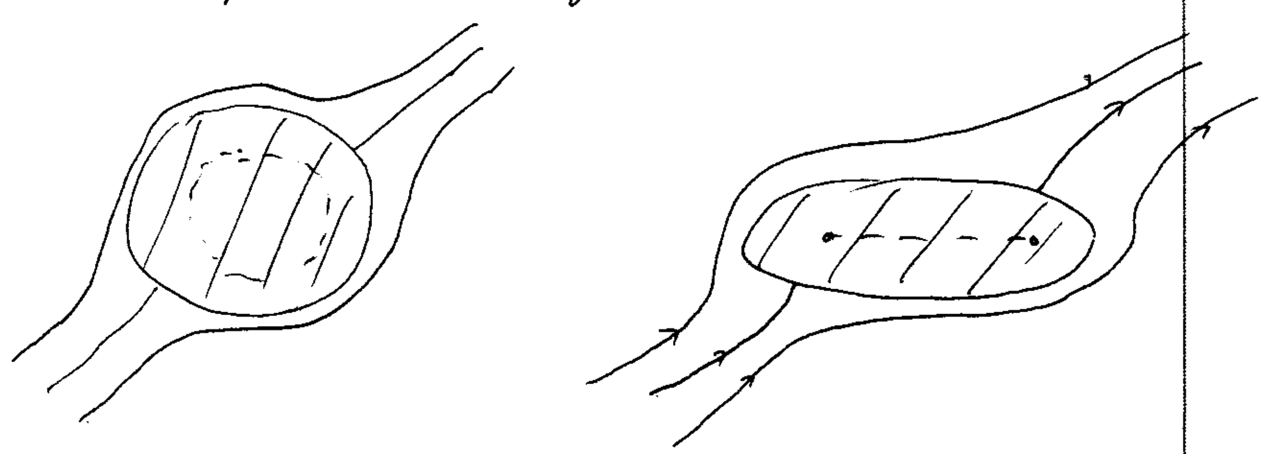
$$\tilde{W}(Z) = \bar{v}_0 F(Z) + v_0 \frac{R^2}{F(Z)} + \frac{\Gamma}{2\pi i} \ln F(Z)$$

$$F(Z) = Z + \sqrt{Z^2 - c^2}$$

$$\frac{1}{F(Z)} = \frac{1}{c^2}\left(Z - \sqrt{Z^2 - c^2}\right)$$

$$\begin{aligned}
\tilde{W}(Z) &= \bar{v}_0 Z + \bar{v}_0 \sqrt{Z^2 - c^2} + \frac{v_0 R^2}{c^2} Z - \frac{v_0 R^2}{c^2} \sqrt{Z^2 - c^2} \\
&\quad + \frac{\Gamma}{2\pi i} \ln\left(Z + \sqrt{Z^2 - c^2}\right) \\
&= \left(\bar{v}_0 + v_0 \frac{R^2}{c^2}\right) Z + \left(\bar{v}_0 - v_0 \frac{R^2}{c^2}\right) \sqrt{Z^2 - c^2} \\
&\quad + \frac{\Gamma}{2\pi i} \ln\left(Z + \sqrt{Z^2 - c^2}\right)
\end{aligned}$$

where we require $|c| < R$. Since $|z| = R$ maps onto an ellipse in the Z plane, we have the complete description of the flow past an elliptical cylinder!



$\Gamma = 0$ streamlines

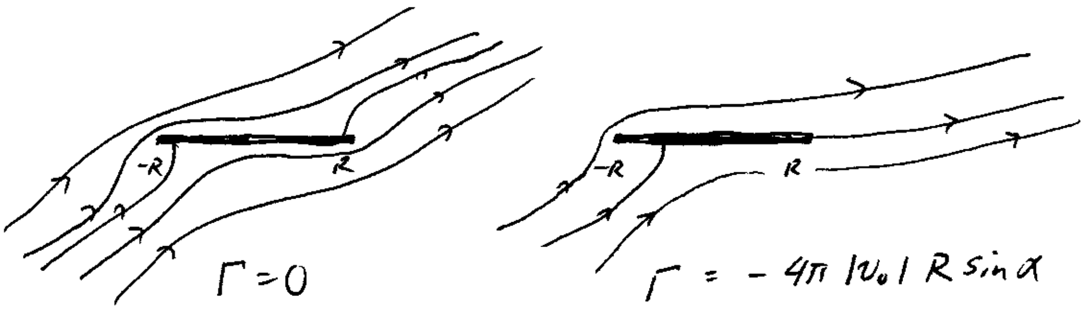
Flow Past a Plate

Let's now take $c = R$, so the circle maps onto a plate. Then with $v_0 = |v_0| e^{+i\alpha}$,

$$\tilde{W}(Z) = 2|v_0| \cos\alpha \, Z - 2i|v_0| \sin\alpha \sqrt{Z^2 - R^2} + \frac{\Gamma}{2\pi i} \ln\left(Z + \sqrt{Z^2 - R^2}\right)$$

$$\overline{V} = V_x - i V_y = \frac{d\tilde{W}}{dz}$$

$$= 2|v_0| \cos\alpha - 2i|v_0| \sin\alpha \frac{Z}{\sqrt{Z^2 - R^2}} + \frac{\Gamma}{2\pi i} \frac{1}{\sqrt{Z^2 - R^2}}$$



$\Gamma = 0$ $\qquad\qquad$ $\Gamma = -4\pi |v_0| R \sin\alpha$

velocity at ∞ is $2|v_0|$!

Note that with $\Gamma = \mp 4\pi |v_0| R \sin\alpha$, the singularity at $Z = \pm R$ is removed. In the first case, the flow leaves the trailing edge smoothly, with a singularity remaining at the leading edge

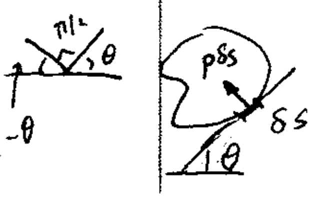
Forces : Blasius' Theorem

Claim :

$$\bar{\mathcal{F}} = \mathcal{F}_x - \mathcal{F}_y = \frac{i}{2} \rho \oint_C \left(\frac{dW}{dz}\right)^2 dz$$

$=$ force per unit length $\mathcal{F} = F/L$

where $C = \partial\Sigma$ is the closed contour of a body of cross-sectional shape Σ. To prove this, let s be the arc length along C, and θ the angle made by the tangent to C wrt the x-axis. Then for a small element δs of C,



$$\delta\bar{\mathcal{F}} = -p \sin\theta \, \delta s - i p \cos\theta \, \delta s$$

$$\delta\mathcal{F}_x = -p \sin\theta \, \delta s ; \quad \delta\mathcal{F}_y = p \cos\theta \, \delta s$$

Since C must be a streamline,

$$v_x = |v| \cos\theta \quad , \quad |v_y| = |v| \sin\theta \qquad \text{on } C$$

$$\frac{dW}{dz} = v_x - i v_y = |v| e^{-i\theta} \quad \text{on } C$$

Now Bernoulli says $p + \frac{1}{2}\rho \vec{v}^2 = A$ = constant

$$\delta \bar{\mathcal{F}} = \delta \mathcal{F}_x - i \delta \mathcal{F}_y = \left(\tfrac{1}{2}\rho \vec{v}^2 - A\right) i e^{-i\theta} \delta s$$

But we have

$$\left(\frac{dW}{dz}\right)^2 = (v_x - i v_y)^2 = |v|^2 e^{-2i\theta}$$

so

$$\delta \bar{\mathcal{F}} = \frac{i}{2}\rho \left(\frac{dW}{dz}\right)^2 e^{+i\theta} \delta s - iA e^{-i\theta} \delta s$$

But $\delta s e^{i\theta} = \delta z$, so

$$\delta \bar{\mathcal{F}} = \frac{i}{2}\rho \left(\frac{dW}{dz}\right)^2 \delta z - iA\, \delta \bar{z}$$

$$\bar{\mathcal{F}}_C = \oint_C d\bar{\mathcal{F}} = \frac{i}{2}\rho \oint_C dz \left(\frac{dW}{dz}\right)^2$$

as the $iA\,d\bar{z}$ term integrates to zero. Similarly, one can establish

$$N = -\frac{1}{2} \mathrm{Re}\left\{\rho \oint_C dz\, z \left(\frac{dW}{dz}\right)^2\right\}$$ = torque about axis through $z=0$

Symmetric Aerofoil

Consider the transformation

$$Z = f(z) = \frac{1}{2}\left(z + a + \frac{(R+a)^2}{z+a}\right)$$

so

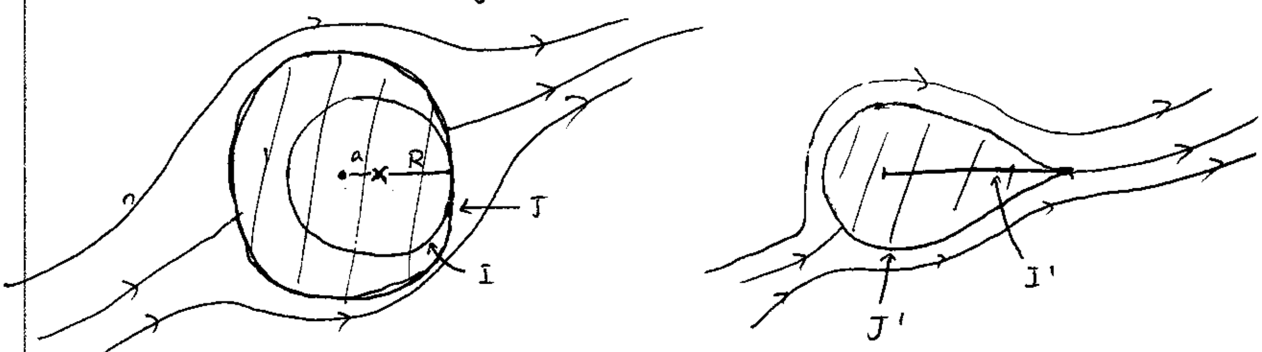
$$z + a = Z + \sqrt{Z^2 - (R+a)^2}$$

$$W(z) = Ue^{-i\alpha} z + Ue^{i\alpha}\frac{R^2}{z} + \frac{\Gamma}{2\pi i}\ln z$$

and

$$\begin{aligned} \widetilde{W}'(Z) &= W'(z)\, z'(Z) \\ &= \left(Ue^{-i\alpha} - Ue^{i\alpha}\frac{R^2}{z^2} + \frac{\Gamma}{2\pi i z}\right)\cdot\left(1 + \frac{Z}{\sqrt{Z^2 - (R+a)^2}}\right) \\ &= \left(Ue^{-i\alpha} - Ue^{i\alpha}\frac{R^2}{z^2} + \frac{\Gamma}{2\pi i z}\right)\cdot\frac{2}{1 - \left(\frac{R+a}{z+a}\right)^2} \end{aligned}$$

It is convenient to analyze this as a f$^{\underline{c}}$ of z.



z - plane Z - plane

Now the only singularity occurs for $z = R$, since $z = -R$ is inside the aerofoil. We can eliminate this singularity by choosing

$$\Gamma = -4\pi U R \sin\alpha$$

NB: A <u>boundary layer</u> may accomodate the <u>slip</u> only if there is no <u>singularity</u> in the velocity field. Else we definitely have <u>separation</u>.

