Chapter 2

Lagrangian Mechanics

2.1 The Equations of Mechanics

The motion of a mechanical system is described by a set of functions, the *generalized coordinates*, $\{q_\sigma(t)\}$. I will often write q for the entire set $\{q_1, q_2, \ldots, q_n\}$, where n is the number of generalized coordinates.

Of course we are not given the motion $q(t)$ directly. Rather, it is encoded in a set of ordinary differential equations, known as the *equations of motion*. Consider a mechanical system such that $q(t_a) = q_a$ and $q(t_b) = q_b$ (see Fig. 2.1). The equations of motion select a particular path, $q^*(t)$, from the infinite-dimensional space of all possible paths $q(t)$ connecting these endpoints. The equations of motion for a given system follow from *Hamilton's Principle*, economically expressed as

$$\delta S = 0 \ . \tag{2.1}$$

Here, $S[q(t)]$ is the *action functional*, which maps paths $q(t)$ to real numbers. Hamilton's principle, in words, says that the motion of a mechanical system corresponds to an *extremum* of the action functional.

The action functional is a time integral of the Lagrangian function $L(q, \dot{q}, t)$:

$$S[q(t)] = \int_{t_a}^{t_b} dt \, L(q, \dot{q}, t) \ . \tag{2.2}$$

We shall discuss the properties of L presently, but first let us digress and discuss some properties of functionals.

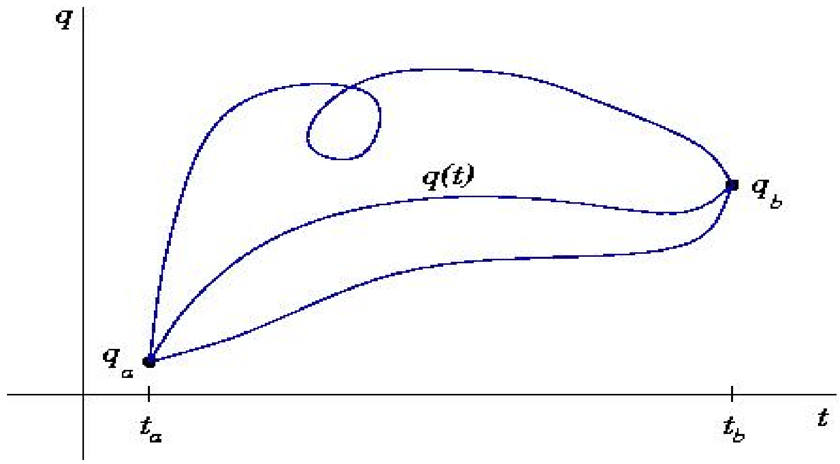

Figure 2.1: Examples of paths $q(t)$ connecting $q(t_a) = q_a$ to $q(t_b) = q_b$.

2.1.1 Digression on Functionals

You all know that a function f is an animal which gets fed a real number x and excretes a real number $f(x)$. We say f maps the reals to the reals, or

$$f\colon \mathbf{R} \to \mathbf{R} \tag{2.3}$$

Of course we also have functions $g\colon \mathbf{C} \to \mathbf{C}$ which eat and excrete complex numbers, multivariable functions $h\colon \mathbf{R}^N \to \mathbf{R}$ which eat N-tuples of numbers and excrete a single number, *etc.*

A *functional* $F[f(x)]$ eats entire functions (!) and excretes numbers. That is,

$$F\colon \Big\{ f(x) \;\Big|\; x \in \mathbf{R} \Big\} \to \mathbf{R} \tag{2.4}$$

This says that F operates on the set of real-valued functions of a single real variable, yielding a real number. Some examples:

$$F[f(x)] = \tfrac{1}{2}\int\limits_{-\infty}^{\infty}\!dx\,\big[f(x)\big]^2 \tag{2.5}$$

$$F[f(x)] = \tfrac{1}{2}\int\limits_{-\infty}^{\infty}\!dx\int\limits_{-\infty}^{\infty}\!dx'\,K(x,x')\,f(x)\,f(x') \tag{2.6}$$

$$F[f(x)] = \int\limits_{-\infty}^{\infty}\!dx\,\bigg\{\tfrac{1}{2}A\,f^2(x) + \tfrac{1}{2}B\left(\frac{df}{dx}\right)^2\bigg\}\ . \tag{2.7}$$

In classical mechanics, the action S is a functional of the path $q(t)$:

$$S[q(t)] = \int\limits_{t_\mathrm{a}}^{t_\mathrm{b}}\!dt\,\Big\{\tfrac{1}{2}m\dot{q}^2 - U(q)\Big\}\ . \tag{2.8}$$

We can also have functionals which feed on functions of more than one independent variable, such as

$$S[y(x,t)] = \int\limits_{t_\mathrm{a}}^{t_\mathrm{b}} dt \int\limits_{x_\mathrm{a}}^{x_\mathrm{b}} dx \left\{ \tfrac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 - \tfrac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2 \right\} , \tag{2.9}$$

which just happens to be the functional for a string of mass density μ under uniform tension τ. Another example comes from electrodynamics:

$$S[A^\mu(\boldsymbol{x},t)] = -\int d^3x \int dt \left\{ \frac{1}{16\pi} F_{\mu\nu}\, F^{\mu\nu} + J_\mu\, A^\mu \right\} , \tag{2.10}$$

which is a functional of the four fields $\{A^0, A^1, A^2, A^3\}$, where $A^0 = c\phi$. These are the components of the 4-potential, each of which is itself a function of four independent variables (x^0, x^1, x^2, x^3), with $x^0 = ct$. The field strength tensor is written in terms of derivatives of the A^μ: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where we use a metric $g_{\mu\nu} = \mathrm{diag}(+,-,-,-)$ to raise and lower indices. The 4-potential couples linearly to the source term J_μ, which is the electric 4-current $(c\rho, \boldsymbol{J})$.

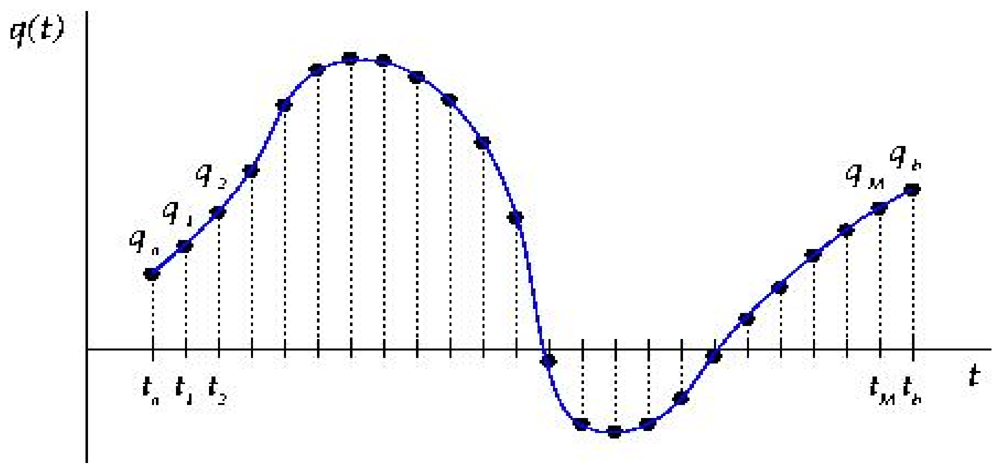

Figure 2.2: A functional $S[q(t)]$ is the continuum limit of a function of a large number of variables, $S(q_1, \ldots, q_M)$.

We extremize functions by sending the independent variable x to $x + dx$ and demanding that the variation $df = 0$ to first order in dx. That is,

$$f(x + dx) = f(x) + f'(x)\, dx + \tfrac{1}{2} f''(x)(dx)^2 + \ldots\ , \tag{2.11}$$

whence $df = f'(x)\, dx + \mathcal{O}\big((dx)^2\big)$ and thus

$$f'(x^*) = 0 \quad \Longleftrightarrow \quad x^* \text{ an extremum.} \tag{2.12}$$

We extremize *functionals* by sending

$$f(x) \to f(x) + \delta f(x) \tag{2.13}$$

and demanding that the variation δF in the functional $F[f(x)]$ vanish to first order in $\delta f(x)$. The variation $\delta f(x)$ must sometimes satisfy certain boundary conditions. For example, if $F[f(x)]$ only operates on functions which vanish at a pair of endpoints, *i.e.* $f(x_a) = f(x_b) = 0$, then when we extremize the functional F we must do so *within the space of allowed functions.* Thus, we would in this case require $\delta f(x_a) = \delta f(x_b) = 0$. We may expand the functional $F[f + \delta f]$ in a *functional Taylor series*,

$$\begin{aligned} F[f+\delta f] = F[f] + \int\! dx_1\, K_1(x_1)\,\delta f(x_1) + \tfrac{1}{2!}\int\! dx_1\!\int\! dx_2\, K_2(x_1,x_2)\,\delta f(x_1)\,\delta f(x_2) \\ + \tfrac{1}{3!}\int\! dx_1\!\int\! dx_2\!\int\! dx_3\, K_3(x_1,x_2,x_3)\,\delta f(x_1)\,\delta f(x_2)\,\delta f(x_3) + \ldots \end{aligned} \tag{2.14}$$

and we write

$$K_n(x_1,\ldots,x_n) \equiv \frac{\delta^n F}{\delta f(x_1)\cdots\delta f(x_n)}\ . \tag{2.15}$$

In a more general case, $F = F[\{f_i(\boldsymbol{x})\}$ is a functional of several functions, each of which is a function of several independent variables.[1] We then write

$$\begin{aligned} F[\{f_i+\delta f_i\}] = F[\{f_i\}] &+ \int\! d\boldsymbol{x}_1\, K_1^i(\boldsymbol{x}_1)\,\delta f_i(\boldsymbol{x}_1) \\ &+ \tfrac{1}{2!}\int\! d\boldsymbol{x}_1\!\int\! d\boldsymbol{x}_2\, K_2^{ij}(\boldsymbol{x}_1,\boldsymbol{x}_2)\,\delta f_i(\boldsymbol{x}_1)\,\delta f_j(\boldsymbol{x}_2) \\ &+ \tfrac{1}{3!}\int\! d\boldsymbol{x}_1\!\int\! d\boldsymbol{x}_2\!\int\! dx_3\ K_3^{ijk}(\boldsymbol{x}_1,\boldsymbol{x}_2,x_3)\,\delta f_i(\boldsymbol{x}_1)\,\delta f_j(\boldsymbol{x}_2)\,\delta f_k(\boldsymbol{x}_3) + \ldots\ , \end{aligned} \tag{2.16}$$

with

$$K_n^{i_1 i_2 \cdots i_n}(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_n) = \frac{\delta^n F}{\delta f_{i_1}(\boldsymbol{x}_1)\,\delta f_{i_2}(\boldsymbol{x}_2)\,\delta f_{i_n}(\boldsymbol{x}_n)}\ . \tag{2.17}$$

Another way to compute functional derivatives is to send

$$f(x) \to f(x) + \epsilon_1\,\delta(x-x_1) + \ldots + \epsilon_n\,\delta(x-x_n) \tag{2.18}$$

and then differentiate n times with respect to ϵ_1 through ϵ_n. That is,

$$\frac{\delta^n F}{\delta f(x_1)\cdots\delta f(x_n)} = \frac{\partial^n}{\partial\epsilon_1\cdots\partial\epsilon_n}\Bigg|_{\epsilon_1=\epsilon_2=\cdots\epsilon_n=0} F\big[f(x) + \epsilon_1\,\delta(x-x_1) + \ldots + \epsilon_n\,\delta(x-x_n)\big]\ . \tag{2.19}$$

Let's see how this works. As an example, we'll take the action functional from classical mechanics,

$$S[q(t)] = \int\limits_{t_\mathrm{a}}^{t_\mathrm{b}}\! dt\,\Big\{\tfrac{1}{2}m\dot{q}^2 - U(q)\Big\}\ . \tag{2.20}$$

[1]It may be also be that different functions depend on a different number of independent variables. *E.g.* $F = F[f(x),\, g(x,y),\, h(x,y,z)]$.

To compute the first functional derivative, we replace the function $q(t)$ with $q(t)+\epsilon\,\delta(t-t_1)$, and expand in powers of ϵ:

$$
\begin{aligned}
S\big[q(t)+\epsilon\delta(t-t_1)\big] &= S[q(t)] + \epsilon\int\limits_{t_\mathrm{a}}^{t_\mathrm{b}} dt\,\Big\{m\,\dot q\,\delta'(t-t_1) - U'(q)\,\delta(t-t_1)\Big\} \\
&= -\epsilon\,\Big\{m\,\ddot q(t_1) + U'\big(q(t_1)\big)\Big\}\ , \qquad (2.21)
\end{aligned}
$$

hence

$$
\frac{\delta S}{\delta q(t)} = -\Big\{m\,\ddot q(t) + U'\big(q(t)\big)\Big\} \qquad (2.22)
$$

and setting the first functional derivative to zero yields Newton's Second Law, $m\ddot q = -U'(q)$, for all $t\in[t_\mathrm{a},t_\mathrm{b}]$. Note that we have used the result

$$
\int\limits_{-\infty}^{\infty} dt\,\delta'(t-t_1)\,h(t) = -h'(t_1)\ , \qquad (2.23)
$$

which is easily established upon integration by parts.

To compute the second functional derivative, we replace

$$
q(t)\to q(t) + \epsilon_1\,\delta(t-t_1) + \epsilon_2\,\delta(t-t_2) \qquad (2.24)
$$

and extract the term of order $\epsilon_1\,\epsilon_2$ in the double Taylor expansion. One finds this term to be

$$
\epsilon_1\,\epsilon_2\int\limits_{t_\mathrm{a}}^{t_\mathrm{b}} dt\,\Big\{m\,\delta'(t-t_1)\,\delta'(t-t_2) - U''(q)\,\delta(t-t_1)\,\delta(t-t_2)\Big\}\ . \qquad (2.25)
$$

Note that we needn't bother with terms proportional to ϵ_1^2 or ϵ_2^2 since the recipe is to differentiate once with respect to each of ϵ_1 and ϵ_2 and then to set $\epsilon_1=\epsilon_2=0$. This procedure uniquely selects the term proportional to $\epsilon_1\,\epsilon_2$, and yields

$$
\frac{\delta^2 S}{\delta q(t_1)\,\delta q(t_2)} = -\Big\{m\,\delta''(t_1-t_2) + U''\big(q(t_1)\big)\,\delta(t_1-t_2)\Big\}\ . \qquad (2.26)
$$

In multivariable calculus, the stability of an extremum is assessed by computing the matrix of second derivatives at the extremal point, known as the Hessian matrix. One has

$$
\frac{\partial f}{\partial x_i}\bigg|_{\boldsymbol{x}^*} = 0 \quad \forall\ i \qquad ; \qquad H_{ij} = \frac{\partial^2 f}{\partial x_i\,\partial x_j}\bigg|_{\boldsymbol{x}^*}\ . \qquad (2.27)
$$

The eigenvalues of the Hessian H_{ij} determine the stability of the extremum. Since H_{ij} is a symmetric matrix, its eigenvectors η^α may be chosen to be orthogonal. The associated eigenvalues λ_α, defined by the equation

$$
H_{ij}\,\eta^\alpha_j = \lambda_\alpha\,\eta^\alpha_i\ , \qquad (2.28)
$$

are the respective curvatures in the directions η^α, where $\alpha \in \{1, \ldots, n\}$ where n is the number of variables. The extremum is a local minimum if all the eigenvalues λ_α are positive, a maximum if all are negative, and otherwise is a saddle point. Near a saddle point, there are some directions in which the function increases and some in which it decreases.

In the case of functionals, the second functional derivative $K_2(x_1, x_2)$ defines an eigenvalue problem for $\delta f(x)$:

$$\int\limits_{x_a}^{x_b} dx_2\, K_2(x_1, x_2)\, \delta f(x_2) = \lambda\, \delta f(x_1) \ . \tag{2.29}$$

In general there are an infinite number of solutions to this equation which form a basis in function space, subject to appropriate boundary conditions at x_{a} and x_{b}. For example, in the case of the action functional from classical mechanics, the above eigenvalue equation becomes a differential equation,

$$-\left\{ m\, \frac{d^2}{dt^2} + U''\big(q^*(t)\big) \right\} \delta q(t) = \lambda\, \delta q(t) \ , \tag{2.30}$$

where $q^*(t)$ is the solution to the Euler-Lagrange equations. As with the case of ordinary multivariable functions, the functional extremum is a local minimum (in function space) if every eigenvalue λ_α is positive, a local maximum if every eigenvalue is negative, and a saddle point otherwise.

Consider the simple harmonic oscillator, for which $U(q) = \frac{1}{2}\, m\omega_0^2\, q^2$. Then $U''\big(q^*(t)\big) = m\, \omega_0^2$; note that we don't even need to know the solution $q^*(t)$ to obtain the second functional derivative in this special case. The eigenvectors obey $m(\delta\ddot{q} + \omega_0^2\, \delta q) = -\lambda\, \delta q$, hence

$$\delta q(t) = A \cos\left(\sqrt{\omega_0^2 + (\lambda/m)}\; t + \varphi \right) \ , \tag{2.31}$$

where A and φ are constants. Demanding $\delta q(t_{\mathrm{a}}) = \delta q(t_{\mathrm{b}}) = 0$ requires

$$\sqrt{\omega_0^2 + (\lambda/m)}\; \big(t_{\mathrm{b}} - t_{\mathrm{a}}\big) = n\pi \ , \tag{2.32}$$

where n is an integer. Thus, the eigenfunctions are

$$\delta q_n(t) = A \sin\left(n\pi \cdot \frac{t - t_{\mathrm{a}}}{t_{\mathrm{b}} - t_{\mathrm{a}}} \right) \ , \tag{2.33}$$

and the eigenvalues are

$$\lambda_n = m \left(\frac{n\pi}{T} \right)^2 - m\omega_0^2 \ , \tag{2.34}$$

where $T = t_{\mathrm{b}} - t_{\mathrm{a}}$. Thus, so long as $T > \pi/\omega_0$, there is at least one negative eigenvalue. Indeed, for $\frac{n\pi}{\omega_0} < T < \frac{(n+1)\pi}{\omega_0}$ there will be n negative eigenvalues. This means the action is generally not a minimum, but rather lies at a *saddle point* in the (infinite-dimensional) function space.

To test this explicitly, consider a harmonic oscillator with the boundary conditions $q(0) = 0$ and $q(T) = Q$. The equations of motion, $\ddot{q} + \omega_0^2\, q = 0$, along with the boundary conditions, determine the motion,

$$q^*(t) = \frac{Q\,\sin(\omega_0 t)}{\sin(\omega_0 T)}\ . \tag{2.35}$$

The action for this path is then

$$\begin{aligned} S[q^*(t)] &= \int\limits_0^T\!dt\,\Big\{\tfrac{1}{2}m\,\dot{q}^{*2} - \tfrac{1}{2}m\omega_0^2\,q^{*2}\Big\} \\ &= \frac{m\,\omega_0^2\,Q^2}{2\sin^2\!\omega_0 T}\int\limits_0^T\!dt\,\Big\{\cos^2\!\omega_0 t - \sin^2\!\omega_0 t\Big\} \\ &= \tfrac{1}{2}m\omega_0\,Q^2\ \mathrm{ctn}\,(\omega_0 T)\ . \end{aligned} \tag{2.36}$$

Next consider the path $q(t) = Q\,t/T$ which satisfies the boundary conditions but does not satisfy the equations of motion (it proceeds with constant velocity). One finds the action for this path is

$$S[q(t)] = \tfrac{1}{2}m\omega_0\,Q^2\left(\frac{1}{\omega_0 T} - \tfrac{1}{3}\omega_0 T\right)\ . \tag{2.37}$$

Thus, provided $\omega_0 T \neq n\pi$, in the limit $T \to \infty$ we find that the constant velocity path has lower action.

Finally, consider the general mechanical action,

$$S\big[q(t)\big] = \int\limits_{t_a}^{t_b}\!dt\,L(q,\dot{q},t)\ . \tag{2.38}$$

We now evaluate the first few terms in the functional Taylor series:

$$\begin{aligned} S\big[q^*(t) + \delta q(t)\big] = \int\limits_{t_a}^{t_b}\!dt\,\Bigg\{ & L(q^*,\dot{q}^*,t) + \frac{\partial L}{\partial q_i}\bigg|_{q^*}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\bigg|_{q^*}\delta \dot{q}_i \\ & + \frac{1}{2}\,\frac{\partial^2 L}{\partial q_i\,\partial q_j}\bigg|_{q^*}\delta q_i\,\delta q_j + \frac{\partial^2 L}{\partial q_i\,\partial \dot{q}_j}\bigg|_{q^*}\delta q_i\,\delta \dot{q}_j + \frac{1}{2}\,\frac{\partial^2 L}{\partial \dot{q}_i\,\partial \dot{q}_j}\bigg|_{q^*}\delta \dot{q}_i\,\delta \dot{q}_j + \ldots\Bigg\}\ . \end{aligned} \tag{2.39}$$

To identify the functional derivatives, we integrate by parts. Let $\Phi_{\ldots}(t)$ be an arbitrary function of time. Then

$$\int_{t_a}^{t_b} dt\, \Phi_i(t)\, \delta\dot{q}_i(t) = -\int_{t_a}^{t_b} dt\, \dot{\Phi}_i(t)\, \delta q_i(t) \tag{2.40}$$

$$\begin{aligned}\int_{t_a}^{t_b} dt\, \Phi_{ij}(t)\, \delta q_i(t)\, \delta\dot{q}_j(t) &= \int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt'\, \Phi_{ij}(t)\, \delta(t-t')\, \frac{d}{dt'}\, \delta q_i(t)\, \delta q_j(t') \\ &= \int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt'\, \Phi_{ij}(t))\, \delta'(t-t')\, \delta q_i(t)\, \delta q_j(t') \end{aligned}\tag{2.41}$$

$$\begin{aligned}\int_{t_a}^{t_b} dt\, \Phi_{ij}(t)\, d\dot{q}_i(t)\, \delta\dot{q}_j(t) &= \int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt'\, \Phi_{ij}(t)\, \delta(t-t')\, \frac{d}{dt}\, \frac{d}{dt'}\, \delta q_i(t)\, \delta q_j(t') \\ &= -\int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt' \Big[\dot{\Phi}_{ij}(t)\, \delta'(t-t') + \Phi_{ij}(t)\, \delta''(t-t') \Big]\, \delta q_i(t)\, \delta q_j(t')\ . \end{aligned}\tag{2.42}$$

Thus,

$$\frac{\delta S}{\delta q_i(t)} = \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right]_{q^*(t)} \tag{2.43}$$

$$\begin{aligned}\frac{\delta^2 S}{\delta q_i(t)\, \delta q_j(t')} = &\left\{ \frac{\partial^2 L}{\partial q_i\, \partial q_j}\bigg|_{q^*(t)} \delta(t-t') - \frac{\partial^2 L}{\partial \dot{q}_i\, \partial \dot{q}_j}\bigg|_{q^*(t)} \delta''(t-t') \right. \\ &\left. + \left[2\, \frac{\partial^2 L}{\partial q_i\, \partial \dot{q}_j} - \frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{q}_i\, \partial \dot{q}_j}\right)\right]_{q^*(t)} \delta'(t-t') \right\}\ . \end{aligned}\tag{2.44}$$

2.1.2 Example 1 : Minimal Surface of Revolution

Consider a surface formed by rotating the function $y(x)$ about the x-axis. The area is then

$$A\big[y(x)\big] = \int_{x_1}^{x_2} dx\, 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ , \tag{2.45}$$

and is a functional of the curve $y(x)$. Thus we can define $L(y, y') = 2\pi y\sqrt{1+y'^2}$ and make the identification $y(x) \leftrightarrow q(t)$. We can then apply what we have derived for the mechanical action, with $L = L(q, \dot{q}, t)$, *mutatis mutandis.* Thus, the equation of motion is

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial y}\ , \tag{2.46}$$

which is a second order ODE for $y(x)$. Rather than treat the second order equation, though, we can integrate once to obtain a first order equation, by noticing that

$$\begin{aligned}\frac{d}{dx}\left[y'\,\frac{\partial L}{\partial y'} - L\right] &= y''\,\frac{\partial L}{\partial y'} + y'\,\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y'}\,y'' - \frac{\partial L}{\partial y}\,y' - \frac{\partial L}{\partial x} \\ &= y'\left[\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y}\right] - \frac{\partial L}{\partial x}\ .\end{aligned} \tag{2.47}$$

In the second line above, the term in square brackets vanishes, thus

$$\mathcal{J} = y'\,\frac{\partial L}{\partial y'} - L \quad\Rightarrow\quad \frac{d\mathcal{J}}{dx} = -\frac{\partial L}{\partial x}\ , \tag{2.48}$$

and when L has no explicit x-dependence, $\mathcal{J}$ is conserved. One finds

$$\mathcal{J} = 2\pi y \cdot \frac{y'^2}{\sqrt{1+y'^2}} - 2\pi y\,\sqrt{1+y'^2} = -\frac{2\pi y}{\sqrt{1+y'^2}}\ . \tag{2.49}$$

Solving for y',

$$\frac{dy}{dx} = \pm\sqrt{\left(\frac{2\pi y}{\mathcal{J}}\right)^2 - 1}\ , \tag{2.50}$$

which may be integrated with the substitution $y = \frac{\mathcal{J}}{2\pi}\cosh\chi$, yielding

$$y(x) = b\,\cosh\left(\frac{x-a}{b}\right)\ , \tag{2.51}$$

where a and $b = \frac{\mathcal{J}}{2\pi}$ are constants of integration. Note there are two such constants, as the original equation was second order. This shape is called a *catenary.* As we shall later find, it is also the shape of a uniformly dense rope hanging between two supports, under the influence of gravity. To fix the constants a and b, we invoke the boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$.

Consider the case where $-x_1 = x_2 \equiv x_0$ and $y_1 = y_2 \equiv y_0$. Then clearly $a = 0$, and we have

$$y_0 = b\,\cosh\left(\frac{x_0}{b}\right) \quad\Rightarrow\quad \gamma = \kappa^{-1}\,\cosh\kappa\ , \tag{2.52}$$

with $\gamma \equiv y_0/x_0$ and $\kappa \equiv x_0/b$. One finds that for any $\gamma > 1.5089$ there are two solutions, one of which is a local minimum and one of which is a saddle point of $A[y(x)]$. The solution with the smaller value of κ (*i.e.* the larger value of $\operatorname{sech}\kappa$) yields the smaller value of A, as shown in Fig. 2.3. Note that

$$\frac{y}{y_0} = \frac{\cosh(x/b)}{\cosh(x_0/b)}\ , \tag{2.53}$$

so $y(x=0) = y_0\,\operatorname{sech}(x_0/b)$.