Symmetric Aerofoil

Consider the transformation

$$Z = f(z) = \frac{1}{2}\left(z + a + \frac{c^2}{z+a}\right)$$

with $R + a = c$, i.e.

$$Z_1 = \frac{1}{2}\left(z+a + \frac{(R+a)^2}{z+a}\right)$$

$$z = Re^{i\alpha} \Rightarrow \quad Z_1(\alpha) = \frac{1}{2}\left(Re^{i\alpha} + a + \frac{(R+a)^2}{Re^{i\alpha}+a}\right)$$

$$Z_1(\alpha) = \frac{1}{2}\left(Re^{i\alpha} + a + \frac{(R+a)^2(Re^{-i\alpha}+a)}{R^2+a^2+2Ra\cos\alpha}\right)$$

$$= \frac{1}{2} \; \frac{(Re^{i\alpha}+a)(R^2+a^2+2Ra\cos\alpha) + (R+a)^2(Re^{-i\alpha}+a)}{R^2+a^2+2Ra\cos\alpha}$$

$$X(\alpha) = R\cos\alpha + a + \frac{(R\cos\alpha+a)Ra(1-\cos\alpha)}{R^2+a^2+2Ra\cos\alpha}$$

$$Y(\alpha) = -\frac{R^2a(1-\cos\alpha)}{R^2+a^2+2Ra\cos\alpha}\sin\alpha$$

Let $\epsilon \equiv a/R$, so

$$X(\alpha) = R\left\{\epsilon + \cos\alpha + \frac{(\epsilon+\cos\alpha)\cdot\epsilon(1-\cos\alpha)}{1+\epsilon^2+2\epsilon\cos\alpha}\right\}$$

$$Y(\alpha) = -R\cdot\left\{\frac{\epsilon(1-\cos\alpha)\sin\alpha}{1+\epsilon^2+2\epsilon\cos\alpha}\right\}$$

i.e. a 1-parameter family of shapes.

Uniform Flow Past Cylinder

With

$$W(z) = |v_0|\left(z + \frac{a^2}{z}\right) + \frac{\Gamma}{2\pi i}\ln z$$

we have

$$\bar{\mathcal{F}} = \mathcal{F}_x - i\mathcal{F}_y = \frac{i}{2}\rho \oint_{|z|=a} \left[|v_0|\left(1 - \frac{a^2}{z^2}\right) - \frac{\Gamma}{2\pi i z}\right]^2 dz$$

$$= i\rho\Gamma|v_0|$$

so

$$\mathcal{F}_x = 0, \qquad \mathcal{F}_y = -\rho\Gamma|v_0|$$

as we found earlier.

Uniform Flow Past Elliptical Cylinder

With $\Gamma = 0$, there is no net force, but there is a torque:

$$N = \mathrm{Re}\left\{-\frac{1}{2}\rho \oint_{\text{ellipse}} dZ\; Z\left(\frac{d\tilde{W}}{dZ}\right)^2\right\}$$

$$= \mathrm{Re}\left\{-\frac{\rho}{2}\oint_{|z|=R} dz\left(\frac{dW}{dz}\right)^2 Z\frac{dz}{dZ}\right\}$$

$\Gamma = 0 \Rightarrow W = |v_0|\left(ze^{-i\alpha} + \frac{R^2}{z}e^{i\alpha}\right)$ so with $Z = \frac{1}{2}\left(z + \frac{c^2}{z}\right)$

$$N = \mathrm{Re}\left\{-\frac{\rho}{2}|v_0|^2 \oint_{|z|=R} dz \cdot \left(z + \frac{c^2}{z}\right)\cdot\left(e^{-i\alpha} - \frac{R^2}{z^2}e^{i\alpha}\right)^2 \cdot \left(1 - \frac{c^2}{z^2}\right)^{-1}\right\}$$

$$= -2\pi\rho|v_0|^2 c^2 \sin 2\alpha$$

which tends to align the ellipse broadside to the stream.

Kutta - Joukowski Lift Theorem

Steady flow past a two-dim^{l} body with simple cross-sectional curve C. Let flow be uniform at ∞ with $v_0 = U \in \mathbb{R}$. The theorem says

$$F_x = 0, \quad F_y = -\rho U \Gamma$$

Proof: Assume $z=0$ lies inside the body. Then if flow is free of singularities, $W'(z)$ has a Laurent expansion valid for $R < |z| < \infty$ where R is radius of smallest circle enclosing Σ cross-section. Clearly

$$\frac{dW}{dz} = U + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$$

since $\bar{V}(\infty) = U$. Now we can integrate around <u>any</u> closed surface enclosing Σ to compute $\mathcal{F}$, since $W'(z)$ has no poles in the region $D_R - \Sigma$, where D_R is a disk of radius R. Then

$$\mathcal{F}_x - i\mathcal{F}_y = \frac{i}{2}\rho \oint_{D_R} \left(U + \frac{a_1}{z} + \frac{a_2}{z} + \cdots\right)^2 dz$$

$$= -2\pi\rho U a_1$$

But the circulation is related to a_1: change in quantity

$$2\pi i a_1 = \oint_{C=\partial\Sigma} dz \frac{dW}{dz} = [W]_C = [\phi + i\psi]_C$$

Since C is a streamline, $[\psi]_C = 0$. But $[\phi]_C = \oint_C \vec{v} \cdot d\vec{l} = \oint_C \vec{\nabla}\phi \cdot d\vec{l} = \Gamma$

So $2\pi i a_1 = \Gamma$, and $\mathcal{F} = i\rho U \Gamma$.

Steady Momentum Flow

Euler's equation in the absence of forces, and with steady flow, gives

$$\rho(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}p$$

Thus,

$$\int_{\Omega} \rho v_j \frac{\partial v_i}{\partial x_j}\, dV = \oint_{\partial\Omega} \rho\, v_i v_j n_j\, d\Sigma$$

since

$$v_j \frac{\partial v_i}{\partial x_j} = \frac{\partial}{\partial x_j}(v_i v_j) \qquad \text{with} \quad \vec{\nabla}\cdot\vec{v} = 0$$

Then we have

$$\int_{\Omega} \rho(\vec{v}\cdot\vec{\nabla})\vec{v}\, dV = -\oint_{\partial\Omega} p\,\hat{n}\, d\Sigma = \oint_{\partial\Omega} \rho\,\vec{v}(\hat{n}\cdot\vec{v})\, d\Sigma$$

$$\rho\vec{v} = \text{momentum per unit volume}$$
$$\rho\vec{v}(\hat{n}\cdot\vec{v}) = \text{momentum flux density}$$

Thus,

total force on Ω = rate at which momentum leaves Ω through $\partial\Omega$

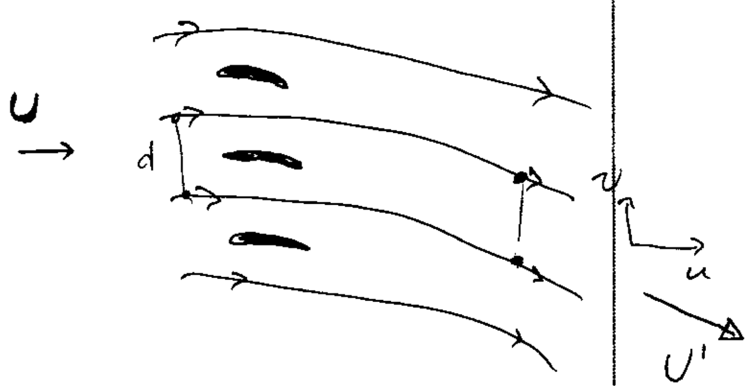
Stack of Aerofoils

$$u_1 d = Ud \Rightarrow u_2 = U$$

$$F_y = -\rho U v_2 d$$

$$\Rightarrow \Gamma = v_2 d \quad \text{circulation}$$

$$F_y = -\rho U \Gamma \quad \checkmark$$



Drag Forces and D'Alambert's Paradox

Consider a moving body in an incompressible fluid, assuming potential flow. We have $\nabla^2\phi = 0$, so at large distances from the body,

$$\phi = \frac{C}{r} + \vec{A}\cdot\vec{\nabla}\frac{1}{r} + \dots \qquad \text{(shift to moving frame centered at body center)}$$

where C, $\vec{A}$ are independent of coordinates, but may depend on the shape of the body. Clearly $C=0$, since

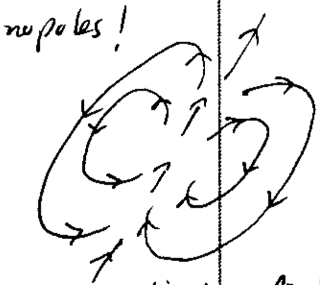
$$\vec{v} = \vec{\nabla}\phi \sim -\frac{C\hat{r}}{r^2} \quad \text{as } r\to\infty$$

$$\rho \oint_{|\vec{r}|=R} d^2r\, \vec{v}\cdot\hat{n} = -4\pi\rho C = \text{mass flux}$$

But since the fluid is incompressible, $C=0$. No monopoles!

Thus, the leading contribution to $\vec{v}$ at $r\to\infty$ is

$$\vec{v} \approx \vec{\nabla}\,\vec{A}\cdot\vec{\nabla}\frac{1}{r} = \frac{3\hat{n}(\vec{A}\cdot\hat{n}) - \vec{A}}{r^3}$$



dipolar field

Total energy:

$$E = \int_{\mathbb{R}^3\backslash\Omega} dV\, \tfrac{1}{2}\rho\vec{v}^2$$

Ω = body

$$= \tfrac{1}{2}\rho\vec{u}^2(V-\Omega) + \tfrac{1}{2}\rho\int_{\mathbb{R}^3\backslash\Omega} dV(\vec{v}-\vec{u})(\vec{v}+\vec{u}) \qquad V = \tfrac{4}{3}\pi R^3 \to \infty$$

$$\vec{\nabla}\cdot\left[(\phi+\vec{u}\cdot\vec{r})(\vec{\nabla}\phi-\vec{u})\right] = (\vec{\nabla}\phi+\vec{u})\cdot(\vec{\nabla}\phi-\vec{u}) = \vec{v}^2-\vec{u}^2 \checkmark$$

So

$$E = \tfrac{1}{2}\rho\vec{u}^2(V-\Omega) + \tfrac{\rho}{2}\int d\hat{n}\,\left\{3(\vec{A}\cdot\hat{n})(\vec{u}\cdot\hat{n}) - (\vec{u}\cdot\hat{n})R^3\right\}$$

$$= 2\pi\rho\vec{A}\cdot\vec{u} - \tfrac{1}{2}\rho\Omega\vec{u}^2 \equiv \tfrac{1}{2}M_{ij}u_iu_j$$

induced mass tensor

All other contributions vanish as $R\to\infty$.

Note $\vec{A}$ must be linear in $\vec{u}$. Now $\vec{u} = \dfrac{\partial E}{\partial\vec{P}}$, whence $P_i = M_{ij}u_j$.

For a sphere, $\vec{A} = \tfrac{1}{2}R^3\vec{u} \Rightarrow M_{ij} = \tfrac{2\pi}{3}\rho R^3\delta_{ij}$.