When extremizing functions that are defined over a finite or semi-infinite interval, one must take care to evaluate the function at the boundary, for it may be that the boundary

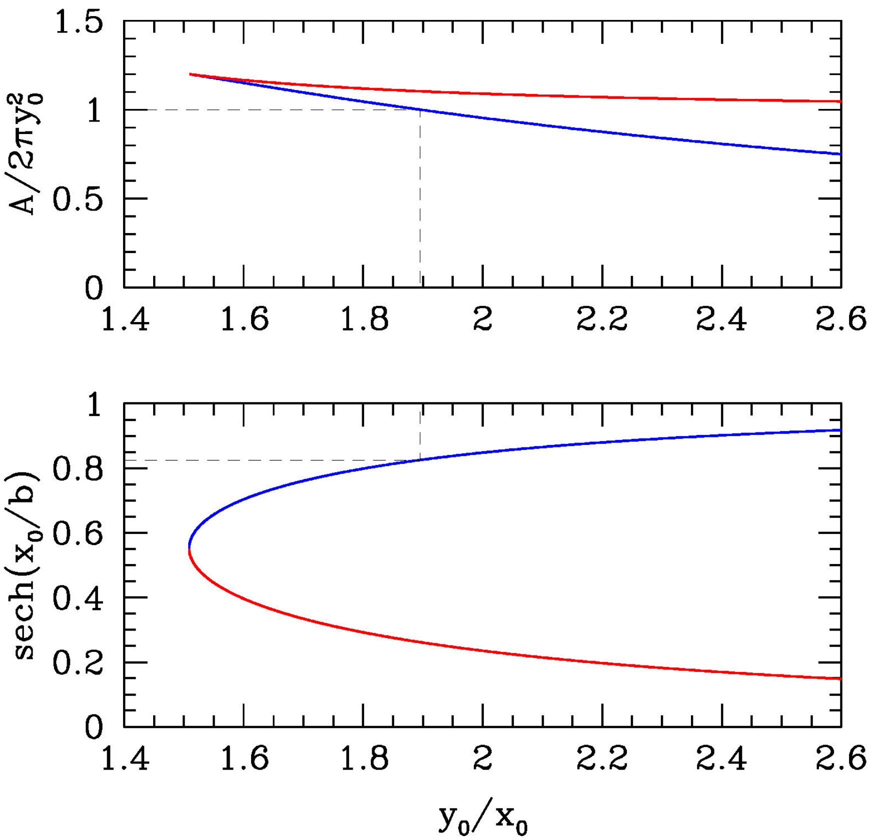

Figure 2.3: Minimal surface solution, with $y(x) = b\cosh(x/b)$ and $y(x_0) = y_0$. Top panel: $A/2\pi y_0^2$ *vs.* y_0/x_0. Bottom panel: $\mathrm{sech}(x_0/b)$ *vs.* y_0/x_0. The blue curve corresponds to a local minimum of $A[y(x)$, and the red curve to a saddle point.

yields a global extremum even though the derivative may not vanish there. Similarly, when extremizing functionals, one must investigate the functions at the boundary of function space. In this case, such a function would be the discontinuous solution, with

$$y(x) = \begin{cases} y_1 & \text{if } x = x_1 \\ \\ 0 & \text{if } x_1 < x < x_2 \\ \\ y_2 & \text{if } x = x_2 \;. \end{cases} \tag{2.54}$$

This solution corresponds to a surface consisting of two discs of radii y_1 and y_2, joined by an infinitesimally thin thread. The area functional evaluated for this particular $y(x)$ is clearly $A = \pi(y_1^2 + y_2^2)$. In Fig. 2.3, we plot $A/2\pi y_0^2$ *versus* the parameter $\gamma = y_0/x_0$. For $\gamma > \gamma_c \approx 1.564$, one of the catenary solutions is the global minimum. For $\gamma < \gamma_c$, the minimum area is achieved by the discontinuous solution.

Note that the functional derivative,

$$K_1(x) = \frac{\delta A}{\delta y(x)} = \left\{ \frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) \right\} = \frac{2\pi\big(1 + y'^2 - yy''\big)}{(1+y'^2)^{3/2}} \ , \tag{2.55}$$

indeed vanishes for the catenary solutions, but does not vanish for the discontinuous solution, where $K_1(x) = 2\pi$ throughout the interval $(-x_0, x_0)$. Since $y = 0$ on this interval, y cannot be decreased. The fact that $K_1(x) > 0$ means that increasing y will result in an increase in A, so the boundary value for A, which is $2\pi y_0^2$, is indeed a local minimum.

We furthermore see in Fig. 2.3 that for $\gamma < \gamma_* \approx 1.5089$ the local minimum and saddle are no longer present. This is the familiar saddle-node bifurcation, here in function space. Thus, for $\gamma \in [0, \gamma_*)$ there are no extrema of $A[y(x)]$, and the minimum area occurs for the discontinuous $y(x)$ lying at the boundary of function space. For $\gamma \in (\gamma_*, \gamma_c)$, two extrema exist, one of which is a local minimum and the other a saddle point. Still, the area is minimized for the discontinuous solution. For $\gamma \in (\gamma_c, \infty)$, the local minimum is the global minimum, and has smaller area than for the discontinuous solution.

2.1.3 Example 2 : Geodesic on a Surface of Revolution

We use cylindrical coordinates (ρ, ϕ, z) on the surface $z = z(\rho)$. Thus,

$$\begin{aligned} ds^2 &= d\rho^2 + \rho^2\, d\phi^2 + dx^2 \\ &= \Big\{1 + \big[z'(\rho)\big]^2\Big\}\, d\rho + \rho^2\, d\phi^2 \ , \end{aligned} \tag{2.56}$$

and the distance functional $D\big[\phi(\rho)\big]$ is

$$D\big[\phi(\rho)\big] = \int\limits_{\rho_1}^{\rho_2} d\rho\, L(\phi, \phi', \rho) \ , \tag{2.57}$$

where

$$L(\phi, \phi', \rho) = \sqrt{1 + z'^2(\rho) + \rho^2\, \phi'^2(\rho)} \ . \tag{2.58}$$

The Euler-Lagrange equation is

$$\frac{\partial L}{\partial \phi} - \frac{d}{d\rho}\left(\frac{\partial L}{\partial \phi'}\right) = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial \phi'} = \text{const.} \tag{2.59}$$

Thus,

$$\frac{\partial L}{\partial \phi'} = \frac{\rho^2\, \phi'}{\sqrt{1 + z'^2 + \rho^2\, \phi'^2}} = a \ , \tag{2.60}$$

where a is a constant. Solving for ϕ', we obtain

$$d\phi = \frac{a\sqrt{1 + \big[z'(\rho)\big]^2}}{\rho\sqrt{\rho^2 - a^2}}\, d\rho \ , \tag{2.61}$$

which we must integrate to find $\phi(\rho)$, subject to boundary conditions $\phi(\rho_i) = \phi_i$, with $i = 1, 2$.

On a cone, $z(\rho) = \lambda\rho$, and we have

$$d\phi = a\,\sqrt{1+\lambda^2}\,\frac{d\rho}{\rho\,\sqrt{\rho^2 - a^2}} = \sqrt{1+\lambda^2}\,d\tan^{-1}\sqrt{\frac{\rho^2}{a^2} - 1}\ , \tag{2.62}$$

which yields

$$\phi(\rho) = \beta + \sqrt{1+\lambda^2}\ \tan^{-1}\sqrt{\frac{\rho^2}{a^2} - 1}\ , \tag{2.63}$$

which is equivalent to

$$\rho\ \cos\left(\frac{\phi - \beta}{\sqrt{1+\lambda^2}}\right) = a\ . \tag{2.64}$$

The constants β and a are determined from $\phi(\rho_i) = \phi_i$.

2.1.4 Example 3 : Brachistochrone

Problem: find the path between (x_1, y_1) and (x_2, y_2) which a particle sliding frictionlessly and under constant gravitational acceleration will traverse in the shortest time. To solve this we first must invoke some elementary mechanics. Assuming the particle is released from (x_1, y_1) at rest, energy conservation says

$$\tfrac{1}{2}mv^2 - mgy = mgy_1\ . \tag{2.65}$$

Then the time, which is a functional of the curve $y(x)$, is

$$\begin{aligned} T\big[y(x)\big] &= \int\limits_{x_1}^{x_2}\frac{ds}{v} = \frac{1}{\sqrt{2g}}\int\limits_{x_1}^{x_2}\!dx\,\sqrt{\frac{1+y'^2}{y_1 - y}} \\ &\equiv \int\limits_{x_1}^{x_2}\!dx\ L(y, y', x)\ , \end{aligned} \tag{2.66}$$

with

$$L(y, y', x) = \sqrt{\frac{1+y'^2}{2g(y_1 - y)}}\ . \tag{2.67}$$

Since L is independent of x, eqn. 2.47, we have that

$$\mathcal{J} = y'\,\frac{\partial L}{\partial y'} - L = -\Big[2g\,(y_1 - y)\,\big(1+y'^2\big)\Big]^{-1/2} \tag{2.68}$$

is conserved. This yields

$$dx = -\sqrt{\frac{y_1 - y}{2a - y_1 + y}}\,dy\ , \tag{2.69}$$

with $a = (4g\mathcal{J}^2)^{-1}$. This may be integrated parametrically, writing

$$y_1 - y = 2a\sin^2(\tfrac{1}{2}\theta) \quad \Rightarrow \quad dx = 2a\sin^2(\tfrac{1}{2}\theta)\,d\theta \ , \tag{2.70}$$

which results in the parametric equations

$$x - x_1 = a\,\big(\theta - \sin\theta\big) \tag{2.71}$$

$$y - y_1 = -a\,(1 - \cos\theta) \ . \tag{2.72}$$

This curve is known as a *cycloid.*

2.2 Lagrangian Mechanics

Setting the first variation of the action

$$S\big[q(t)\big] = \int\limits_{t_a}^{t_b} dt\, L(q,\dot{q},t) \tag{2.73}$$

to zero gives the Euler-Lagrange equations,

$$\frac{d}{dt}\overbrace{\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right)}^{\text{momentum } p_\sigma} = \overbrace{\frac{\partial L}{\partial q_\sigma}}^{\text{force } F_\sigma} \ . \tag{2.74}$$

Thus, we have the familiar $\dot{p}_\sigma = F_\sigma$, also known as Newton's second law. Note, however, that the $\{q_\sigma\}$ are *generalized coordinates*, so p_σ may not have dimensions of momentum, nor F_σ of force. For example, if the generalized coordinate in question is an angle ϕ, then the corresponding generalized momentum is the angular momentum about the axis of ϕ's rotation, and the generalized force is the torque.

Note that the equations of motion are second order in time. This follows from the fact that $L = L(q,\dot{q},t)$. Using the chain rule,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) = \frac{\partial^2 L}{\partial \dot{q}_\sigma\,\partial \dot{q}_{\sigma'}}\,\ddot{q}_{\sigma'} + \frac{\partial^2 L}{\partial \dot{q}_\sigma\,\partial q_{\sigma'}}\,\dot{q}_{\sigma'} + \frac{\partial L}{\partial t} \ . \tag{2.75}$$

That the equations are second order in time can be regarded as an empirical fact. Suppose the Lagrangian also depends on the generalized accelerations $\ddot{q}_\sigma$. Then

$$\begin{aligned} \delta\int\limits_{t_a}^{t_b} dt\, L(q,\dot{q},\ddot{q},t) &= \left[\frac{\partial L}{\partial \dot{q}_\sigma}\,\delta q_\sigma + \frac{\partial L}{\partial \ddot{q}_\sigma}\,\delta \dot{q}_\sigma - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_\sigma}\right)\delta q_\sigma\right]_{t_a}^{t_b} \\ &\qquad + \int\limits_{t_a}^{t_b} dt\,\left\{\frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_\sigma}\right)\right\}\delta q_\sigma \ . \end{aligned} \tag{2.76}$$

The boundary term vanishes if we require $\delta q_\sigma(t_a) = \delta q_\sigma(t_b) = \delta \dot{q}_\sigma(t_a) = \delta \dot{q}_\sigma(t_b) = 0$ for all σ. The equations of motion are now *fourth order* in time.

2.2.1 Invariance of the Equations of Motion

Suppose

$$\tilde{L}(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt} G(q, t) \; . \tag{2.77}$$

Then

$$\tilde{S}[q(t)] = S[q(t)] + G(q_b, t_b) - G(q_a, t_a) \; . \tag{2.78}$$

Since the difference $\tilde{S} - S$ is a function only of the endpoint values $\{q_a, q_b\}$, their variations are identical: $\delta\tilde{S} = \delta S$. This means that L and $\tilde{L}$ result in the same equations of motion. Thus, the equations of motion are invariant under a shift of L by a total time derivative of a function of coordinates and time.