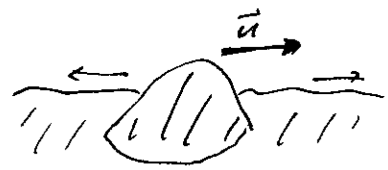
We now have

$$\vec{P} = 4\pi\rho\vec{A} - mV_0\vec{u}$$
$$\vec{F} = -\frac{d\vec{P}}{dt} = \text{force on fluid.}$$

If the flow is uniform, then $\vec{F} = 0$, and

$$F_{drag} = \vec{F}\cdot\hat{u}$$
$$\vec{F}_{lift} = \vec{F} - \hat{u}(\hat{u}\cdot\vec{F}) = 0$$

The pressure forces around the body must balance out. This is known as <u>D'Alambert's paradox</u>. Since there is no dissipation of energy in an ideal fluid, there can be no drag on a moving body. If a body moves while intersecting the free surface of a fluid, energy can be dissipated in the form of surface waves radiating to infinity.



wave drag at a surface

An external force $\vec{F}_{ext}$ on an object in an ideal fluid therefore results in

$$\vec{F}_{ext} = M_0\frac{d\vec{u}}{dt} + \frac{d\vec{P}}{dt}$$

mass of body

$$\Rightarrow \quad (M_0\delta_{ij} + M_{ij})\frac{du_j}{dt} = F_i \qquad \text{eqn of motion}$$

Another related problem: a body immersed in an oscillating fluid, where $\lambda_{osc} \gg L_{body}$. Let $\vec{v}$ be the fluid velocity at the body's position assuming the body were absent. Then

$$\rho V_0\frac{dv_i}{dt} - M_{ij}\frac{d}{dt}(u_j - v_j) = M_0\frac{du_i}{dt}$$

force if body moves with fluid — force on fluid — total force on body

Thus,

$$(M_0 \delta_{ij} + M_{ij}) u_j = (M_{ij} + \rho V_0 \delta_{ij}) v_j$$

$$\vec{u} = (M + M_0 \mathbb{1})^{-1} (M + \rho V_0 \mathbb{1}) \vec{v}$$

Example: sphere immersed in ideal fluid. $M_{ij} = \frac{1}{2} \rho V_0 \delta_{ij}$, so

$$\vec{u} = \frac{\frac{3}{2}\rho}{\frac{1}{2}\rho + \rho_0} \vec{v} = \frac{3\rho}{\rho + 2\rho_0} \vec{v}$$

If $\rho_0 > \rho$, the sphere lags; if $\rho_0 < \rho$ the sphere leads.

Viscous Fluids

We have already derived the relation

$$\frac{\partial}{\partial t} (\rho v_i) + \frac{\partial \Pi_{ij}}{\partial x_j} = 0$$

where Π_{ij} = momentum flux density tensor. For an ideal fluid,

$$\Pi_{ij} = p \delta_{ij} + \rho v_i v_j$$

In a viscous fluid, there is an extra term:

$$\begin{aligned} \Pi_{ij} &= p \delta_{ij} + \rho v_i v_j - \tilde{\sigma}_{ij} \\ &\equiv \rho v_i v_j - \sigma_{ij} \end{aligned}$$

$$\sigma_{ij} = - p \delta_{ij} + \tilde{\sigma}_{ij}$$

stress tensor $\qquad\qquad$ viscosity stress tensor

The stress tensor gives that part of the momentum flux not due to direct transfer of momentum within the mass of the moving fluid.

The viscosity stress tensor is due to internal friction when neighboring fluid particles move at different velocities. So $\tilde{\sigma}_{ij}$ must depend on $\frac{\partial v_i}{\partial x_j}$. The antisymmetric part must vanish, since this corresponds to uniform rotation

$$\vec{v} = \vec{\Omega} \times \vec{r} \Rightarrow v_i = \epsilon_{ijk} \Omega_j x_k$$

$$\frac{\partial v_i}{\partial x_k} = \epsilon_{ijk} \Omega_j = -\frac{\partial v_k}{\partial x_i}$$

So we conclude

$$\tilde{\sigma}_{ij} = \eta \underbrace{\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \vec{\nabla} \cdot \vec{v} \right)}_{\text{traceless}} + \zeta \delta_{ij} \vec{\nabla} \cdot \vec{v}$$

The coefficients, both of which must be positive, are the <u>shear</u> (η) and <u>bulk</u> (ζ) <u>viscosities</u>. Assuming these are constants, we have

$$\rho \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tilde{\sigma}_{ij}}{\partial x_j}$$

$$\Rightarrow \boxed{\rho \frac{\partial \vec{v}}{\partial t} + \rho (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\vec{\nabla} p + \eta \nabla^2 \vec{v} + (\zeta + \tfrac{1}{3}\eta) \vec{\nabla} (\vec{\nabla} \cdot \vec{v})}$$

If the fluid is incompressible, $\vec{\nabla} \cdot \vec{v} = 0$ and

$$\boxed{\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{1}{\rho} \vec{\nabla} p + \nu \nabla^2 \vec{v}}$$

with $\nu = \eta / \rho$ the kinematic viscosity. This is the <u>Navier-Stokes equation</u>. $\nu_{H_2O} = 10^{-2}\ \text{cm}^2/\text{sec}$; $\nu_{air} = 0.15\ \text{cm}^2/\text{sec}$; $\nu_{glycerine} = 6.8\ \text{cm}^2/\text{sec}$ (all @ 20°C).

$$[\eta] = [\zeta] = \text{g}/\text{cm}\cdot\text{sec}$$

Taking the curl, we obtain

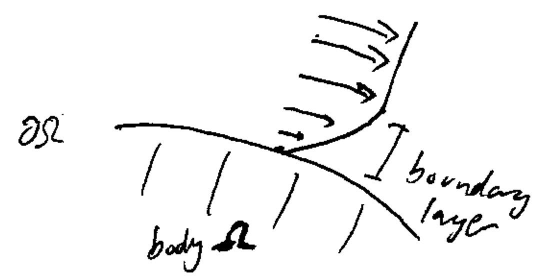
$$\frac{\partial \vec{\omega}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{\omega}) + \nu \nabla^2 \vec{\omega}$$

for the vorticity $\vec{\omega} = \vec{\nabla} \times \vec{v}$. Notice the pressure drops out of this equation.

Since molecular forces pin the closest fluid particles next to a body surface, the boundary conditions for a viscous fluid are

$$\vec{v}\big|_{\partial\Omega} = 0$$

Recall for an ideal fluid only $\hat{n}\cdot\vec{v}\big|_{\partial\Omega} = 0$, and $\hat{n}\times\vec{v}\big|_{\partial\Omega}$ may be finite (slip). The difference, i.e. the slip, is accomodated in a <u>boundary layer</u> (assuming no separation).



The force acting on an element $\hat{n}\,dA$ of surface is given in terms of the momentum flux

$$\Pi_{ij} n_j \, dA = (\rho v_i v_j - \sigma_{ij}) n_j \, dA$$

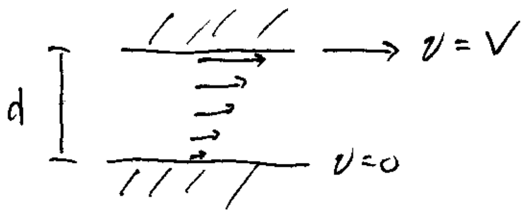
Thus,

$$\frac{dF_i}{dA} = -\sigma_{ij} n_j = p n_i - \tilde{\sigma}_{ij} n_j$$

since $\vec{v} = 0$ on body surfaces. At <u>free surfaces</u> of a fluid,

$$p n_i = \tilde{\sigma}_{ij} n_j$$

Fluids:

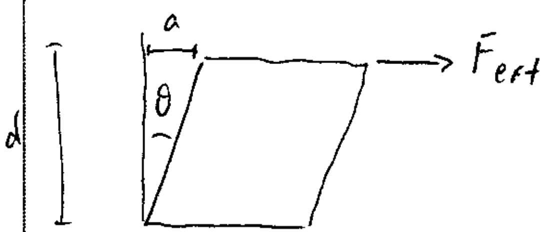


$$F_x = -\eta \frac{\partial v_x}{\partial y} \cdot A = -\eta \frac{V}{d} A$$

drag on surface

$$F_{ext} = \eta \frac{V}{d} \cdot A$$

Solids:



$$F_{ext} = \sigma \cdot \frac{a}{d} \cdot A$$

shear modulus

Reynolds Number

$$R = \frac{UL}{\nu} \qquad \text{(dimensionless)}$$

U = characteristic fluid flow speed (relative to body)

L = characteristic length of body

ν = kinematic viscosity

$$[\text{inertial term}] = [(\vec{v} \cdot \vec{\nabla})\vec{v}] = \mathcal{O}(U^2/L)$$

$$[\text{viscous term}] = [\nu \nabla^2 \vec{v}] = \mathcal{O}(\nu U/L^2)$$

$$\left| \frac{\text{inertial term}}{\text{viscous term}} \right| \sim \frac{UL}{\nu} = R$$

$R \ll 1$: viscosity dominates (e.g. microorganisms)

$R \gg 1$: inertia dominates (e.g. airplanes) ; turbulence

Boundary layer: $\frac{\delta}{L} \sim R^{-1/2}$

Large R is necessary for potential flow, but not sufficient.

Vorticity Diffusion : Example

Suppose $0 < y < \infty$, with

$$v(0,t) = V$$
$$v(\infty,t) = 0$$

i.e. a surface $y=0$ is impulsively jerked to move at $v=V$.
We have $v_x = v(y,t)$, $v_y = v_z = 0$ by assumption. We must solve

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 v}{\partial y^2}$$

so if $p = p(y)$ then $\frac{\partial p}{\partial x} = 0$ and we have diffusion:

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial y^2}$$

(Note for the $\hat{y}$-direction, we have $\frac{\partial p}{\partial y} = -\rho g \Rightarrow p = p_0 - \rho g y$.)
We assume that $v = v(s)$, where $s = y/\sqrt{\nu t}$. Then

$$0 = \frac{d^2 v}{ds^2} + \frac{s}{2}\frac{dv}{ds} = \frac{d}{ds}\left(e^{+s^2/4}\frac{dv}{ds}\right) \cdot e^{-s^2/4}$$

so

$$v'(s) = B e^{-s^2/4}$$
$$v(s) = A + B\int_0^s ds'\, e^{-s'^2/4}$$

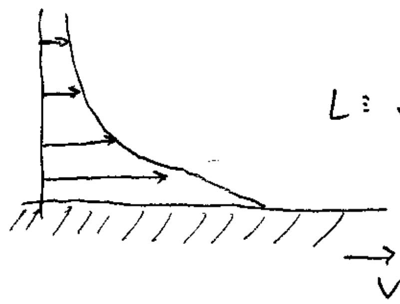
Boundary conditions:

$$v(s=\infty) = 0 \quad , \quad v(0) = V$$

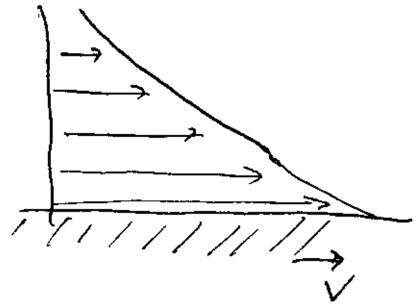
$$\Rightarrow \quad v(y,t) = V\left\{1 - \mathrm{erf}\left(y/\sqrt{4\nu t}\right)\right\}$$
$$= V \mathrm{erfc}\left(y/\sqrt{4\nu t}\right)$$

Thus,

$$\omega = -\frac{\partial v}{\partial y} = \frac{V}{\sqrt{\pi \nu t}}\, e^{-y^2/4\nu t}$$



$L \approx \sqrt{\nu t}$



Thus, the viscous diffusion time is $\Delta t \sim L^2/\nu$, where L is the distance over which viscosity diffuses.

Energy Dissipation

We have for an incompressible fluid

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho \vec{v}^2\right) = -\frac{\partial}{\partial x_i}\underbrace{\left[\rho v_i\left(\tfrac{1}{2}\vec{v}^2 + \frac{P}{\rho}\right) - \vec{v}_j \tilde{\sigma}_{ji}\right]}_{\text{energy flux density in the fluid}} - \tilde{\sigma}_{ij}\frac{\partial v_i}{\partial x_j}$$

Integrating over the entirety of the fluid,

$$\frac{dE_{kin}}{dt} = -\int dV\, \tilde{\sigma}_{ij}\frac{\partial v_i}{\partial x_j}$$

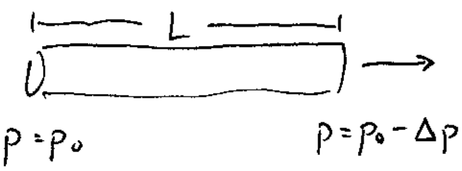
$$= -\tfrac{1}{2}\eta \int dV \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2 < 0$$

So energy is always dissipated if $\eta > 0$.

Pipe Flow

Consider steady flow in a pipe of radius R. We have $v_z = v(r)$, $v_r = v_\phi = 0$. Then

$$\nabla^2 v = \frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) = \frac{1}{\eta}\frac{\partial p}{\partial z} = -\frac{\Delta p}{\eta L}$$



Integrating, we find

$$v(r) = -\frac{\Delta p}{4\eta L} r^2 + C_1 \ln r + C_2$$

Since $v(r=0)$ is finite, $C_1 = 0$. We adjust C_2 by demanding

$$v(r=R) = 0 \Rightarrow C_2 = \frac{\Delta p}{4\eta L} R^2$$

and

$$v(r) = \frac{\Delta p}{4\eta L}(R^2 - r^2)$$

I.e. a parabolic distribution. The total mass flux is

$$\dot{M} \equiv Q = 2\pi\rho \int_0^R dr\, r\, v(r)$$

$$= \frac{\pi \Delta p}{8\nu L} R^4$$

which is known as <u>Poiseuille's formula</u>.

Spin-down in a cylinder

Suppose $v_\phi = \Omega r$ for $r \leq R$ at $t=0$ and suddenly the cylinder is brought to rest. Then

$$\frac{\partial v_\phi}{\partial t} = \nu\left(\frac{\partial^2 v_\phi}{\partial r^2} + \frac{1}{r}\frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r^2}\right)$$

where we've written this in polar coordinates. Boundary conditions are : $v_\theta = 0$ at $r = R$ for $t \geq 0$. Find

$$v_\phi(r,t) = -2\Omega R \sum_{n=1}^{\infty} \frac{J_1(x_{1,n} r/R)}{x_{1,n} J_0(x_{1,n})} e^{-x_{1,n}^2 \nu t/R^2}$$

where $J_\nu(x_{\nu,n}) = 0$, i.e. $x_{\nu,n}$ = n^{th} zero of $J_\nu(x)$.

$x_{1,n=1} \approx 3.83$. The viscous diffusion time is $\Delta t = R^2/\nu x_{1,n}^2$.

Teacup: $R \simeq 4\,cm$, $\nu \sim 10^{-2}\,cm^2/s$ $\Rightarrow$ $\Delta t \approx 2$ minutes!

Too long! (15 sec is closer to truth.) What have we neglected? The bottom of the teacup!

Line Vortex : Viscous Decay

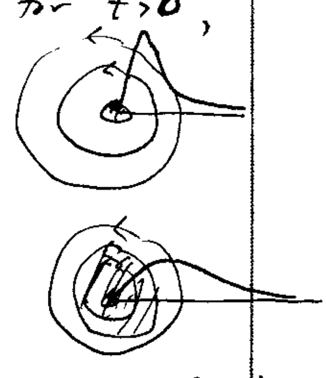
Let $\vec{v} = \frac{\Gamma}{2\pi r}\hat{\phi}$. Let $\Gamma(r,t) = 2\pi r\, v_\phi(r,t)$ be the quantity of interest. Then

$$\frac{\partial \Gamma}{\partial t} = \nu\left(\frac{\partial^2 \Gamma}{\partial r^2} - \frac{1}{r}\frac{\partial \Gamma}{\partial r}\right)$$

with $\Gamma(r, t=0) = \Gamma_0$, and $\Gamma(r=0, t) = 0$ for $t>0$, i.e. no singularity in v_ϕ at $r=0$. Find

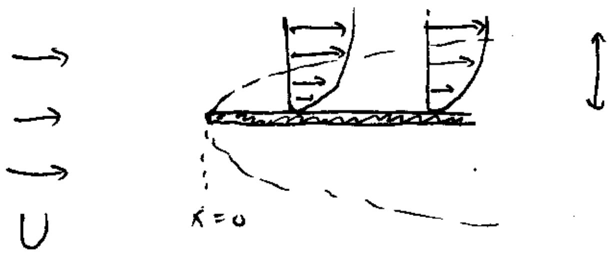
$$\Gamma(r,t) = \Gamma_0\left(1 - e^{-r^2/4\nu t}\right)$$

$$v_\phi(r,t) = \frac{\Gamma_0}{2\pi r}\left(1 - e^{-r^2/4\nu t}\right)$$



viscous diffusion of a vortex

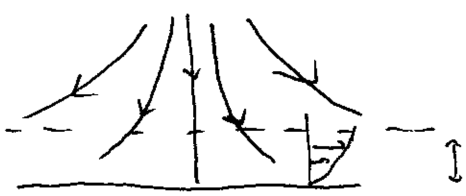
High R flow past a flat plate



$$\delta \sim \left(\frac{\nu x}{U}\right)^{1/2}$$

Stagnation Point

$$v_x = \alpha x \quad , \quad v_y = -\alpha y$$



$\delta \propto (\nu/\alpha)^{1/2}$ ⇒ flow toward wall balances viscous diffusion effects

Stokes' Formula

Consider now small R flow. For steady flow,

$$(\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p + \nu\nabla^2\vec{v}$$

If $R \ll 1$, the inertial term may be neglected, and

$$\nu\nabla^2\vec{v} = -\frac{1}{\rho}\vec{\nabla}p$$

This eqn, along with $\vec{\nabla}\cdot\vec{v} = 0$ (continuity), completely determines the fluid motion. Note that $\nabla^2\vec{\omega} = 0$, independent of p.

Now suppose $\vec{v} = \vec{v}_\infty$ at infinity, and write

$$\vec{v} = \vec{v}_\infty + \vec{\nabla}\times\vec{A}$$

For a sphere in a steady flow, we must have

$$\vec{A} = \vec{\nabla}f \times \vec{v}_\infty \qquad \text{axial vector}$$

Then, since $\vec{v}_\infty$ is constant,

$$\vec{\nabla}\times\vec{A} = \vec{\nabla}\times\vec{\nabla}\times(f\vec{v}_\infty)$$

Then

$$\vec{\omega} = \vec{\nabla}\times\vec{v} = \vec{\nabla}\times\left(\vec{\nabla}\,\vec{\nabla}\cdot(f\vec{v}_\infty) - \nabla^2 f\vec{v}_\infty\right)$$

$$= -\nabla^2\,\vec{\nabla}\times(f\vec{v}_\infty)$$

and since $\vec{\nabla}^2\vec{\omega} = 0$, we have $\nabla^2(\vec{\nabla}f\times\vec{v}_\infty) = 0$

$$\Rightarrow \nabla^4 f = \text{constant} \qquad \nabla^4 \equiv (\vec{\nabla}^2)^2$$

In order for $\vec{v} - \vec{\nabla}\times\vec{A} = \vec{v}_\infty$ at infinity, we must have

$$\nabla^4 f = 0$$

The allowed sol$\underline{\text{ns}}$ are

$$f = C_1 r + \frac{C_2}{r}$$

Matching at $r = R$ gives

$$v_r = v_\infty\cos\theta\left\{1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right\}$$

$$v_\theta = -v_\infty\sin\theta\left\{1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right\}$$

The pressure is

$$\vec{\nabla}p = \eta\nabla^2\vec{v} = \eta\,\vec{v}_\infty\cdot\vec{\nabla}\,\nabla^2 f + p_0$$

$$= p_0 - \frac{3}{2}\frac{\hat{n}\cdot\vec{v}_\infty}{r^2}R$$

The drag force :

$$F = \oint dA\left(-p\cos\theta + \tilde{\sigma}_{rr}\cos\theta - \tilde{\sigma}_{r\theta}\sin\theta\right)$$

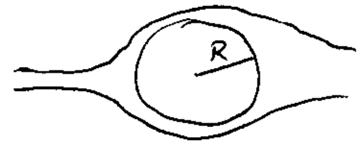
$$\tilde{\sigma}_{rr} = 2\eta\frac{\partial v_r}{\partial r}, \qquad \tilde{\sigma}_{r\theta} = \eta\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)$$

$$= 0 \qquad\qquad = -\frac{3\eta}{2R}v_\infty\sin\theta$$

$$\Rightarrow \quad p = p_0 - \frac{3\eta}{2R}v_\infty\cos\theta$$

$$F = 6\pi R\eta v_\infty \qquad \underline{\text{Stokes' formula}}$$

Stokes' Formula : Viscous Drag on a Sphere



We have in steady state

if gravity present

$$(\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p + \nu\nabla^2\vec{v} + \vec{g}$$

Suppose $Re = UL/\nu \ll 1$. Then we should assume that the LHS is parametrically small, and begin our analysis by neglecting it. In the absence of body forces, then,

$$0 = -\vec{\nabla}p + \eta\nabla^2\vec{v} = -\vec{\nabla}p - \eta\vec{\nabla}\times\vec{\omega} + \eta\vec{\nabla}(\vec{\nabla}\cdot\vec{v})$$

We seek a sol$^{\underline{n}}$ of the form

$$\vec{v}(r,\theta) = v_r\hat{r} + v_\theta\hat{\theta} \qquad \text{(axisymmetric)}$$

Since

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

we write

$$v_r = \frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}, \quad v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r}$$

which then satisfies $\vec{\nabla}\cdot\vec{v} = 0$. The curl is

$$\vec{\omega} = \vec{\nabla}\times\vec{v} = \frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi}\\ \partial_r & \partial_\theta & \partial_\phi\\ v_r & r v_\theta & r\sin\theta\, v_\phi\end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & \hat{\theta} & r\sin\theta\,\hat{\phi}\\ \partial_r & \partial_\theta & \partial_\phi\\ \frac{1}{r^2\sin\theta}\partial_\theta\Psi & -\frac{1}{\sin\theta}\partial_r\Psi & 0\end{vmatrix}$$

$$= -\left(\frac{1}{r\sin\theta}\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r^3}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right)\right)\hat{\phi}$$

$$\equiv -\frac{\hat{\phi}}{r\sin\theta}L\Psi$$

with

$$L \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$$

Thus, $\vec{\omega} = \omega_\phi \hat{\phi}$ with $\omega_\phi = -\frac{1}{r\sin\theta} L\Psi$

$$\vec{\nabla} \times \vec{\omega} = \frac{\hat{r}}{r\sin\theta} \frac{\partial(\omega_\phi \sin\theta)}{\partial\theta} - \frac{\hat{\theta}}{r} \frac{\partial(r\omega_\phi)}{\partial r}$$

$$= -\frac{\hat{r}}{r^2\sin\theta} \frac{\partial}{\partial\theta} L\Psi + \frac{\hat{\theta}}{r\sin\theta} \frac{\partial}{\partial r} L\Psi$$

and since $\vec{\nabla} p = -\eta \vec{\nabla} \times \vec{\omega}$,

$$\frac{\partial p}{\partial r} = + \frac{\eta}{r^2\sin\theta} \frac{\partial}{\partial\theta} L\Psi$$

$$\frac{1}{r} \frac{\partial p}{\partial\theta} = - \frac{\eta}{r\sin\theta} \frac{\partial}{\partial r} L\Psi$$

We can now eliminate p:

$$\frac{\partial^2 p}{\partial r \partial\theta} = \frac{\eta}{r^2} \frac{\partial}{\partial\theta} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} L\Psi$$

$$= -\frac{\eta}{\sin\theta} \frac{\partial^2}{\partial r^2} L\Psi$$

$$\Rightarrow \quad \frac{\eta}{r^2\sin\theta} L^2\Psi = 0$$

So we must solve $L^2\Psi = 0$. The boundary conditions are no slip at $r = R$, and that $\vec{v} = U\hat{z} = U\cos\theta\,\hat{r} - U\sin\theta\,\hat{\theta}$ as $r \to \infty$.