2.2.2 Lagrangian for a Free Particle

For a free particle, we can use Cartesian coordinates for each particle as our system of generalized coordinates. For a single particle, the Lagrangian $L(\boldsymbol{x}, \boldsymbol{v}, t)$ must be a function solely of $\boldsymbol{v}^2$. This is because homogeneity with respect to space and time preclude any dependence of L on $\boldsymbol{x}$ or on t, and isotropy of space means L must depend on $\boldsymbol{v}^2$. We next invoke Galilean relativity, which says that the equations of motion are invariant under transformation to a reference frame moving with constant velocity. Let $\boldsymbol{V}$ be the velocity of the new reference frame $\mathcal{K}'$ relative to our initial reference frame $\mathcal{K}$. Then $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{V}t$, and $\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{V}$. In order that the equations of motion be invariant under the change in reference frame, we demand

$$L'(\boldsymbol{v}') = L(\boldsymbol{v}) + \frac{d}{dt} G(\boldsymbol{x}, t) \; . \tag{2.79}$$

The only possibility is $L = \frac{1}{2}m\boldsymbol{v}^2$, where the constant m is the mass of the particle. Note:

$$L' = \tfrac{1}{2}m(\boldsymbol{v} - \boldsymbol{V})^2 = \tfrac{1}{2}m\boldsymbol{v}^2 + \frac{d}{dt}\Big(\tfrac{1}{2}m\boldsymbol{V}^2\, t - m\boldsymbol{V}\cdot\boldsymbol{x}\Big) = L + \frac{dG}{dt} \; . \tag{2.80}$$

For K interacting particles,

$$L = \tfrac{1}{2}\sum_a m_a \left(\frac{d\boldsymbol{x}_a}{dt}\right)^2 - U\big(\{\boldsymbol{x}_a\}, \{\dot{\boldsymbol{x}}_a\}\big) \; . \tag{2.81}$$

Here, U is the *potential energy*. Generally, U is of the form

$$U = \sum_a U_1(\boldsymbol{x}_a) + \sum_{a<a'} v(\boldsymbol{x}_a - \boldsymbol{x}_{a'}) \; , \tag{2.82}$$

however, as we shall see, velocity-dependent potentials appear in the case of charged particles interacting with electromagnetic fields. In general, though,

$$L = T - U \; , \tag{2.83}$$

where T is the kinetic energy, and U is the potential energy.

2.2.3 Example: The Double Pendulum

As an example of the generalized coordinate approach to Lagrangian dynamics, consider the double pendulum system, sketched in Fig. **??**. We choose as generalized coordinates the two angles θ_1 and θ_2. In order to evaluate the Lagrangian, we must obtain the kinetic and potential energies in terms of the generalized coordinates $\{\theta_1, \theta_2\}$ and their corresponding velocities $\{\dot{\theta}_1, \dot{\theta}_2\}$.

In Cartesian coordinates,

$$T = \tfrac{1}{2} m_1 \, (\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2} m_2 \, (\dot{x}_2^2 + \dot{y}_2^2) \tag{2.84}$$

$$U = m_1 \, g \, y_1 + m_2 \, g \, y_2 \ . \tag{2.85}$$

We therefore express the Cartesian coordinates $\{x_1, y_1, x_2, y_2\}$ in terms of the generalized coordinates $\{\theta_1, \theta_2\}$:

$$x_1 = \ell_1 \, \sin\theta_1 \qquad\qquad x_2 = \ell_1 \, \sin\theta_1 + \ell_2 \, \sin\theta_2 \tag{2.86}$$

$$y_1 = -\ell_1 \, \cos\theta_1 \qquad\qquad y_2 = -\ell_1 \, \cos\theta_1 - \ell_2 \, \cos\theta_2 \ . \tag{2.87}$$

Thus, the velocities are

$$\dot{x}_1 = \ell_1 \, \dot{\theta}_1 \, \cos\theta_1 \qquad\qquad \dot{x}_2 = \ell_1 \, \dot{\theta}_1 \, \cos\theta_1 + \ell_2 \, \dot{\theta}_2 \, \cos\theta_2 \tag{2.88}$$

$$\dot{y}_1 = \ell_1 \, \dot{\theta}_1 \, \sin\theta_1 \qquad\qquad \dot{y}_2 = \ell_1 \, \dot{\theta}_1 \, \sin\theta_1 + \ell_2 \, \dot{\theta}_2 \, \sin\theta_2 \ . \tag{2.89}$$

Thus,

$$T = \tfrac{1}{2} m_1 \, \ell_1^2 \, \dot{\theta}_1^2 + \tfrac{1}{2} m_2 \Big\{ \ell_1^2 \, \dot{\theta}_1^2 + 2 \ell_1 \, \ell_2 \, \cos(\theta_1 - \theta_2) \, \dot{\theta}_1 \, \dot{\theta}_2 + \ell_2^2 \, \dot{\theta}_2^2 \Big\} \tag{2.90}$$

$$U = -m_1 \, g \, \ell_1 \, \cos\theta_1 - m_2 \, g \, \ell_1 \, \cos\theta_1 - m_2 \, g \, \ell_2 \, \cos\theta_2 \ , \tag{2.91}$$

and

$$\begin{aligned} L = T - U = \tfrac{1}{2} (m_1 + m_2) \, \ell_1^2 \, \dot{\theta}_1^2 + m_2 \, \ell_1 \, \ell_2 \, \cos(\theta_1 - \theta_2) \, \dot{\theta}_1 \, \dot{\theta}_2 + \tfrac{1}{2} m_2 \, \ell_2^2 \, \dot{\theta}_2^2 \\ + (m_1 + m_2) \, g \, \ell_1 \, \cos\theta_1 + m_2 \, g \, \ell_2 \, \cos\theta_2 \ . \end{aligned} \tag{2.92}$$

The generalized (canonical) momenta are

$$p_1 = \frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2) \, \ell_1^2 \, \dot{\theta}_1 + m_2 \, \ell_1 \, \ell_2 \, \cos(\theta_1 - \theta_2) \, \dot{\theta}_2 \tag{2.93}$$

$$p_2 = \frac{\partial L}{\partial \dot{\theta}_2} = m_2 \, \ell_1 \, \ell_2 \, \cos(\theta_1 - \theta_2) \, \dot{\theta}_1 + m_2 \, \ell_2^2 \, \dot{\theta}_2 \ , \tag{2.94}$$

and the equations of motion are

$$\begin{aligned} \dot{p}_1 &= (m_1 + m_2) \, \ell_1^2 \, \ddot{\theta}_1 + m_2 \, \ell_1 \, \ell_2 \, \cos(\theta_1 - \theta_2) \, \ddot{\theta}_2 - m_2 \, \ell_1 \, \ell_2 \, \sin(\theta_1 - \theta_2) \, (\dot{\theta}_1 - \dot{\theta}_2) \, \dot{\theta}_2 \\ &= -(m_1 + m_2) \, g \, \ell_1 \, \sin\theta_1 - m_2 \, \ell_1 \, \ell_2 \, \sin(\theta_1 - \theta_2) \, \dot{\theta}_1 \, \dot{\theta}_2 = \frac{\partial L}{\partial \theta_1} \end{aligned} \tag{2.95}$$

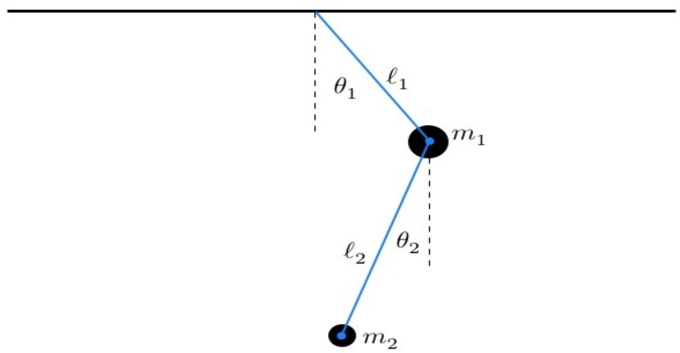

Figure 2.4: The double pendulum, with generalized coordinates θ_1 and θ_2. All motion is confined to a single plane.

and

$$\begin{aligned}\dot{p}_2 &= m_2\,\ell_1\,\ell_2\,\cos(\theta_1-\theta_2)\,\ddot{\theta}_1 - m_2\,\ell_1\,\ell_2\,\sin(\theta_1-\theta_2)\,(\dot{\theta}_1-\dot{\theta}_2)\,\dot{\theta}_1 + m_2\,\ell_2^2\,\ddot{\theta}_2 \\ &= -m_2\,g\,\ell_2\,\sin\theta_2 + m_2\,\ell_1\,\ell_2\,\sin(\theta_1-\theta_2)\,\dot{\theta}_1\,\dot{\theta}_2 = \frac{\partial L}{\partial \theta_2}\ . \end{aligned} \tag{2.96}$$

We therefore find

$$\ell_1\,\ddot{\theta}_1 + \frac{m_2\,\ell_2}{m_1+m_2}\,\cos(\theta_1-\theta_2)\,\ddot{\theta}_2 + \frac{m_2\,\ell_2}{m_1+m_2}\,\sin(\theta_1-\theta_2)\,\dot{\theta}_2^2 + g\,\sin\theta_1 = 0 \tag{2.97}$$

$$\ell_1\,\cos(\theta_1-\theta_2)\,\ddot{\theta}_1 + \ell_2\,\ddot{\theta}_2 - \ell_1\,\sin(\theta_1-\theta_2)\,\dot{\theta}_1^2 + g\,\sin\theta_2 = 0\ . \tag{2.98}$$

These are coupled, nonlinear second order ODEs. When the system is close to equilibrium, we may assume that the amplitudes of the motion are small, and expand in powers of the deviations from equilibrium. The linearized equations of motion are then

$$\ell_1\,\ddot{\theta}_1 + \frac{m_2\,\ell_2}{m_1+m_2}\,\ddot{\theta}_2 + g\,\theta_1 = 0 \tag{2.99}$$

$$\ell_1\,\ddot{\theta}_1 + \ell_2\,\ddot{\theta}_2 + g\,\theta_2 = 0\ . \tag{2.100}$$

We can solve this coupled set of equations by a nifty trick. Let's take a linear combination of the first equation plus an undetermined coefficient, α, times the second:

$$(1+\alpha)\,\ell_1\,\ddot{\theta}_1 + \left(\frac{m_2}{m_1+m_2}+\alpha\right)\ell_2\,\ddot{\theta}_2 + g\,(\theta_1+\alpha\,\theta_2) = 0\ . \tag{2.101}$$

We now demand that the ratio of the coefficients of θ_2 and θ_1 is the same as the ratio of the coefficients of $\ddot{\theta}_2$ and $\ddot{\theta}_1$:

$$\alpha = \frac{\left(\frac{m_2}{m_1+m_2}+\alpha\right)\ell_2}{(1+\alpha)\,\ell_1}\ . \tag{2.102}$$

This is a quadratic equation, with solutions

$$\alpha_\pm = \frac{\ell_2 - \ell_1 \pm \sqrt{(\ell_2-\ell_1)^2 + \frac{4\ell_1\ell_2 m_2}{m_1+m_2}}}{2\,\ell_1} \ . \tag{2.103}$$

When α takes on either of these values, the equation of motion becomes

$$(1+\alpha_\pm)\,\ell_1\,\frac{d^2}{dt^2}\big(\theta_1 + \alpha\,\theta_2\big) + g\,\big(\theta_1 + \alpha\,\theta_2\big) = 0\ , \tag{2.104}$$

and defining the (unnormalized) *normal modes*

$$\xi_\pm \equiv \big(\theta_1 + \alpha\,\theta_2\big)\ , \tag{2.105}$$

we find

$$\ddot{\xi}_\pm + \omega_\pm^2\,\xi_\pm = 0\ , \tag{2.106}$$

with

$$\omega_\pm = \sqrt{\frac{g}{(1+\alpha_\pm)\,\ell_1}}\ . \tag{2.107}$$

Thus, by switching to the normal coordinates, we decoupled the equations of motion, and identified the two *normal frequencies of oscillation*. We shall have much more to say about small oscillations further below.

Note that for $\ell_1 = \ell_2 = \ell$ and $m_1 = m_2 = m$, that

$$\alpha_\pm = \pm\tfrac{1}{\sqrt{2}} \quad , \quad \xi_\pm = \theta_1 \pm \tfrac{1}{\sqrt{2}}\,\theta_2 \quad , \quad \omega_\pm = \sqrt{2\mp\sqrt{2}}\,\sqrt{\frac{g}{\ell}}\ . \tag{2.108}$$

Note that the oscillation frequency for the 'in-phase' mode ξ_+ is low, and that for the 'out of phase' mode ξ_- is high.

2.3 Conservative Mechanical Systems in One Dimension

For one-dimensional 'conservative' mechanical systems, Newton's second law reads

$$m\ddot{x} = -\frac{dU}{dx}\ , \tag{2.109}$$

where $F = -dU/dx$. This may be written as an $N = 2$ system,

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\frac{1}{m}\,U'(x) \end{pmatrix}\ . \tag{2.110}$$

The total energy is conserved:

$$E = T + U = \tfrac{1}{2}m\dot{x}^2 + U(x)\ . \tag{2.111}$$

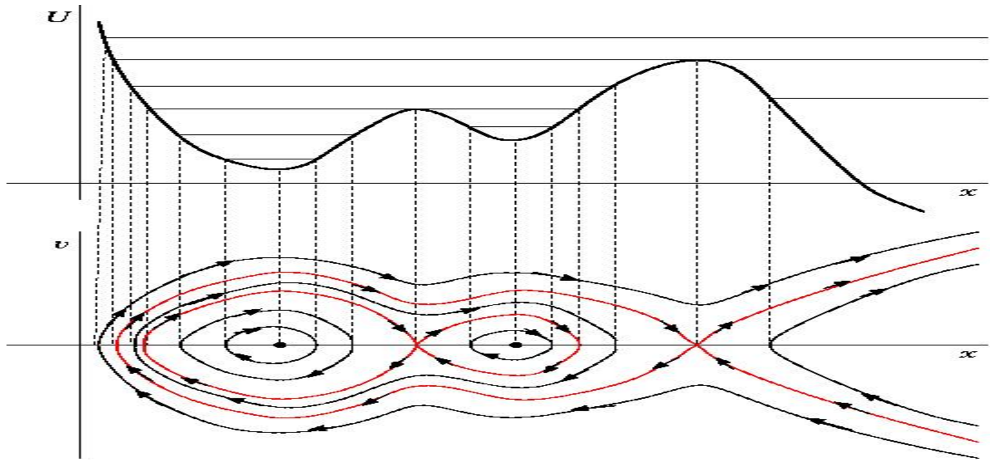

Figure 2.5: A potential $U(x)$ and the corresponding phase portraits. Separatrices are shown in red.