Thus we have four boundary conditions,

- $\left.\frac{\partial\Psi}{\partial r}\right|_{r=R} = 0$ • $\left.\frac{\partial\Psi}{\partial\theta}\right|_{r=R} = 0$
- $\left.\frac{\partial\Psi}{\partial r}\right|_{r=\infty} = +Ur\sin^2\theta$ • $\left.\frac{\partial\Psi}{\partial\theta}\right|_{r=\infty} = Ur^2\sin\theta\cos\theta$

which is appropriate for the fourth order eqn $L^2\Psi = 0$.

We therefore have

$$\Psi \sim \frac{1}{2} U r^2 \sin^2\theta \quad \text{as} \quad r \to \infty$$

which suggests a general sol$^{\underline{n}}$ of the form

$$\Psi(r,\theta) = U f(r) \sin^2\theta \qquad f(r\to\infty) = \frac{1}{2} r^2$$

Applying L to Ψ,

$$L\Psi = \left(\frac{\partial^2}{\partial r^2} - \frac{2}{r^2}\right)\Psi$$

$$L^2\Psi = \left(\frac{\partial^2}{\partial r^2} - \frac{2}{r^2}\right)^2 f(r) \cdot U \sin^2\theta$$

Try a homogeneous sol^n :

$$f(r) \propto r^\alpha$$

$$\Rightarrow \quad \big(\alpha(\alpha-1) - 2\big)\big((\alpha-2)(\alpha-3) - 2\big) = 0$$

From $\alpha(\alpha-1) = 2$ we obtain $\alpha = -1, +2$. From $(\alpha-2)(\alpha-3) = 2$,

we obtain $\alpha - 2 = -1, +2 \Rightarrow \alpha = +1, +4$. Thus,

$$f(r) = C_{-1} r^{-1} + C_{+1} r + C_{+2} r^2 + C_{+4} r^4$$

Clearly $C_{+2} = \frac{1}{2}$, $C_{+4} = 0$, so

$$f(r) = \frac{A}{r} + Br + \frac{1}{2} r^2$$

Now apply

$$\left.\frac{\partial \Psi}{\partial r}\right|_{r=R} = \left(-\frac{A}{R^2} + B + R\right) U \sin^2\theta = 0$$

$$\left.\frac{\partial \Psi}{\partial \theta}\right|_{r=R} = \left(\frac{A}{R} + BR + \frac{1}{2}R^2\right) \cdot 2U \sin\theta \cos\theta = 0$$

This fixes A and B to be $A = \frac{1}{4}R^3$, $B = -\frac{3}{4}R$, so

$$\Psi(r,\theta) = \frac{1}{4}U\left(\frac{R^3}{r} - 3Rr + 2r^2\right)\sin^2\theta$$

We've now solved for the flow. Let's compute the drag on the sphere.

We have

$$L\,\Psi = +\frac{3}{2}UR \cdot \frac{\sin^2\theta}{r}$$

$$\frac{\partial p}{\partial r} = \frac{\eta}{r^2\sin\theta}\frac{\partial}{\partial\theta}L\,\Psi = +3\eta UR \cdot \frac{\cos\theta}{r^3}$$

$$\frac{\partial p}{\partial\theta} = -\frac{\eta}{\sin\theta}\frac{\partial}{\partial r}L\,\Psi = +\frac{3}{2}\eta UR\frac{\sin\theta}{r^2}$$

$$\Rightarrow \quad p(r,\theta) = p_\infty - \frac{3}{2}\eta UR\frac{\cos\theta}{r^2}$$

The force on the sphere is not simply

$$\int_0^{2\pi} d\theta \sin\theta \int_0^{\pi} d\phi \; p(R,\theta)\,\hat{r}$$

due to pressure forces alone. The reason is that viscous forces also act on the sphere. The force per unit area acting on a surface element of fluid is

$$P_i = p\,n_i - \tilde{\sigma}_{ij}n_j = -\sigma_{ij}n_j$$

$$\sigma_{ij} = -p\,\delta_{ij} + \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

In spherical coordinates,

$$\sigma_{rr} = -p + 2\eta \frac{\partial v_r}{\partial r}$$

$$\sigma_{\theta\theta} = -p + 2\eta\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)$$

$$\sigma_{\phi\phi} = -p + 2\eta\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \operatorname{ctn}\theta}{r}\right)$$

$$\sigma_{r\theta} = \eta\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)$$

$$\sigma_{r\phi} = \eta\left(\frac{\partial v_\phi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r}\right)$$

$$\sigma_{\theta\phi} = \eta\left(\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{1}{r}\frac{\partial v_\phi}{\partial \theta} - \frac{v_\phi \operatorname{ctn}\theta}{r}\right)$$

Thus, with $\hat{n} = \hat{r}$,

$$\vec{P} = -\sigma_{rr}\,\hat{r} - \sigma_{r\theta}\,\hat{\theta}$$

$$\begin{cases} \hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z} \\ \hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z} \\ \hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \end{cases}$$

We have

$$v_r = \frac{1}{r^2\sin\theta}\frac{\partial \Psi}{\partial \theta} = \frac{1}{2}U\left(\frac{R^3}{r^3} - \frac{3R}{r} + 2\right)\cos\theta$$

$$v_\theta = -\frac{1}{r\sin\theta}\frac{\partial \Psi}{\partial r} = \frac{1}{4}U\left(+\frac{R^3}{r^3} + \frac{3R}{r} \quad 4\right)\sin\theta$$

$$\frac{\partial v_r}{\partial r} = \frac{1}{2}U\left(-\frac{3R^3}{r^4} + \frac{3R}{r^2}\right)\cos\theta = 0 \quad @ \ r = R$$

$$\frac{\partial v_r}{\partial \theta} = -\frac{1}{2}U\left(\frac{R^3}{r^3} - \frac{3R}{r} + 2\right)\sin\theta = 0 \ @ \ r = R$$

$$\frac{\partial v_\theta}{\partial r} = \frac{1}{4}U\left(\frac{3R^3}{r^4} + \frac{3R}{r^2}\right)\sin\theta = \frac{3}{2}\frac{U}{r}\sin\theta \ @ \ r = R$$

$$\sigma_{rr} = -p = -p_\infty + \frac{3}{2}\frac{\eta U}{R}\cos\theta$$

$$\sigma_{r\theta} = -\frac{3}{2}\frac{\eta U}{R}\sin\theta$$

Therefore,

$$\vec{F}_d = \int_{r=R} d^2r\, \vec{P} = -2\pi R^2 \int_0^\pi d\theta \sin\theta \left\{ -\sigma_{rr}\cos\theta + \sigma_{r\theta}\sin\theta \right\}_{r=R} \hat{z}$$

$$= -2\pi R^2 \cdot \frac{3}{2}\frac{\eta U}{R} \int_0^\pi d\theta \sin\theta \left(-\sin^2\theta - \cos^2\theta \right)$$

$$= +6\pi\eta U R\, \hat{z} \qquad \text{(Stokes, 1851)}$$

Note that $\vec{F}_d = 0$ when $\eta = 0$, in agreement with D'Alambert.

Remarks

Stokes attempted a similar treatment of viscous flow past a cylinder. The method fails in this case: no sol^{n} can be found which satisfies the boundary conditions. In this case, it is crucially important to include the effects of the inertial term $(\vec{v}\cdot\vec{\nabla})\vec{v}$, i.e. to deal with finite Re. This is a delicate matter, for as Whitehead showed in 1889, a naive expansion in Re leads to divergences, even in the case of the sphere. The problem is that L becomes larger as r/R increases, so we can't drop the $(\vec{v}\cdot\vec{\nabla})\vec{v}$ term.

The modern treatment of this problem dates to Proudman and Pearson (1957), who formulated a sol^{n} in terms of a "matched asymptotic expansion". This allows, among other things, a determination of the (finite!) first corrections to Stokes' formula:

$$F_d = 6\pi\eta U R \left(1 + \tfrac{3}{8}Re + \cdots\right)$$

Uniqueness and Reversibility of Slow Flows

The slow flow eqns are

$$\vec{\nabla} p = \eta \nabla^2 \vec{v}$$
$$\vec{\nabla} \cdot \vec{v} = 0$$

Suppose the boundary conditions on a viscous flow are

$$\vec{v}(\vec{x}) = \vec{v}_B(\vec{x}) \quad \text{on} \quad \partial\Omega$$

Now suppose another flow $\vec{v}^*(\vec{x})$ also satisfies $\vec{v}^*(\vec{x}) = \vec{v}_B(\vec{x})$ on $\partial\Omega$, with pressure p^*. Let $\delta\vec{v} = \vec{v} - \vec{v}^*$, $\delta p = p - p^*$.

Then

$$\vec{\nabla}\delta p = \eta \nabla^2 \delta\vec{v} \quad ; \quad \delta\vec{v}(\vec{x}) = 0 \quad \text{on} \quad \partial\Omega$$

We have

$$\begin{aligned} 0 &= \delta\vec{v} \cdot \left\{ -\vec{\nabla}\delta p + \eta \nabla^2 \delta\vec{v} \right\} \\ &= -\vec{\nabla}\cdot(p\,\delta\vec{v}) + \eta\, \delta\vec{v}\cdot\nabla^2\delta\vec{v} \\ &= -\vec{\nabla}\cdot(p\,\delta\vec{v}) + \eta\, \vec{\nabla}\cdot(\delta v^\alpha \vec{\nabla}\delta v^\alpha) - \eta\, \vec{\nabla}\delta v^\alpha \cdot \vec{\nabla}\delta v^\alpha \end{aligned}$$

Integrate over Ω:

$$0 = -\eta \int_\Omega dV \left(\vec{\nabla}\delta v^\alpha\right)^2 + \int_\Omega dV\, \vec{\nabla}\cdot\left(\eta\, \delta v^\alpha \vec{\nabla}\delta v^\alpha - p\,\delta\vec{v}\right)$$

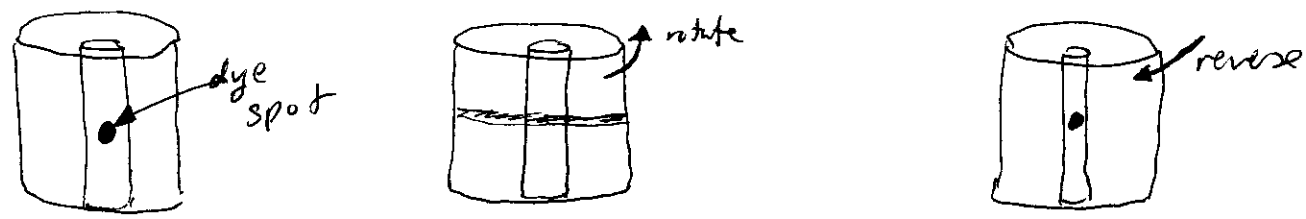
The second term integrates to zero since $\delta\vec{v}\big|_{\partial\Omega} = 0$. The first term is negative definite if $\delta\vec{v} \neq 0$ anywhere. Thus, we conclude $\delta\vec{v} = 0$, i.e. the flow is unique!

Next, consider the reversed BCs with $\vec{u}(\vec{x})|_{\partial\Omega} = -\vec{v}_B(\vec{x})$.

Clearly $-\vec{u}(\vec{x})$ is a sol$^{\underline{n}}$ to the reversed problem, and by the theorem it is unique. The pressure is $\tilde{p} = c - p$.

Thus, reversed boundary conditions lead to reversed flow.

This explains a simple experiment:



This means that swimming at low Reynolds number must proceed via temporally asymmetric sequences. A clam can't swim in maple syrup.

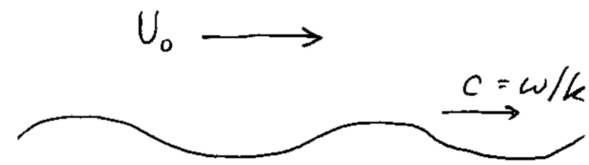
Thin Flexible Sheet

As a crude model, consider a thin flexible sheet which flexes according to

$$y(x,t) = a\sin(kx - \omega t) \quad ; \quad \omega = ck$$

I.e. $(x_s, a\sin(kx_s - \omega t))$ are the coordinates of a point on the sheet. The velocity is only vertical. This motion is temporally asymmetric: reversing time sends a right-moving wave to a left-moving wave.

We'll find that the flexing sheet induces a steady flow component to the velocity field



Thus, in the rest frame of the fluid, the sheet moves to the left.

We solve by writing

$$u_x = \frac{\partial \psi}{\partial y} \quad , \quad v_y = -\frac{\partial \psi}{\partial x} \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{v} = 0$$

$$\left.\begin{aligned} \frac{\partial p}{\partial x} &= \eta \nabla^2 u_x \\ \frac{\partial p}{\partial y} &= \eta \nabla^2 u_y \end{aligned}\right\} \Rightarrow \quad \nabla^4 \psi = (\vec{\nabla}^2)^2 \psi = 0$$

(biharmonic eqn)

Boundary conditions:

$$\frac{\partial \psi}{\partial y} = 0 \quad , \quad \frac{\partial \psi}{\partial x} = -v_y = \omega a \cos(kx - \omega t)$$

on $y = a \sin(kx - \omega t)$

Define

$$\left.\begin{aligned} x' &\equiv x - ct \\ y' &\equiv y \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial y'} \end{aligned}$$

I.e. t is just a parameter. Thus,

$$\frac{\partial \psi}{\partial y} = 0 \quad , \quad \frac{\partial \psi}{\partial x'} = \omega a \cos kx' \quad \text{on} \quad y = a \sin kx'$$

It is convenient to rescale:

$$x' \equiv kx - \omega t \quad , \quad y' \equiv ky \quad , \quad \psi' \equiv \psi / ac \quad ; \quad \varepsilon \equiv ka << 1$$

Then we have (dropping primes),

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \psi = 0$$

$$\left.\frac{\partial \psi}{\partial x}\right|_{y=\varepsilon \sin x} = \cos x \quad , \quad \left.\frac{\partial \psi}{\partial y}\right|_{y=\varepsilon \sin x} = 0$$

Now expand:

$$\left.\frac{\partial \psi}{\partial x}\right|_{y=\varepsilon \sin x} = \left.\frac{\partial \psi}{\partial x}\right|_{y=0} + \varepsilon \sin x \left.\frac{\partial^2 \psi}{\partial x \partial y}\right|_{y=0} + \frac{1}{2}\varepsilon^2 \sin^2 x \left.\frac{\partial^3 \psi}{\partial x \partial y^2}\right|_{y=0} + \ldots = \cos x$$

$$\left.\frac{\partial \psi}{\partial y}\right|_{y=\varepsilon \sin x} = \left.\frac{\partial \psi}{\partial y}\right|_{y=0} + \varepsilon \sin x \left.\frac{\partial^2 \psi}{\partial y^2}\right|_{y=0} + \frac{1}{2}\varepsilon^2 \sin^2 x \left.\frac{\partial^3 \psi}{\partial y^3}\right|_{y=0} + \ldots = 0$$

Next write

$$\psi = \psi_1 + \varepsilon \psi_2 + \varepsilon^2 \psi_3 + \ldots$$

and obtain:

$$(\nabla^2)^2 \psi_1 = 0 \quad ; \quad \left.\frac{\partial \psi_1}{\partial x}\right|_{y=0} = \cos x \quad ; \quad \left.\frac{\partial \psi_1}{\partial y}\right|_{y=0} = 0$$

$$(\nabla^2)^2 \psi_2 = 0 \quad ; \quad \left.\frac{\partial \psi_2}{\partial x}\right|_{y=0} = -\sin x \left.\frac{\partial^2 \psi_1}{\partial x \partial y}\right|_{y=0} \quad ; \quad \frac{\partial \psi_2}{\partial y} = -\sin x \left.\frac{\partial^2 \psi_1}{\partial y^2}\right|_{y=0}$$

etc. The general sol$^{\underline{n}}$ to $(\nabla^2)^2 \chi = 0$ is

$$\chi(x,y) = (B + Cx + Dy)\, e^{\lambda_1 x} e^{\lambda_2 y} + 3^{rd} \text{ order polynomials}$$

with $\lambda_1^2 + \lambda_2^2 = 0$. In our case, then, we have

$$\psi_1(x,y) = \left\{(A+By)\,e^{-y} + (C+Dy)\,e^{y}\right\}\sin x$$

must have $C = D = 0$ so $|v|_\infty < \infty$

Matching boundary conditions yields

$$\psi_1(x,y) = (1+y)\,e^{-y}\sin x$$

The next order set of eqns yield

$$\left.\frac{\partial \psi_2}{\partial x}\right|_{y=0} = -\sin x \left.\frac{\partial^2 \psi_1}{\partial x \partial y}\right|_{y=0} = 0 \quad ; \quad \left.\frac{\partial \psi_2}{\partial y}\right|_{y=0} = -\sin x \left.\frac{\partial^2 \psi_1}{\partial y^2}\right|_{y=0} = \sin^2 x$$

The appropriate sol$^{\underline{n}}$ is

$$\psi_2(x,y) = \frac{1}{2}y - \frac{1}{2}y\,e^{-2y}\cos 2x$$

Thus, restoring units,

$$\psi(x,y) = \frac{\omega a}{k}(1+ky)\,e^{-ky}\sin(kx-\omega t) + \frac{1}{2}\omega a^2\left(ky - ky\,e^{-2ky}\cos(2kx-2\omega t)\right)$$
$$+ \cdots$$

$$u_x = \frac{\partial \psi}{\partial y} = -\omega k a\, y e^{-ky}\sin(kx-\omega t) + \frac{1}{2}\omega k a^2\left(1 + (2ky-1)\,e^{-2ky}\cos(2kx-2\omega t)\right)$$

$$u_y = -\frac{\partial \psi}{\partial x} = \omega a\,(1+ky)\,e^{-ky}\cos(kx-\omega t) + \omega k^2 a^2\, y e^{-2ky}\sin(2kx-2\omega t)$$

Note that

$$\langle \vec{u} \rangle = \frac{1}{2}c\,(ka)^2\,\hat{x}$$

Thin Films

Consider steady flow between two rigid boundaries at $z=0$ and $z=h(x,y)$. Suppose that $h \ll L$, where L is a typical horizontal length scale for the flow. If U is the typical horizontal flow speed, then $\frac{\partial v}{\partial z} \sim \frac{U}{h}$, $\frac{\partial^2 v}{\partial z^2} \sim \frac{U}{h^2}$, both of which are larger than $\frac{\partial v}{\partial x,y} \sim \frac{U}{L}$ and $\frac{\partial^2 v}{\partial x^2, y^2} \sim \frac{U}{L^2}$. Thus,

$$\nu \nabla^2 \vec{v} \approx \nu \frac{\partial^2 \vec{v}}{\partial z^2}$$

Now

$$(\vec{v}\cdot\vec{\nabla})\vec{v} \sim \frac{U^2}{L}\left(1, 1, \frac{h}{L}\right)$$

$$\nu \nabla^2 \vec{v} \sim \frac{\nu U}{h^2}\left(1, 1, \frac{h}{L}\right)$$

with $\vec{\nabla}\cdot\vec{v} = 0$ implying

$$\frac{\partial v_z}{\partial z} \sim \frac{U}{L} \Rightarrow v_z \sim \frac{Uh}{L}$$

Thus, provided

$$\frac{UL}{\nu}\cdot\left(\frac{h}{L}\right)^2 \ll 1$$

we may neglect the inertial term $(\vec{v}\cdot\vec{\nabla})\vec{v}$. Thus, Re needn't necessarily be small -- only $Re(h/L)^2$.

We then obtain

$$\vec{\nabla} p = \eta \frac{\partial^2 \vec{v}}{\partial z^2} \quad , \quad \vec{\nabla} \cdot \vec{v} = 0$$

Since $v_z \sim \frac{h}{L} v_{x,y}$, we have $\frac{\partial p}{\partial z} \sim \frac{h}{L} \frac{\partial p}{\partial x_{,y}}$, so to lowest order we may assume $p = p(x,y)$, in which case

$$\frac{\partial^2 v_x}{\partial z^2} = \frac{1}{\eta} \frac{\partial p}{\partial x} \Rightarrow v_x(x,y,z) = A + Bz + \frac{1}{2\eta} \frac{\partial p}{\partial x} z^2$$

$$\frac{\partial^2 v_y}{\partial z^2} = \frac{1}{\eta} \frac{\partial p}{\partial y} \Rightarrow v_y(x,y,z) = C + Dz + \frac{1}{2\eta} \frac{\partial p}{\partial y} z^2$$

with A, B, C, D and $\frac{\partial p}{\partial x}$, $\frac{\partial p}{\partial y}$ all functions of x and y.

Another important result is

$$\sigma_{ij} = -\overset{\mathcal{O}(\eta U L / h^2)}{p \delta_{ij}} + \overset{\mathcal{O}(\eta U / h)}{\eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)}$$

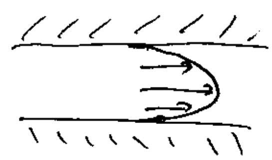
$$\simeq -p \delta_{ij}$$

Hele-Shaw Cells

Suppose $h(x,y) = h$ = constant. No-slip at $z = 0$, $z = h$ means

$$v_x = -\frac{1}{2\eta} \frac{\partial p}{\partial x} z(h-z)$$

$$v_y = -\frac{1}{2\eta} \frac{\partial p}{\partial y} z(h-z)$$



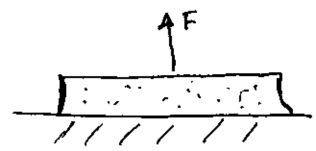
Eliminating p gives $\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} = 0$, i.e. the 2D flow is irrotational.

Furthermore, the circulation around *any* closed curve must vanish:

$$\Gamma = \oint_{\partial\Sigma} \vec{v}\cdot d\vec{\ell}$$
$$= -\frac{1}{2\eta} z(h-z) \oint_{\partial\Sigma} dp = 0$$

since p is single-valued.

<u>Adhesion</u>



Consider changing thickness $h(t)$, but sufficiently slowly that $\frac{\partial\vec{v}}{\partial t} \simeq 0$ is OK. The BCs do change, though.

We assume radial symmetry, in which case

$$\frac{\partial p}{\partial r} = \eta \frac{\partial^2 v_r}{\partial z^2}$$

$$\Rightarrow \quad v_r = \frac{1}{2\eta}\frac{\partial p}{\partial r} z\cdot(z-h)$$

Now

$$\vec{\nabla}\cdot\vec{v} = 0 \quad \Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial v_z}{\partial z} = 0$$

$$\Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)\cdot\frac{1}{2\eta} z(z-h) + \frac{\partial v_z}{\partial z} = 0$$

Thus,

$$v_z = -\frac{1}{2\eta r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)\cdot\left(\frac{1}{3}z^3 - \frac{1}{2}hz^2\right)$$

But $v_z = \dot{h}$ at $z = h(t)$, so

$$\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) = \frac{12\eta r}{h^3}\frac{dh}{dt}$$

Integrating,

$$\frac{\partial p}{\partial r} = \frac{6\eta r}{h^3}\frac{dh}{dt} + \frac{C(t)}{r}$$ (need $C(t)=0$ for no $r=0$ singularity)

Thus,

$$p(r) = \frac{3\eta}{h^3} r^2 \frac{dh}{dt} + D(t)$$

But at $r=a$ we must have $p = p_0$ (atmospheric pressure).