This may be verified explicitly:

$$\begin{aligned}\frac{dE}{dt} &= \frac{d}{dt}\Big[\tfrac{1}{2}m\dot{x}^2 + U(x)\Big] \\ &= \Big[m\ddot{x} + U'(x)\Big]\,\dot{x} = 0\ . \end{aligned} \tag{2.112}$$

The phase curves are thus curves of constant energy. Examples of phase curves are sketched in Fig. 2.5.

A fixed point (x^*, v^*) of the dynamics satisfies $U'(x^*) = 0$ and $v^* = 0$. linearizing in the vicinity of such a fixed point, we write $\delta x = x - x^*$ and $\delta v = v - v^*$, obtaining

$$\frac{d}{dt}\begin{pmatrix}\delta x\\ \delta v\end{pmatrix} = \begin{pmatrix}0 & 1\\ -\frac{1}{m}\,U''(x^*) & 0\end{pmatrix}\begin{pmatrix}\delta x\\ \delta v\end{pmatrix} + \ldots, \tag{2.113}$$

The trace and determinant of the above matrix are $T = 0$ and $D = \frac{1}{m}\,U''(x^*)$. Thus, there are only two (generic) possibilities: centers, when $U''(x^*) > 0$, and saddles, when $U''(x^*) < 0$. Examples of each are shown in Fig. 2.5.

Conservation of energy allows us to reduce the dynamics to those of an $N = 1$ system:

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}\Big(E - U(x)\Big)}\ . \tag{2.114}$$

The $\pm$ sign above depends on the direction of motion. Points $x(E)$ which satisfy

$$E = U(x) \quad \Rightarrow \quad x(E) = U^{-1}(E)\ , \tag{2.115}$$

where U^{-1} is the inverse function, are called *turning points*. We can integrate eqn. 2.114 to obtain

$$t(x) - t(x_0) = \pm\sqrt{\tfrac{m}{2}}\int\limits_{x_0}^{x}\frac{dx'}{\sqrt{E - U(x')}}\ . \tag{2.116}$$

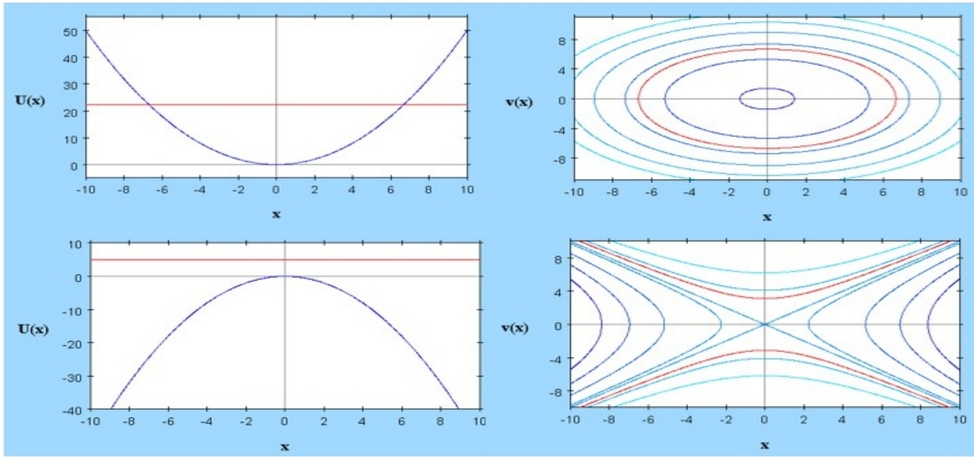

Figure 2.6: Phase curves in the vicinity of centers and saddles.

This is to be inverted to obtain the function $x(t)$. For motion confined between two turning points $x_\pm(E)$, the period of the motion is given by

$$T(E) = \sqrt{2m} \int\limits_{x_-(E)}^{x_+(E)} \frac{dx'}{\sqrt{E - U(x')}} \ . \tag{2.117}$$

This has a geometric interpretation. The area $\mathcal{A}$ in phase space enclosed by a bounded phase curve is

$$\mathcal{A}(E) = \oint\limits_E v\, dx = \sqrt{\tfrac{8}{m}} \int\limits_{x_-(E)}^{x_+(E)} dx' \sqrt{E - U(x')} \ . \tag{2.118}$$

Thus, the period is proportional to the rate of change of $A(E)$ with E:

$$T = m\, \frac{\partial \mathcal{A}}{\partial E} \ . \tag{2.119}$$

2.3.1 Small Oscillations

If we expand about a local minimum of $U(x)$, we have

$$\frac{d}{dt}\begin{pmatrix} \delta x \\ \delta v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix}\begin{pmatrix} \delta x \\ \delta v \end{pmatrix} \ , \tag{2.120}$$

with $\omega_0^2 = U''(x^*)/m > 0$. Here $\delta x = x - x^*$ and $\delta v = v - v^*$ with $v^* = 0$. This is a harmonic oscillator: $\delta\ddot{x} = -\omega_0^2\, \delta x$, with solution

$$\delta x(t) = \delta x_0 \, \cos(\omega_0 t) + \omega_0^{-1}\, \delta v_0 \, \sin(\omega_0 t) \tag{2.121}$$

$$\delta v(t) = \delta v_0 \, \cos(\omega_0 t) - \omega_0 \, \delta x_0 \, \sin(\omega_0 t) \ . \tag{2.122}$$

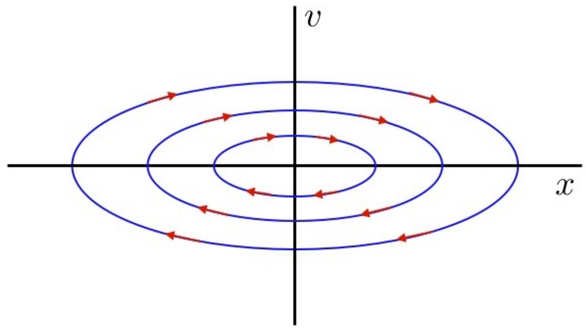

Figure 2.7: Phase curves for the harmonic oscillator.

The phase curves are ellipses:

$$\omega_0 \big(\delta x(t)\big)^2 + \omega_0^{-1} \big(\delta v(t)\big)^2 = C \ , \tag{2.123}$$

where C is a constant, independent of time. A sketch of the phase curves and of the phase flow is shown in Fig. 2.7. Note that the x and v axes have different dimensions. Energy is conserved:

$$E = \tfrac{1}{2} m \, (\delta v)^2 + \tfrac{1}{2} m \omega_0^2 \, (\delta x)^2 \ , \tag{2.124}$$

Therefore we may find the length of the semimajor and semiminor axes by setting $\delta v = 0$ or $\delta x = 0$, which gives

$$\delta x_{\text{max}} = \sqrt{\frac{2E}{k}} \quad , \quad \delta v_{\text{max}} = \sqrt{\frac{2E}{m}} \ . \tag{2.125}$$

The area of the elliptical phase curves is thus

$$\mathcal{A}(E) = \pi \, \delta x_{\text{max}} \, \delta v_{\text{max}} = \frac{2\pi E}{\sqrt{mk}} \ . \tag{2.126}$$

The period of motion is therefore

$$T(E) = m \, \frac{\partial \mathcal{A}}{\partial E} = 2\pi \sqrt{\frac{m}{k}} \ , \tag{2.127}$$

which is independent of E.

2.3.2 Pendulum

Next, consider the simple pendulum, composed of a mass point m affixed to a massless rigid rod of length ℓ. The potential is $U(\theta) = -mg\ell \cos\theta$, hence

$$m\ell^2 \, \ddot\theta = -\frac{dU}{d\theta} = -mg\ell \sin\theta \ . \tag{2.128}$$

This is equivalent to

$$\frac{d}{dt} \begin{pmatrix} \theta \\ \omega \end{pmatrix} = \begin{pmatrix} \omega \\ -\omega_0^2 \sin\theta \end{pmatrix} \ , \tag{2.129}$$

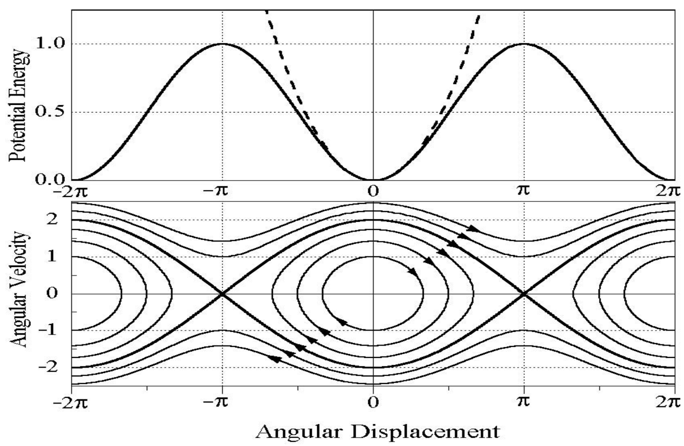

Figure 2.8: Phase curves for the simple pendulum. The *separatrix* divides phase space into regions of vibration and libration.

where $\omega = \dot{\theta}$ is the angular velocity, and where $\omega_0 = \sqrt{g/\ell}$ is the natural frequency of small oscillations.

The conserved energy is

$$E = \tfrac{1}{2}\, m\ell^2\, \dot{\theta}^2 + U(\theta) \ . \tag{2.130}$$

Assuming the pendulum is released from rest at $\theta = \theta_0$,

$$\frac{2E}{m\ell^2} = \dot{\theta}^2 - 2\omega_0^2\, \cos\theta = -2\omega_0^2\, \cos\theta_0 \ . \tag{2.131}$$

The period for motion of amplitude θ_0 is then

$$T\big(\theta_0\big) = \frac{\sqrt{8}}{\omega_0} \int\limits_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \frac{4}{\omega_0}\, \mathrm{K}\big(\sin^2 \tfrac{1}{2}\theta_0\big) \ , \tag{2.132}$$

where $\mathrm{K}(z)$ is the complete elliptic integral of the first kind. Expanding $K(z)$, we have

$$T\big(\theta_0\big) = \frac{2\pi}{\omega_0} \left\{ 1 + \tfrac{1}{4}\, \sin^2\big(\tfrac{1}{2}\theta_0\big) + \tfrac{9}{64}\, \sin^4\big(\tfrac{1}{2}\theta_0\big) + \ldots \right\} . \tag{2.133}$$

For $\theta_0 \to 0$, the period approaches the usual result $2\pi/\omega_0$, valid for the linearized equation $\ddot{\theta} = -\omega_0^2\, \theta$. As $\theta_0 \to \frac{\pi}{2}$, the period diverges logarithmically.

The phase curves for the pendulum are shown in Fig. 2.8. The small oscillations of the pendulum are essentially the same as those of a harmonic oscillator. Indeed, within the small angle approximation, $\sin\theta \approx \theta$, and the pendulum equations of motion are exactly those of the harmonic oscillator. These oscillations are called *librations*. They involve a back-and-forth motion in real space, and the phase space motion is contractable to a point, in the topological sense. However, if the initial angular velocity is large enough, a

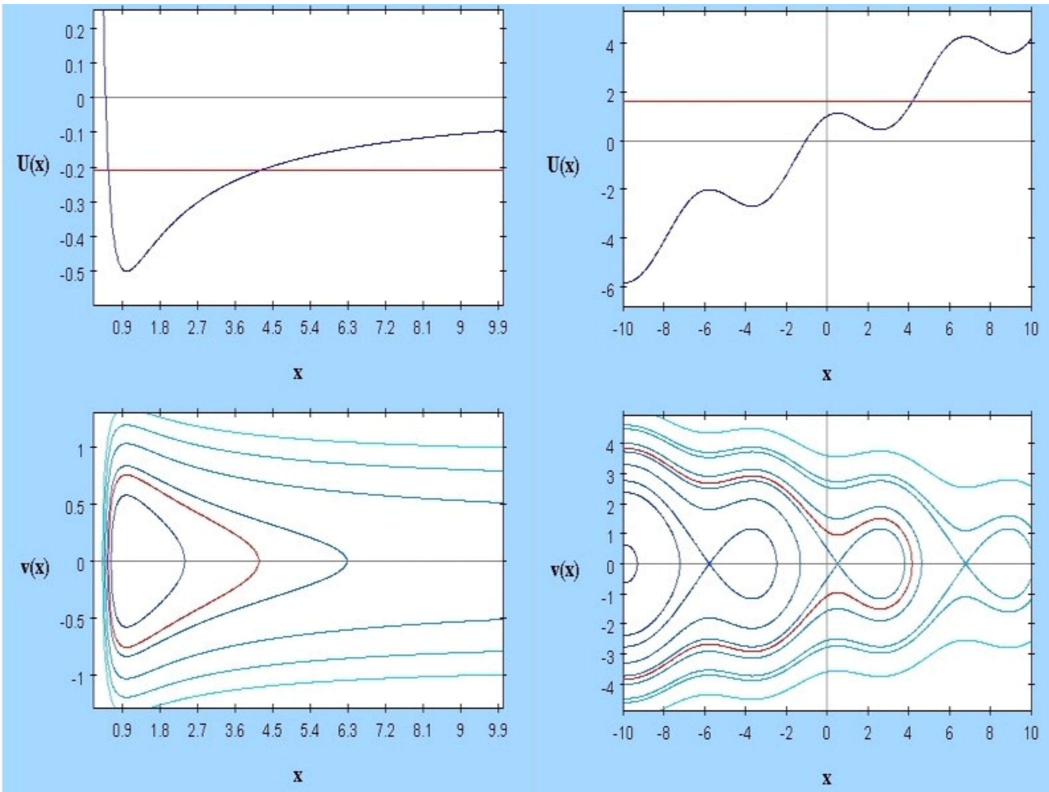

Figure 2.9: Phase curves for two potentials. Left: Kepler effective potential $\mathcal{U}(x) = -x^{-1} + \frac{1}{2}x^{-2}$. Right: 'tilted washboard' potential $\mathcal{U}(x) = \cos(x) + \frac{1}{2}x$.

qualitatively different kind of motion is observed, whose phase curves are *rotations*. In this case, the pendulum bob keeps swinging around in the same direction, because, as we'll see in a later lecture, the total energy is sufficiently large. The phase curve which separates these two topologically distinct motions is called a *separatrix*.