Thus,

$$p - p_0 = \frac{3\eta}{h^3}\frac{dh}{dt}(r^2 - a^2)$$

$$F = \int_0^a dr\, r \int_0^{2\pi} d\phi\, (p - p_0) = \frac{3\pi}{2}\frac{\eta a^2}{h^3}\frac{dh}{dt}$$

which is big if h is small!

Valid when $h \ll a$, $h\frac{dh}{dt} \ll \nu$.

Waves in Fluids

Consider first a bulk inviscid fluid, satisfying

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla} p + \vec{g}$$

We linearize, writing $p = p_0(\vec{x}) + \delta p$, $\rho = \rho_0 + \delta\rho$, so

$$\frac{\partial \vec{v}}{\partial t} = -\frac{1}{\rho_0}\vec{\nabla} p_0 - \frac{1}{\rho_0}\vec{\nabla}\delta p + \vec{g}$$

$$= -\frac{1}{\rho_0}\vec{\nabla}\delta p$$

provided $\vec{\nabla} p_0 = \rho_0 g$. We also have $\partial_t \rho + \vec{\nabla}\cdot(\rho\vec{v}) = 0$, i.e.

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \vec{\nabla}\cdot\vec{v} = 0$$

Our first step is to use local thermodynamics to relate δp to $\delta\rho$:

$$\delta p = \left.\frac{\partial p}{\partial \rho}\right|_s \delta\rho + \left.\frac{\partial p}{\partial s}\right|_\rho \delta s$$

where s is the specific entropy. We assume $\delta s = 0$, so

$$\delta p = c^2 \delta\rho$$

$$c^2 = \left.\frac{\partial p}{\partial \rho}\right|_s$$

Thus,

$$① \quad \frac{\partial \vec{v}}{\partial t} + \frac{c^2}{\rho_0}\vec{\nabla}\delta\rho = 0$$

$$② \quad \frac{\partial \delta\rho}{\partial t} + \rho_0 \vec{\nabla}\cdot\vec{v} = 0$$

Taking the divergence of (1) and the time derivative of (2) yields

$$\frac{c^2}{\rho_0} \nabla^2 \delta\rho = -\frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{v} = \frac{1}{\rho_0} \frac{\partial^2 \delta\rho}{\partial t^2}$$

$$\vec{\nabla}^2 \delta\rho = \frac{1}{c^2} \frac{\partial^2 \delta\rho}{\partial t^2}$$

which is the Helmholtz eqn. We could also have taken the time derivative of (1) and the gradient of (2), yielding

$$\frac{\partial^2 \vec{v}}{\partial t^2} = -\frac{c^2}{\rho_0} \frac{\partial}{\partial t} \vec{\nabla} \delta\rho = c^2 \vec{\nabla}(\vec{\nabla} \cdot \vec{v})$$
$$= c^2 (\vec{\nabla} \times \vec{\omega} + \nabla^2 \vec{v})$$

Thus,

$$\nabla^2 \vec{v} + \vec{\nabla} \times \vec{\omega} = \frac{1}{c^2} \frac{\partial^2 \vec{v}}{\partial t^2}$$

Note that $\vec{\nabla} \times$ (1) gives $\partial \vec{\omega}/\partial t = 0$. If $\vec{\omega} = 0$, or $\vec{\omega} = \text{const.}$ independent of space, then $\nabla^2 \vec{v} = \frac{1}{c^2} \frac{\partial^2 \vec{v}}{\partial t^2}$. For irrotational flow, $\vec{v} = \vec{\nabla}\phi$, with

$$\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}$$

Boundary conditions:

$$\vec{\nabla}\phi \cdot \hat{n}\Big|_{\Sigma} = \vec{V} \cdot \hat{n}\Big|_{\Sigma} \qquad \vec{V} = \text{velocity of surface } \Sigma$$

i.e. $\hat{n} \cdot (\vec{v} - \vec{V})\Big|_{\Sigma} = 0$

Recall also the result

$$\vec{\nabla}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2} v^2 + h + \underset{\text{extenal potential}}{\chi}\right) = 0$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial t} + \frac{1}{2}(\vec{\nabla}\phi)^2 + h + \chi = 0 \qquad (\text{Setting zero of } \chi)$$

To lowest order, then,

$$\frac{\partial \phi}{\partial t} + \frac{\partial h}{\partial \rho}\bigg|_s \delta\rho = 0$$

$$dh = T\,ds + \frac{1}{\rho}\,dp \quad \Rightarrow \quad \frac{\partial h}{\partial \rho}\bigg|_s = \frac{1}{\rho}\left(\frac{\partial p}{\partial \rho}\right)_s = \frac{c^2}{\rho_0} + \mathcal{O}(\delta\rho)$$

and we have

$$\delta\rho = -\frac{\rho_0}{c^2}\frac{\partial \phi}{\partial t}$$

(Any constant term in the original eqn can be gauged away.)

Thus, on <u>free surfaces</u>, where $\delta p = 0$, we have $\dot{\phi}\big|_{\substack{\text{free}\\ \text{surface}}} = 0$.

<u>Waves in Cavities</u>

We solve $\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$ subject to boundary conditions $\vec{\nabla}\phi\cdot\hat{n}\big|_{\partial\Omega} = 0$.

This is standard electrodynamics fare. Write $\phi(\vec{x},t) = \mathrm{Re}\,\phi(\vec{x})\,e^{-i\omega t}$, with $\nabla^2\phi + k^2\phi = 0 \quad ; \quad k = \frac{\omega}{c}$. **Say no more.**

<u>Surface Waves</u>

Let $z = \zeta(x,y,t)$ be the eqn of a free surface, displaced from equilibrium $z = 0$. We assume $\vec{\nabla}\cdot\vec{v} = 0$, $\vec{\nabla}\times\vec{v} = 0$, so we have $\vec{v} = \vec{\nabla}\phi$ and $\nabla^2\phi = 0$. We now invoke Newton's second law,

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p + \vec{g}$$

which leads to Bernoulli's eqn,

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{p}{\rho} + \chi = 0$$

with $\vec{g} = -\vec{\nabla}\chi$.

At the free surface, $p = p_0$ (atmospheric pressure). We also neglect the inertial term $(\vec{v}\cdot\vec{\nabla})\vec{v}$ which is second order in $\vec{v}$. We write $\vec{g} = -g\hat{z} \Rightarrow \chi = gz$. Then at the free surface,

$$\frac{\partial\phi}{\partial t} + g\zeta + \frac{p_0}{\rho} = 0 \qquad \text{(set } z=0 \text{ at surface equilibrium)}$$

with $\zeta \equiv z - h_0$. Since ρ is constant, we may write

$$\phi = \tilde{\phi} - \frac{p_0 t}{\rho}$$

and transform away the pressure term. Henceforth we drop the tilde and write simply

$$\frac{\partial\phi}{\partial t} + g\zeta = 0 \qquad \text{on } z = \zeta(x,y,t)$$

Thus,

$$\zeta = -\frac{1}{g}\left.\frac{\partial\phi}{\partial t}\right|_{\zeta}$$

$$= -\frac{1}{g}\dot{\phi}(x,y,0,t) - \frac{1}{g}\zeta\,\partial_z\dot{\phi}(x,y,0,t) + \cdots$$

To lowest order, then,

$$\zeta = \frac{1}{g}\left.\frac{\partial\phi}{\partial t}\right|_{z=0}$$

Next, consider a fluid particle on the surface. Clearly

$$z = \zeta(x, y, t)$$

$$z + dz = \zeta + \frac{\partial \zeta}{\partial x} dx + \frac{\partial \zeta}{\partial y} dy + \frac{\partial \zeta}{\partial t} dt$$

$$dz = \frac{\partial \zeta}{\partial x} dx + \frac{\partial \zeta}{\partial y} dy + \frac{\partial \zeta}{\partial t} dt$$

$$\Rightarrow \quad v_z = \frac{\partial \zeta}{\partial x} v_x + \frac{\partial \zeta}{\partial y} v_y + \frac{\partial \zeta}{\partial t} \qquad \text{on} \quad z = \zeta(x, y, t)$$

For small amplitude motion,

$$v_z = \frac{\partial \zeta}{\partial t} = \frac{\partial \phi}{\partial z} \qquad \text{at free surface}$$

$$\hat{n} \cdot \vec{\nabla} \phi = 0 \qquad \text{at fixed surface}$$

<u>Dispersion of Surface Waves</u>

Consider a one-dimensional surface wave of the form

$$\phi(x, z, t) = Z(z) \cos(kx - \omega t)$$

Laplace's equation becomes

$$\nabla^2 \phi = Z''(z) \cos(kx - \omega t) - k^2 Z \cos(kx - \omega t) = 0$$

$$\Rightarrow \quad \frac{d^2 Z}{dz^2} - k^2 Z = 0$$

We have then

$$Z(z) = A e^{kz} + B e^{-kz}$$

Assuming a flat bottom at $z = -h$,

$$v_z \big|_{z=-h} = 0 \quad \Rightarrow \quad Z'(-h) = 0$$

Thus,

$$Z(z) = A \cosh k(z+h)$$

We now must satisfy conditions at the free surface:

$$\left.\begin{aligned} \zeta &= -\frac{1}{g}\frac{\partial \phi}{\partial t} \\ \frac{\partial \phi}{\partial z} &= \frac{\partial \zeta}{\partial t} \end{aligned}\right\} \Rightarrow \quad \frac{\partial \phi}{\partial z} = -\frac{1}{g}\left.\frac{\partial^2 \phi}{\partial t^2}\right|_{z=0}$$

Thus,

$$\phi = A \cosh k(z+h) \cos(kx - \omega t)$$

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=0} = +kA \sinh kh \cos(kx-\omega t)$$

$$\frac{1}{g}\left.\frac{\partial^2 \phi}{\partial t^2}\right|_{z=0} = -\frac{1}{g}\omega^2 A \cosh kh \cos(kx-\omega t)$$

and we obtain

$$\omega^2 = gk \tanh(kh)$$

Two useful limits:

- deep channel ($\lambda \ll h$)

$$\omega^2 \approx gk \Rightarrow \omega = \pm\sqrt{gk}$$

- shallow channel ($\lambda \gg h$)

$$\omega^2 \approx ghk^2 \rightarrow \omega = \pm \underbrace{\sqrt{gh}}_{c}\, k$$

If we let ζ_0 = maximum displacement of surface,

$$\phi(x,y,z,t) = \frac{g\zeta_0}{\omega}\frac{\cosh k(z+h)}{\cosh kh}\cos(kx-\omega t)$$

$$\omega = \pm\sqrt{gk \tanh(kh)}$$

Surface Tension

In addition to gravity, there is another force acting on the free surface of a fluid : surface tension. This modifies the boundary conditions :

p_b

p_a ζ

$$p_b = p_a + \tau \nabla^2 \zeta$$

$$\left.\begin{aligned} &\frac{\partial \phi}{\partial t} + \frac{p_a}{\rho} + g\zeta = 0 \\ &p_a = p_0 - \tau \nabla^2 \zeta \end{aligned}\right\} \Rightarrow \quad \frac{\partial \phi}{\partial t} + \frac{p_0}{\rho} - \frac{\tau}{\rho}\nabla^2\zeta + g\zeta = 0$$

Send $\phi \to \phi - \frac{p_0}{\rho} t$ again, whence

$$\frac{\partial \phi}{\partial t} + g\zeta - \frac{\tau}{\rho}\nabla^2 \zeta = 0 \qquad \text{on} \quad z = \zeta(x,y,t)$$