2.3.3 Other Potentials

Using the phase plotter application written by Ben Schmidel, available at

http : //physics.ucsd.edu/students/courses/fall2005/physics110a/PhasePlotter/index.html

it is possible to explore the phase curves for a wide variety of potentials. Two examples are shown in Fig. 2.9. The first is the effective potential for the Kepler problem,

$$U_{\text{eff}}(r) = -\frac{k}{r} + \frac{\ell^2}{2\mu r^2} \;, \tag{2.134}$$

about which we shall have much more to say when we study central forces. Here r is the separation between two gravitating bodies of masses $m_{1,2}$, $\mu = m_1 m_2/(m_1 + m_2)$ is the 'reduced mass', and $k = Gm_1 m_2$, where G is the Cavendish constant. We can then write

$$U_{\rm eff}(r) = U_0 \bigg\{ -\frac{1}{x} + \frac{1}{2x^2} \bigg\} \ , \tag{2.135}$$

where $r_0 = \ell^2/\mu k$ has the dimensions of length, and $x \equiv r/r_0$, and where $U_0 = k/r_0 = \mu k^2/\ell^2$. Thus, if distances are measured in units of r_0 and the potential in units of U_0, the potential may be written in dimensionless form as $\mathcal{U}(x) = -\frac{1}{x} + \frac{1}{2x^2}$.

The second example is the 'tilted washboard' potential

$$U(x) = U_0 \left\{ \cos\left(\frac{x}{a}\right) + \frac{x}{2a} \right\} \ . \tag{2.136}$$

Again measuring x in units of a and U in units of U_0, we arrive at $\mathcal{U}(x) = \cos(x) + \frac{1}{2}x$. This potential arises in the theory of current-biased Josephson junctions.

2.4 The Hamiltonian

The Lagrangian is a function of generalized coordinates, velocities, and time. The canonical momentum conjugate to the generalized coordinate q_σ is

$$p_\sigma = \frac{\partial L}{\partial \dot q_\sigma} \ . \tag{2.137}$$

The Hamiltonian is a function of coordinates, *momenta*, and time. It is defined as the Legendre transform of L:

$$H(q,p,t) = \sum_\sigma p_\sigma \, \dot q_\sigma - L \ . \tag{2.138}$$

Let's examine the differential of H:

$$\begin{aligned} dH &= \sum_\sigma \left(\dot q_\sigma \, dp_\sigma + p_\sigma \, d\dot q_\sigma - \frac{\partial L}{\partial q_\sigma} \, dq_\sigma - \frac{\partial L}{\partial \dot q_\sigma} \, d\dot q_\sigma \right) - \frac{\partial L}{\partial t} \, dt \\ &= \sum_\sigma \left(\dot q_\sigma \, dp_\sigma - \frac{\partial L}{\partial q_\sigma} \, dq_\sigma \right) - \frac{\partial L}{\partial t} \, dt \ , \end{aligned} \tag{2.139}$$

where we have invoked the definition of p_σ to cancel the coefficients of $d\dot q_\sigma$. Since $\dot p_\sigma = \partial L/\partial q_\sigma$, we have *Hamilton's equations of motion*,

$$\dot q_\sigma = \frac{\partial H}{\partial p_\sigma} \quad , \quad \dot p_\sigma = -\frac{\partial H}{\partial q_\sigma} \ . \tag{2.140}$$

Thus, we can write

$$dH = \sum_\sigma \Big(\dot q_\sigma \, dp_\sigma - \dot p_\sigma \, dq_\sigma \Big) - \frac{\partial L}{\partial t} \, dt \ . \tag{2.141}$$

Dividing by dt, we obtain

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \ , \tag{2.142}$$

which says that the Hamiltonian is *conserved* (*i.e.* it does not change with time) whenever there is no *explicit* time dependence to L.

2.5 Is $H = T + U$?

The most general form of the kinetic energy is

$$\begin{aligned} T &= T_2 + T_1 + T_0 \\ &= \tfrac{1}{2} M_{\sigma\sigma'}(q,t)\,\dot q_\sigma\,\dot q_{\sigma'} + B_\sigma(q,t)\,\dot q_\sigma + W(q,t) \ , \end{aligned} \tag{2.143}$$

where $T_n(q,\dot q,t)$ is homogeneous of degree n in the velocities[2]. The Lagrangian is

$$L = T - U = \tfrac{1}{2} M_{\sigma\sigma'}(q,t)\,\dot q_\sigma\,\dot q_{\sigma'} + B_\sigma(q,t)\,\dot q_\sigma + W(q,t) - U(q,t) \ . \tag{2.144}$$

We have assumed $U(q,t)$ is velocity-independent, but the above form for $L = T - U$ is quite general. (*E.g.* any velocity-dependence in U can be absorbed into the $B_\sigma\,\dot q_\sigma$ term.) The canonical momentum conjugate to q_σ is

$$p_\sigma = \frac{\partial L}{\partial \dot q_\sigma} = M_{\sigma\sigma'}\,\dot q_{\sigma'} + B_\sigma \ , \tag{2.145}$$

which is inverted to give

$$\dot q_\sigma = M^{-1}_{\sigma\sigma'}\big(p_{\sigma'} - B_{\sigma'}\big) \ . \tag{2.146}$$

The Hamiltonian is then

$$\begin{aligned} H &= p_\sigma\,\dot q_\sigma - L \\ &= p_\sigma\, M^{-1}_{\sigma\sigma'}\big(p_{\sigma'} - B_{\sigma'}\big) - \tfrac{1}{2} M_{\sigma\sigma'}\, M^{-1}_{\sigma\alpha}\big(p_\alpha - B_\alpha\big)\, M^{-1}_{\sigma'\beta}\big(p_\beta - B_\beta\big) \\ &\qquad - B_\sigma\, M^{-1}_{\sigma\sigma'}\big(p_{\sigma'} - B_{\sigma'}\big) - W + U \\ &= \tfrac{1}{2}\, M^{-1}_{\sigma\sigma'}(q,t)\,\big(p_\sigma - B_\sigma\big)\,\big(p_{\sigma'} - B_{\sigma'}\big) - W(q,t) + U(q,t) && (2.147) \\ &= T_2 - T_0 + U \ . && (2.148) \end{aligned}$$

If T_0 and T_1 vanish, *i.e.* if $T(q,\dot q,t)$ is a homogeneous function of degree two in the generalized velocities, and $U(q,t)$ is velocity-independent, then $H = T + U$. But if T_0 or T_1 is nonzero, then $H \neq T + U$.

---

[2]A homogeneous function of degree k satisfies $f(\lambda x_1, \ldots, \lambda x_n) = \lambda^k f(x_1, \ldots, x_n)$. It is then easy to prove *Euler's theorem*, $\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i} = kf$.

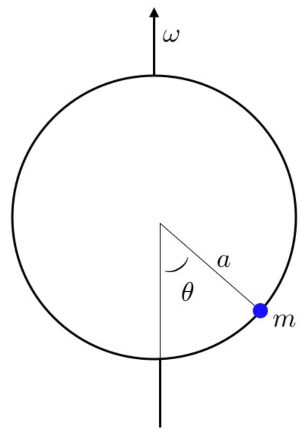

Figure 2.10: A bead of mass m on a rotating hoop of radius a.

2.5.1 Example: A Bead on a Rotating Hoop

Consider a bead of mass m constrained to move along a hoop of radius a. The hoop is further constrained to rotate with angular velocity ω about the $\hat{\boldsymbol{z}}$-axis, as shown in Fig. 2.10.

The most convenient set of generalized coordinates is spherical polar (r, θ, ϕ), in which case

$$
\begin{aligned}
T &= \tfrac{1}{2}m\big(\dot{r}^2 + r^2\,\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\big) \\
&= \tfrac{1}{2}ma^2\big(\dot{\theta}^2 + \omega^2\sin^2\theta\big)\ . \qquad (2.149)
\end{aligned}
$$

Thus, $T_2 = \frac{1}{2}ma^2\dot{\theta}^2$ and $T_0 = \frac{1}{2}ma^2\omega^2\sin^2\theta$. The potential energy is $U(\theta) = mga(1-\cos\theta)$. The momentum conjugate to θ is $p_\theta = ma^2\dot{\theta}$, and thus

$$
\begin{aligned}
H(\theta, p) &= T_2 - T_0 + U \\
&= \tfrac{1}{2}ma^2\dot{\theta}^2 - \tfrac{1}{2}ma^2\omega^2\sin^2\theta + mga(1-\cos\theta) \\
&= \frac{p_\theta^2}{2ma^2} - \tfrac{1}{2}ma^2\omega^2\sin^2\theta + mga(1-\cos\theta)\ . \qquad (2.150)
\end{aligned}
$$

For this problem, we can define the *effective potential*

$$
\begin{aligned}
U_{\text{eff}}(\theta) \equiv U - T_0 &= mga(1-\cos\theta) - \tfrac{1}{2}ma^2\omega^2\sin^2\theta \\
&= mga\bigg(1 - \cos\theta - \frac{\omega^2}{2\omega_0^2}\,\sin^2\theta\bigg)\ , \qquad (2.151)
\end{aligned}
$$

where $\omega_0 \equiv g/a^2$. The Lagrangian may then be written

$$
L = \tfrac{1}{2}ma^2\dot{\theta}^2 - U_{\text{eff}}(\theta)\ , \qquad (2.152)
$$

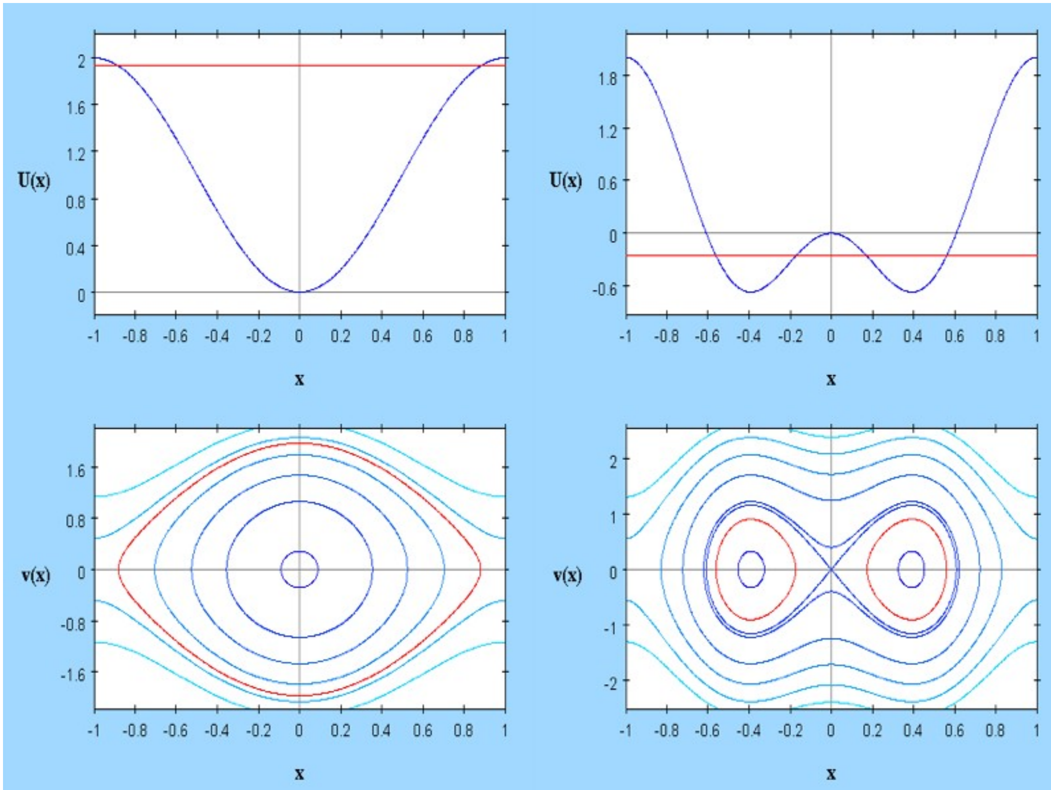

Figure 2.11: The effective potential $U_{\text{eff}}(\theta) = mga\big[1 - \cos\theta - \frac{\omega^2}{2\omega_0^2}\,\sin^2\theta\big]$. (The dimensionless potential $\tilde{U}_{\text{eff}}(x) = U_{\text{eff}}/mga$ is shown, where $x = \theta/\pi$.) Left panels: $\omega = \frac{1}{2}\sqrt{3}\,\omega_0$. Right panels: $\omega = \sqrt{3}\,\omega_0$.

and thus the equations of motion are

$$ma^2\ddot{\theta} = -\frac{\partial U_{\text{eff}}}{\partial\theta} \ . \tag{2.153}$$

Equilibrium is achieved when $U'_{\text{eff}}(\theta) = 0$, which gives

$$\frac{\partial U_{\text{eff}}}{\partial\theta} = mga\sin\theta\,\bigg\{1 - \frac{\omega^2}{\omega_0^2}\cos\theta\bigg\} = 0 \ , \tag{2.154}$$

i.e. $\theta^* = 0$, $\theta^* = \pi$, or $\theta^* = \pm\cos^{-1}(\omega_0^2/\omega^2)$, where the last pair of equilibria are present only for $\omega^2 > \omega_0^2$. The stability of these equilibria is assessed by examining the sign of $U''_{\text{eff}}(\theta^*)$. We have

$$U''_{\text{eff}}(\theta) = mga\,\bigg\{\cos\theta - \frac{\omega^2}{\omega_0^2}\big(2\cos^2\theta - 1\big)\bigg\} \ . \tag{2.155}$$

Thus,

$$U''_{\text{eff}}(\theta^*) = \begin{cases} mga\left(1 - \frac{\omega^2}{\omega_0^2}\right) & \text{at } \theta^* = 0 \\[2ex] -mga\left(1 + \frac{\omega^2}{\omega_0^2}\right) & \text{at } \theta^* = \pi \\[2ex] mga\left(\frac{\omega^2}{\omega_0^2} - \frac{\omega_0^2}{\omega^2}\right) & \text{at } \theta^* = \pm\cos^{-1}\left(\frac{\omega_0^2}{\omega^2}\right) . \end{cases} \tag{2.156}$$

Thus, $\theta^* = 0$ is stable for $\omega^2 < \omega_0^2$ but becomes unstable when the rotation frequency ω is sufficiently large, *i.e.* when $\omega^2 > \omega_0^2$. In this regime, there are two new equilibria, at $\theta^* = \pm\cos^{-1}(\omega_0^2/\omega^2)$, which are both stable. The equilibrium at $\theta^* = \pi$ is always unstable, independent of the value of ω. The situation is depicted in Fig. 2.11.

2.6 Charged Particle in a Magnetic Field

Consider next the case of a charged particle moving in the presence of an electromagnetic field. The particle's potential energy is

$$U(\boldsymbol{r}) = q\,\phi(\boldsymbol{r},t) - \frac{q}{c}\,\boldsymbol{A}(\boldsymbol{r},t)\cdot\dot{\boldsymbol{r}} \ , \tag{2.157}$$

which *is* velocity-dependent. The kinetic energy is $T = \frac{1}{2}m\,\dot{\boldsymbol{r}}^2$, as usual. Here $\phi(\boldsymbol{r})$ is the scalar potential and $\boldsymbol{A}(\boldsymbol{r})$ the vector potential. The electric and magnetic fields are given by

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t} \quad , \quad \boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} \ . \tag{2.158}$$

The canonical momentum is

$$\boldsymbol{p} = \frac{\partial L}{\partial\dot{\boldsymbol{r}}} = m\,\dot{\boldsymbol{r}} + \frac{q}{c}\,\boldsymbol{A} \ , \tag{2.159}$$

and hence the Hamiltonian is

$$\begin{aligned} H(\boldsymbol{r},\boldsymbol{p},t) &= \boldsymbol{p}\cdot\dot{\boldsymbol{r}} - L \\ &= m\dot{\boldsymbol{r}}^2 + \frac{q}{c}\,\boldsymbol{A}\cdot\dot{\boldsymbol{r}} - \tfrac{1}{2}m\,\dot{\boldsymbol{r}}^2 - \frac{q}{c}\,\boldsymbol{A}\cdot\dot{\boldsymbol{r}} + q\,\phi \\ &= \tfrac{1}{2}m\,\dot{\boldsymbol{r}}^2 + q\,\phi \\ &= \frac{1}{2m}\Big(\boldsymbol{p} - \frac{q}{c}\,\boldsymbol{A}(\boldsymbol{r},t)\Big)^2 + q\,\phi(\boldsymbol{r},t) \ . \end{aligned} \tag{2.160}$$

If $\boldsymbol{A}$ and ϕ are time-independent, then $H(\boldsymbol{r},\boldsymbol{p})$ is conserved.

Let's work out the equations of motion. We have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\boldsymbol{r}}}\right) = \frac{\partial L}{\partial\boldsymbol{r}} \tag{2.161}$$

which gives

$$m\,\ddot{\boldsymbol{r}} + \frac{q}{c}\,\frac{d\boldsymbol{A}}{dt} = -q\,\boldsymbol{\nabla}\phi + \frac{q}{c}\,\boldsymbol{\nabla}(\boldsymbol{A}\cdot\dot{\boldsymbol{r}}) \ , \tag{2.162}$$

or, in component notation,

$$m\,\ddot{x}_i + \frac{q}{c}\,\frac{\partial A_i}{\partial x_j}\,\dot{x}_j + \frac{q}{c}\,\frac{\partial A_i}{\partial t} = -q\,\frac{\partial \phi}{\partial x_i} + \frac{q}{c}\,\frac{\partial A_j}{\partial x_i}\,\dot{x}_j \ , \tag{2.163}$$

which is to say

$$m\,\ddot{x}_i = -q\,\frac{\partial \phi}{\partial x_i} - \frac{q}{c}\,\frac{\partial A_i}{\partial t} + \frac{q}{c}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)\dot{x}_j \ . \tag{2.164}$$

It is convenient to express the cross product in terms of the completely antisymmetric tensor of rank three, ϵ_{ijk}:

$$B_i = \epsilon_{ijk}\,\frac{\partial A_k}{\partial x_j} \ , \tag{2.165}$$

and using the result

$$\epsilon_{ijk}\,\epsilon_{imn} = \delta_{jm}\,\delta_{kn} - \delta_{jn}\,\delta_{km} \ , \tag{2.166}$$

we have $\epsilon_{ijk}\,B_i = \partial_j\,A_k - \partial_k\,A_j$, and

$$m\,\ddot{x}_i = -q\,\frac{\partial \phi}{\partial x_i} - \frac{q}{c}\,\frac{\partial A_i}{\partial t} + \frac{q}{c}\,\epsilon_{ijk}\,\dot{x}_j\,B_k \ , \tag{2.167}$$

or, in vector notation,

$$\begin{aligned} m\,\ddot{\boldsymbol{r}} &= -q\,\boldsymbol{\nabla}\phi - \frac{q}{c}\,\frac{\partial \boldsymbol{A}}{\partial t} + \frac{q}{c}\,\dot{\boldsymbol{r}}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) \\ &= q\,\boldsymbol{E} + \frac{q}{c}\,\dot{\boldsymbol{r}}\times\boldsymbol{B} \ , \end{aligned} \tag{2.168}$$

which is, of course, the Lorentz force law.

2.7 Noether's Theorem: Continuous Symmetry Implies Conserved Charges

Consider a particle moving in two dimensions under the influence of an external potential $U(r)$. The potential is a function only of the magnitude of the vector $\boldsymbol{r}$. The Lagrangian is then

$$L = T - U = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\,\dot{\phi}^2\right) - U(r) \ , \tag{2.169}$$

where we have chosen generalized coordinates (r,ϕ). The momentum conjugate to ϕ is $p_\phi = m\,r^2\,\dot{\phi}$. The generalized force F_ϕ clearly vanishes, since L does not depend on the coordinate ϕ. (One says that L is 'cyclic' in ϕ.) Thus, although $r = r(t)$ and $\phi = \phi(t)$ will in general be time-dependent, the combination $p_\phi = m\,r^2\,\dot{\phi}$ is constant. This is the conserved angular momentum about the $\hat{\boldsymbol{z}}$ axis.

In general, whenever the system exhibits a *continuous symmetry*, there is an associated *conserved charge*. (The terminology 'charge' is from field theory.) Indeed, this is a rigorous result, known as *Noether's Theorem*. Consider a one-parameter family of transformations,

$$q_\sigma \longrightarrow \tilde{q}_\sigma(q, \zeta) \ , \tag{2.170}$$

where ζ is the continuous parameter. Suppose further (without loss of generality) that at $\zeta = 0$ this transformation is the identity, *i.e.* $\tilde{q}_\sigma(q, \zeta) = q_\sigma$. The transformation may be nonlinear in the generalized coordinates. Suppose further that the Lagrangian L s invariant under the replacement $q \to \tilde{q}$. Then we must have

$$\begin{aligned} 0 = \frac{d}{d\zeta}\bigg|_{\zeta=0} L(\tilde{q}, \dot{\tilde{q}}, t) &= \frac{\partial L}{\partial q_\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial \dot{\tilde{q}}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} \\ &= \frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{q}_\sigma}\bigg)\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{d}{dt}\bigg(\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg)_{\zeta=0} \\ &= \frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg)_{\zeta=0} \ . \end{aligned} \tag{2.171}$$

Thus, there is an associated conserved quantity

$$\Lambda = \frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} \ . \tag{2.172}$$

If there are several one-parameter families of transformations which leave L invariant, then to each such family there corresponds a conserved quantity

$$\Lambda_a = \frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta_a}\bigg|_{\zeta=0} \ . \tag{2.173}$$

Suppose that the Lagrangian of a mechanical system is invariant under a uniform translation of all particles in the $\hat{\boldsymbol{n}}$ direction. Then our one-parameter family of transformations is given by

$$\tilde{\boldsymbol{x}}_a = \boldsymbol{x}_a + \zeta\,\hat{\boldsymbol{n}} \ , \tag{2.174}$$

and the associated conserved Noether charge is

$$\Lambda = \sum_a \frac{\partial L}{\partial \dot{\boldsymbol{x}}_a} \cdot \hat{\boldsymbol{n}} = \hat{\boldsymbol{n}} \cdot \boldsymbol{P} \ , \tag{2.175}$$

where $\boldsymbol{P} = \sum_a \boldsymbol{p}_a$ is the *total momentum* of the system.

If the Lagrangian of a mechanical system is invariant under rotations about an axis $\hat{\boldsymbol{n}}$, then

$$\begin{aligned} \tilde{\boldsymbol{x}}_a &= R(\zeta, \hat{\boldsymbol{n}})\,\boldsymbol{x}_a \\ &= \boldsymbol{x}_a + \zeta\,\hat{\boldsymbol{n}} \times \boldsymbol{x}_a + \mathcal{O}(\zeta^2) \ , \end{aligned} \tag{2.176}$$

where we have expanded the rotation matrix $R(\zeta, \hat{\boldsymbol{n}})$ in powers of ζ. The conserved Noether charge associated with this symmetry is

$$\Lambda = \sum_a \frac{\partial L}{\partial \dot{\boldsymbol{x}}_a} \cdot \hat{\boldsymbol{n}} \times \boldsymbol{x}_a = \hat{\boldsymbol{n}} \cdot \sum_a \boldsymbol{x}_a \times \boldsymbol{p}_a = \hat{\boldsymbol{n}} \cdot \boldsymbol{L} \ , \tag{2.177}$$

where $\boldsymbol{L}$ is the *total angular momentum* of the system.

2.7.1 Advanced Discussion

Observant readers might object that demanding invariance of L is too strict. We should instead be demanding invariance of the action S[3]. Suppose S is invariant under

$$t \rightarrow \tilde{t}(q, t, \zeta) \tag{2.178}$$

$$q_\sigma(t) \rightarrow \tilde{q}_\sigma(q, t, \zeta) \ . \tag{2.179}$$

Then invariance of S means

$$S = \int\limits_{t_a}^{t_b} \! dt \, L(q, \dot{q}, t) = \int\limits_{\tilde{t}_a}^{\tilde{t}_b} \! dt \, L(\tilde{q}, \dot{\tilde{q}}, t) \ . \tag{2.180}$$

Note that t is a dummy variable of integration, so it doesn't matter whether we call it t or $\tilde{t}$. The endpoints of the integral, however, do change under the transformation. Now consider an infinitesimal transformation, for which $\delta t = \tilde{t} - t$ and $\delta q = \tilde{q}\big(\tilde{t}\big) - q(t)$ are both small. Invariance of S means

$$S = \int\limits_{t_a}^{t_b} \! dt \, L(q, \dot{q}, t) = \int\limits_{t_a + \delta t_a}^{t_b + \delta t_b} \! dt \, \bigg\{ L(q, \dot{q}, t) + \frac{\partial L}{\partial q_\sigma} \, \bar{\delta} q_\sigma + \frac{\partial L}{\partial \dot{q}_\sigma} \, \bar{\delta} \dot{q}_\sigma + \ldots \bigg\} \ , \tag{2.181}$$

where

$$\begin{aligned} \bar{\delta} q_\sigma(t) &\equiv \tilde{q}_\sigma(t) - q_\sigma(t) \\ &= \tilde{q}_\sigma\big(\tilde{t}\big) - \tilde{q}_\sigma\big(\tilde{t}\big) + \tilde{q}_\sigma(t) - q_\sigma(t) \\ &= \delta q_\sigma - \dot{q}_\sigma \, \delta t + \mathcal{O}(\delta q \, \delta t) \end{aligned} \tag{2.182}$$

Subtracting the top line from the bottom, we obtain

$$\begin{aligned} 0 &= L_b \, \delta t_b - L_a \, \delta t_a + \frac{\partial L}{\partial \dot{q}_\sigma}\bigg|_b \bar{\delta} q_{\sigma,b} - \frac{\partial L}{\partial \dot{q}_\sigma}\bigg|_a \bar{\delta} q_{\sigma,a} + \int\limits_{t_a + \delta t_a}^{t_b + \delta t_b} \! dt \, \bigg\{ \frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{q}_\sigma}\bigg) \bigg\} \, \bar{\delta} q(t) \\ &= \int\limits_{t_a}^{t_b} \! dt \, \frac{d}{dt} \bigg\{ \bigg(L - \frac{\partial L}{\partial \dot{q}_\sigma} \, \dot{q}_\sigma \bigg) \, \delta t + \frac{\partial L}{\partial \dot{q}_\sigma} \, \delta q_\sigma \bigg\} \ . \end{aligned} \tag{2.183}$$

[3] Indeed, we should be demanding that S only change by a function of the endpoint values.