Now let's redo our calculation for the surface waves. We still have

$$\phi(x,y,z,t) = A \cosh k(z+h) \cos(kx - \omega t)$$

but with

$$\left.\begin{aligned} \frac{\partial \phi}{\partial t} &= \frac{\tau}{\rho}\nabla^2\zeta - g\zeta = -\left(g + \frac{\tau k^2}{\rho}\right)\zeta \\ \frac{\partial \phi}{\partial z} &= \frac{\partial \zeta}{\partial t} = -\frac{1}{g + \frac{\tau k^2}{\rho}} \frac{\partial^2 \phi}{\partial t^2} \end{aligned}\right\} \quad \text{at } z = \zeta$$

so that

$$kA \sinh(kh) = \frac{\omega^2}{g + \frac{\tau k^2}{\rho}} A \cosh(kh)$$

$$\omega^2 = gk\left(1 + \frac{\tau k^2}{\rho g}\right)\tanh(kh)$$

If $\lambda \ll h$ (deep channel), then

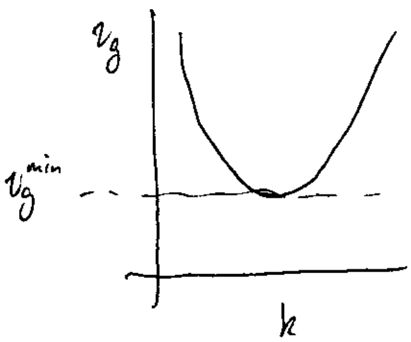
$$\omega^2 = gk + \frac{\tau k^3}{\rho}$$

$$\omega = \left(gk + \frac{\tau k^3}{\rho}\right)^{1/2}$$

$$v_g = \frac{\partial \omega}{\partial k} = \frac{1}{2}\left(g + \frac{3\tau k^2}{\rho}\right) \Big/ \left(gk + \frac{\tau k^3}{\rho}\right)^{1/2} \sim \begin{cases} \frac{1}{2}\sqrt{g/k}\,, & k \to 0 \\ \frac{3}{2}\sqrt{\frac{\tau k}{\rho}}\,, & k \to \infty \end{cases}$$

We can define $k \equiv \left(\frac{g\rho}{\tau}\right)^{1/2} s$, in which case

$$\omega = \left(\frac{g^3\rho}{\tau}\right)^{1/4} (s+s^3)^{1/2}$$



$$v_g = \frac{1}{2}\left(\frac{g\tau}{\rho}\right)^{1/4} \frac{1+3s^2}{(s+s^3)^{1/2}}$$

$$\frac{\partial v_g}{\partial s} = \frac{1}{2}\left(\frac{g\tau}{\rho}\right)^{1/4} \cdot \left\{ \frac{6s(s+s^3)^{1/2} - (1+3s^2)\cdot\frac{1}{2}(1+3s^2)(s+s^3)^{-1/2}}{s+s^3} \right\}$$

$$= \frac{1}{4}\left(\frac{g\tau}{\rho}\right)^{1/4} \frac{12s(s+s^3) - (1+3s^2)^2}{(s+s^3)^{3/2}}$$

So

$$\frac{\partial v_g}{\partial s} = 0 \quad \Rightarrow \quad 12s^2 + 12s^4 - 1 - 6s^2 - 9s^4 = 0$$

$$3s^4 + 6s^2 - 1 = 0$$

$$s^2 = \frac{-6 + \sqrt{36+12}}{6} = -1 + \frac{2}{\sqrt{3}}$$

Thus,

$$v_g^{min} \simeq 1.086 \left(\frac{g\tau}{\rho}\right)^{1/4}$$

No signal propagates more slowly than v_g^{min}. For water,

$$g = 980 \text{ cm/s}^2, \quad \tau = 74 \text{ erg/cm}^2, \quad \rho = 1 \text{ g/cm}^3$$

$$\Rightarrow \begin{cases} v_g^{min} = 17.8 \text{ cm/sec} \\ \lambda_{min} = 4.39 \text{ cm} \end{cases}$$

Sound Waves in Viscous Fluids

Start with

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}p + \eta\nabla^2\vec{v} + (\zeta + \tfrac{1}{3}\eta)\vec{\nabla}(\vec{\nabla}\cdot\vec{v})$$

We also must consider thermal balance, since viscous friction leads to internal dissipation of energy. The equation of local entropy balance is

$$\rho T\left(\frac{\partial s}{\partial t} + \vec{v}\cdot\vec{\nabla}s\right) = -\vec{\nabla}\cdot\vec{J}_{th} + \rho\dot{q}_{ex} + \frac{1}{2}\eta\sum_{i,j}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\vec{\nabla}\cdot\vec{v}\right)^2 + \zeta(\vec{\nabla}\cdot\vec{v})^2$$

and is derived in chapter 12 of FW. Here, s is the local specific entropy (i.e. entropy per unit mass), $\dot{q}_{ex}$ is due to external sources/sinks of heat, and

$$\vec{J}_{th} = -k_{th}\vec{\nabla}T$$

is the heat current; k_{th} is the <u>thermal</u> <u>conductivity</u>.

We also have continuity,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$

OK, now we linearize:

$$T = T_0 + \delta T \qquad s = s_0 + \delta s \qquad \vec{v} = 0 + \delta\vec{v}$$

$$\rho = \rho_0 + \delta\rho \qquad p = p_0 + \delta p$$

This yields

① $\frac{\partial}{\partial t}\delta\rho + \rho_0\vec{\nabla}\cdot\delta\vec{v} = 0$

② $\frac{\partial \delta\vec{v}}{\partial t} = -\frac{1}{\rho_0}\vec{\nabla}\delta p + \frac{\eta}{\rho_0}\nabla^2\delta\vec{v} + \frac{1}{\rho_0}(\zeta + \tfrac{1}{3}\eta)\vec{\nabla}(\vec{\nabla}\cdot\delta\vec{v})$

③ $T_0\rho_0\frac{\partial \delta s}{\partial t} = k_{th}\nabla^2 T$

We now appeal to thermodynamics on the local scale, and write

$$\delta p = \left(\frac{\partial p}{\partial s}\right)_\rho \delta s + \left(\frac{\partial p}{\partial \rho}\right)_s \delta \rho$$

$$\delta T = \left(\frac{\partial T}{\partial s}\right)_\rho \delta s + \left(\frac{\partial T}{\partial \rho}\right)_s \delta \rho$$

Now

$$d\varepsilon = T\,ds + \frac{p}{\rho^2}\,d\rho = T\,ds - p\,d\rho^{-1}$$

$$\Rightarrow \quad -\left.\frac{\partial p}{\partial s}\right)_\rho = \left.\frac{\partial T}{\partial \rho^{-1}}\right)_s = -\rho^2 \left.\frac{\partial T}{\partial \rho}\right)_s$$

So

$$\left(\frac{\partial T}{\partial \rho}\right)_s = \frac{1}{\rho^2}\left(\frac{\partial p}{\partial s}\right)_\rho$$

Thus,

① $$\frac{\partial \delta \rho}{\partial t} + \rho_0 \vec{\nabla} \cdot \delta \vec{v} = 0$$

② $$\frac{\partial \delta \vec{v}}{\partial t} + \frac{c^2}{\rho_0} \vec{\nabla} \delta \rho = -\rho_0 \left(\frac{\partial T}{\partial \rho}\right)_s \vec{\nabla} \delta s + \frac{\eta}{\rho_0} \nabla^2 \delta \vec{v} + \frac{1}{\rho_0}\left(\zeta + \tfrac{1}{3}\eta\right) \vec{\nabla}(\vec{\nabla} \cdot \delta \vec{v})$$

③ $$\frac{\partial \delta s}{\partial t} - \gamma \kappa \nabla^2 \delta s = \frac{c_p \kappa}{T_0}\left(\frac{\partial T}{\partial \rho}\right)_s \nabla^2 \delta \rho$$

where

$$\gamma \equiv \frac{c_p}{c_v}, \qquad \kappa \equiv \frac{k_{th}}{\rho c_p} = \text{thermal diffusivity}$$

$$\left(\frac{\partial T}{\partial t} = \kappa \nabla^2 T + \frac{\dot{q}}{c_p} \quad \text{in solids}\right)$$

We seek a wave sol$^{\underline{n}}$ to these eqns,

$$\begin{pmatrix} \delta\rho \\ \delta\vec{v} \\ \delta s \end{pmatrix} = \mathrm{Re} \begin{pmatrix} \delta\rho \\ \delta\vec{v} \\ \delta s \end{pmatrix} e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

This gives

$$\left.\begin{aligned}
&① \quad \omega\,\delta\rho - \rho_0\,\vec{k}\cdot\delta\vec{v} = 0 \\
&② \quad \omega\,\delta\vec{v} - \frac{c^2\vec{k}}{\rho_0}\,\delta\rho = \rho_0\left(\frac{\partial T}{\partial\rho}\right)_s \vec{k}\,\delta s - \frac{i\eta k^2}{\rho_0}\,\delta\vec{v} \\
&\qquad\qquad - \frac{i}{\rho_0}\left(\zeta+\tfrac{1}{3}\eta\right)\vec{k}\,(\vec{k}\cdot\delta\vec{v}) \\
&③ \quad (\omega + i\gamma\kappa k^2)\,\delta s = -\frac{i c_p \kappa}{T_0}\left(\frac{\partial T}{\partial\rho}\right)_s k^2\,\delta\rho
\end{aligned}\right\} \text{5 modes}$$

There are two types of solutions: transverse and longitudinal modes. Transverse modes have $\vec{k}\cdot\delta\vec{v} = 0$, and longitudinal modes have $\vec{k}\times\delta\vec{v} = 0$. Taking the dot product of ② with $\vec{k}$ yields

$$0 = \rho_0\left(\frac{\partial T}{\partial\rho}\right)_s k^2\,\delta s$$

From ① we have $\delta\rho = 0$. Thus, only $\delta\vec{v}_\perp \neq 0$, and we conclude from ② that

$$\omega = -i\nu k^2 \qquad \text{(2 modes)}$$

with $\nu = \eta/\rho_0$. Thus, the mode is damped, with

$$e^{i(\vec{k}\cdot\vec{x}-\omega t)} = e^{i\vec{k}\cdot\vec{x}}\,e^{-\nu k^2 t}$$

The longitudinal mode is more interesting. Taking the dot product of ② with k, and eliminating $k \cdot \delta\vec{v}$, we find

$$\left.\begin{aligned} \left[\omega^2 - c^2k^2 + \frac{i\omega k^2}{\rho_0}\left(\frac{4}{3}\eta + \zeta\right)\right]\delta\rho - \rho_0 k^2 \left(\frac{\partial T}{\partial \rho}\right)_s \delta s &= 0 \\ \frac{i\kappa k^2 c_p}{T_0}\left(\frac{\partial T}{\partial \rho}\right)_s \delta\rho + (\omega + i\gamma\kappa k^2)\delta s &= 0 \end{aligned}\right\} \text{3 modes}$$

For a nontrivial sol$\underline{n}$, the determinant of the coefficients must vanish:

$$(\omega + i\gamma\kappa k^2)\left(\omega^2 - c^2k^2 + i\frac{\omega k^2}{\rho_0}\left(\frac{4}{3}\eta + \zeta\right)\right) + \frac{i\kappa c_p \rho_0}{T_0}\left(\frac{\partial T}{\partial \rho}\right)_s^2 (k^2)^2 = 0$$

Clearly if $\kappa = 0$ and $\eta = \zeta = 0$, we have $\omega = ck$, $\omega = -ck$.

Let's write

$$\omega = ck - i\beta$$

Find

$$\beta = + \frac{\omega^2}{2\rho_0 c^2}\left(\frac{4}{3}\eta + \zeta\right) + \frac{\rho_0^2 \kappa c_p \omega^2}{2T_0 c^4}\left(\frac{\partial T}{\partial \rho}\right)_s^2 > 0$$

So longitudinal modes also are damped, with $\beta \propto \omega^2$.

From thermodynamics, we may write

$$\beta = \frac{\omega^2}{2\rho_0 c^2}\left\{\frac{4}{3}\eta + \zeta + k_{th}\left(\frac{1}{c_v} - \frac{1}{c_p}\right)\right\} > 0$$