Thus, if $\zeta \equiv \delta\zeta$ is infinitesimal, and

$$\delta t = A(q,t)\,\delta\zeta \tag{2.184}$$

$$\delta q_\sigma = B_\sigma(q,t)\,\delta\zeta\ , \tag{2.185}$$

then the conserved charge is

$$\begin{aligned} \varLambda &= \left(L - \frac{\partial L}{\partial \dot q_\sigma}\,\dot q_\sigma\right) A(q,t) + \frac{\partial L}{\partial \dot q_\sigma}\,B_\sigma(q,t) \\ &= -\,H(q,p,t)\,A(q,t) + p_\sigma\,B_\sigma(q,t)\ . \end{aligned} \tag{2.186}$$

Thus, when $A = 0$, we recover our earlier results, obtained by assuming invariance of L. Note that conservation of H follows from time translation invariance: $t \to t + \zeta$, for which $A = 1$ and $B_\sigma = 0$.

2.8 Field Theory: Systems with Several Independent Variables

Suppose $\phi_a(\boldsymbol{x})$ depends on several independent variables: $\{x^1, x^2, \ldots, x^n\}$. Furthermore, suppose

$$S\big[\{\phi_a(\boldsymbol{x})\big] = \int\limits_\Omega d\boldsymbol{x}\,\mathcal{L}(\phi_a\,\partial_\mu\phi_a, \boldsymbol{x})\ , \tag{2.187}$$

i.e. the *Lagrangian density* $\mathcal{L}$ is a function of the fields ϕ_a and their partial derivatives $\partial\phi_a/\partial x^\mu$. Here $\varOmega$ is a region in R^K. Then the first variation of S is

$$\begin{aligned} \delta S &= \int\limits_\Omega d\boldsymbol{x} \left\{ \frac{\partial\mathcal{L}}{\partial\phi_a}\,\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\frac{\partial\,\delta\phi_a}{\partial x^\mu} \right\} \\ &= \oint\limits_{\partial\Omega} d\varSigma\, n^\mu\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\delta\phi_a - \int\limits_\Omega d\boldsymbol{x} \left\{ \frac{\partial\mathcal{L}}{\partial\phi_a} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right) \right\} \delta\phi_a\ , \end{aligned} \tag{2.188}$$

where $\partial\varOmega$ is the $(n-1)$-dimensional boundary of $\varOmega$, $d\varSigma$ is the differential surface area, and n^μ is the unit normal. If we demand $\partial\mathcal{L}/\partial(\partial_\mu\phi_a)\big|_{\partial\Omega} = 0$ of $\delta\phi_a\big|_{\partial\Omega} = 0$, the surface term vanishes, and we conclude

$$\frac{\delta S}{\delta\phi_a(\boldsymbol{x})} = \frac{\partial\mathcal{L}}{\partial\phi_a} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\ . \tag{2.189}$$

As an example, consider the case of a stretched string of linear mass density μ and tension τ. The action is a functional of the height $y(x,t)$, where the coordinate along the string, x, and time, t, are the two independent variables. The Lagrangian density is

$$\mathcal{L} = \tfrac{1}{2}\mu \left(\frac{\partial y}{\partial t}\right)^2 - \tfrac{1}{2}\tau \left(\frac{\partial y}{\partial x}\right)^2 \ , \tag{2.190}$$

whence the Euler-Lagrange equations are

$$\begin{aligned} 0 = \frac{\delta S}{\delta y(x,t)} &= -\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) \\ &= \tau\, \frac{\partial^2 y}{\partial x^2} - \mu\, \frac{\partial^2 y}{\partial t^2} \ , \end{aligned} \tag{2.191}$$

where $y' = \frac{\partial y}{\partial x}$ and $\dot{y} = \frac{\partial y}{\partial t}$. Thus, $\mu \ddot{y} = \tau y''$, which is the Helmholtz equation. We've assumed boundary conditions where $\delta y(x_a, t) = \delta y(x_b, t) = \delta y(x, t_a) = \delta y(x, t_b) = 0$.

The Lagrangian density for an electromagnetic field with sources is

$$\mathcal{L} = -\tfrac{1}{16\pi}\, F_{\mu\nu}\, F^{\mu\nu} - J_\mu\, A^\mu \ . \tag{2.192}$$

The equations of motion are then

$$\frac{\partial \mathcal{L}}{\partial A^\nu} - \frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}\right) = 0 \quad \Rightarrow \quad \partial_\mu\, F^{\mu\nu} = 4\pi J^\nu \ , \tag{2.193}$$

which are Maxwell's equations.

2.8.1 Conserved Currents in Field Theory

Recall the result of Noether's theorem for mechanical systems:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\, \frac{\partial \tilde{q}_\sigma}{\partial \zeta}\right)_{\zeta=0} = 0 \ , \tag{2.194}$$

where $\tilde{q}_\sigma = \tilde{q}_\sigma(q, \zeta)$ is a one-parameter (ζ) family of transformations of the generalized coordinates which leaves L invariant. We generalize to field theory by replacing

$$q_\sigma(t) \longrightarrow \phi_a(\boldsymbol{x}, t) \ , \tag{2.195}$$

where $\{\phi_a(\boldsymbol{x}, t)\}$ are a set of fields, which are functions of the independent variables $\{x, y, z, t\}$. We will adopt covariant relativistic notation and write for four-vector $x^\mu = (ct, x, y, z)$. The generalization of $dQ/dt = 0$ is

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\, \frac{\partial \tilde{\phi}_a}{\partial \zeta}\right)_{\zeta=0} = 0 \ , \tag{2.196}$$

where there is an implied sum on both μ and a. We can write this as $\partial_\mu J^\mu = 0$, where

$$J^\mu \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)} \frac{\partial \tilde{\phi}_a}{\partial \zeta}\bigg|_{\zeta=0} . \tag{2.197}$$

We call $Q = J^0/c$ the *total charge*. If we assume $\boldsymbol{J} = 0$ at the spatial boundaries of our system, then integrating the conservation law $\partial_\mu J^\mu$ over the spatial region Ω gives

$$\frac{dQ}{dt} = \int\limits_\Omega d^3\!x\, \partial_0 J^0 = -\int\limits_\Omega d^3\!x\, \boldsymbol{\nabla}\cdot\boldsymbol{J} = -\oint\limits_{\partial\Omega} d\Sigma\, \hat{\boldsymbol{n}}\cdot\boldsymbol{J} = 0 \ , \tag{2.198}$$

assuming $\boldsymbol{J} = 0$ at the boundary $\partial\Omega$.

As an example, consider the case of a complex scalar field, with Lagrangian density[4]

$$\mathcal{L}(\psi, , \psi^*, \partial_\mu\psi, \partial_\mu\psi^*) = \tfrac{1}{2} K \left(\partial_\mu \psi^*\right)(\partial^\mu \psi) - U\big(\psi^*\psi\big) \ . \tag{2.199}$$

This is invariant under the transformation $\psi \to e^{i\zeta}\,\psi$, $\psi^* \to e^{-i\zeta}\,\psi^*$. Thus,

$$\frac{\partial\tilde{\psi}}{\partial\zeta} = i\, e^{i\zeta}\,\psi \qquad , \qquad \frac{\partial\tilde{\psi}^*}{\partial\zeta} = -i\, e^{-i\zeta}\,\psi^* \ , \tag{2.200}$$

and, summing over both ψ and ψ^* fields, we have

$$\begin{aligned} J^\mu &= \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)} \cdot (i\psi) + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi^*\right)} \cdot (-i\psi^*) \\ &= \frac{K}{2i}\big(\psi^*\partial^\mu\psi - \psi\,\partial^\mu\psi^*\big) \ . \end{aligned} \tag{2.201}$$

The potential, which depends on $|\psi|^2$, is independent of ζ. Hence, this form of conserved 4-current is valid for an entire class of potentials.

2.8.2 Gross-Pitaevskii Model

As one final example of a field theory, consider the Gross-Pitaevskii model, with

$$\mathcal{L} = i\hbar\,\psi^*\,\frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\psi - g\left(|\psi|^2 - n_0\right)^2 \ . \tag{2.202}$$

[4]We raise and lower indices using the Minkowski metric $g_{\mu\nu} = \mathrm{diag}\,(+,-,-,-)$.

This describes a Bose fluid with repulsive short-ranged interactions. Here $\psi(\boldsymbol{x},t)$ is again a complex scalar field, and ψ^* is its complex conjugate. Using the Leibniz rule, we have

$$
\begin{aligned}
\delta S[\psi^*,\psi] &= S[\psi^*+\delta\psi^*,\psi+\delta\psi] \\
&= \int\!dt\!\int\!d^dx\,\bigg\{ i\hbar\,\psi^*\,\frac{\partial\delta\psi}{\partial t} + i\hbar\,\delta\psi^*\,\frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\delta\psi - \frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\delta\psi^*\cdot\boldsymbol{\nabla}\psi \\
&\qquad\qquad - 2g\left(|\psi|^2-n_0\right)\left(\psi^*\delta\psi+\psi\delta\psi^*\right)\bigg\} \\
&= \int\!dt\!\int\!d^dx\,\bigg\{ \bigg[-i\hbar\,\frac{\partial\psi^*}{\partial t} + \frac{\hbar^2}{2m}\,\nabla^2\psi^* - 2g\left(|\psi|^2-n_0\right)\psi^* \bigg]\,\delta\psi \\
&\qquad\qquad + \bigg[i\hbar\,\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\,\nabla^2\psi - 2g\left(|\psi|^2-n_0\right)\psi \bigg]\,\delta\psi^* \bigg\}\ ,
\end{aligned} \tag{2.203}
$$

where we have integrated by parts where necessary and discarded the boundary terms. Extremizing $S[\psi^*,\psi]$ therefore results in the *nonlinear Schrödinger equation* (NLSE),

$$
i\hbar\,\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\,\nabla^2\psi + 2g\left(|\psi|^2-n_0\right)\psi \tag{2.204}
$$

as well as its complex conjugate,

$$
-i\hbar\,\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\,\nabla^2\psi^* + 2g\left(|\psi|^2-n_0\right)\psi^*\ . \tag{2.205}
$$

Note that these equations are indeed the Euler-Lagrange equations:

$$
\frac{\delta S}{\delta\psi} = \frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\psi}\right) \tag{2.206}
$$

$$
\frac{\delta S}{\delta\psi^*} = \frac{\partial\mathcal{L}}{\partial\psi^*} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\psi^*}\right)\ , \tag{2.207}
$$

with $x^\mu = (t,\boldsymbol{x})$[5] Plugging in

$$
\frac{\partial\mathcal{L}}{\partial\psi} = -2g\left(|\psi|^2-n_0\right)\psi^* \quad , \quad \frac{\partial\mathcal{L}}{\partial\,\partial_t\psi} = i\hbar\,\psi^* \quad , \quad \frac{\partial\mathcal{L}}{\partial\,\boldsymbol{\nabla}\psi} = -\frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi^* \tag{2.208}
$$

and

$$
\frac{\partial\mathcal{L}}{\partial\psi^*} = i\hbar\,\psi - 2g\left(|\psi|^2-n_0\right)\psi \quad , \quad \frac{\partial\mathcal{L}}{\partial\,\partial_t\psi^*} = 0 \quad , \quad \frac{\partial\mathcal{L}}{\partial\,\boldsymbol{\nabla}\psi^*} = -\frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi\ , \tag{2.209}
$$

we recover the NLSE and its conjugate.

The Gross-Pitaevskii model also possesses a U(1) invariance, under

$$
\psi(\boldsymbol{x},t)\to\tilde{\psi}(\boldsymbol{x},t) = e^{i\zeta}\,\psi(\boldsymbol{x},t) \quad , \quad \psi^*(\boldsymbol{x},t)\to\tilde{\psi}^*(\boldsymbol{x},t) = e^{-i\zeta}\,\psi^*(\boldsymbol{x},t)\ . \tag{2.210}
$$

[5] In the nonrelativistic case, there is no utility in defining $x^0 = ct$, so we simply define $x^0 = t$.

Thus, the conserved Noether current is then

$$J^\mu = \frac{\partial \mathcal{L}}{\partial\, \partial_\mu \psi} \, \frac{\partial \tilde{\psi}}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial \mathcal{L}}{\partial\, \partial_\mu \psi^*} \, \frac{\partial \tilde{\psi}^*}{\partial \zeta}\bigg|_{\zeta=0}$$

$$J^0 = -\hbar \, |\psi|^2 \tag{2.211}$$

$$\boldsymbol{J} = -\frac{\hbar^2}{2im} \left(\psi^* \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \psi^* \right) . \tag{2.212}$$

Dividing out by $\hbar$, taking $J^0 \equiv -\hbar\rho$ and $\boldsymbol{J} \equiv -\hbar\boldsymbol{j}$, we obtain the continuity equation,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0 \; , \tag{2.213}$$

where

$$\rho = |\psi|^2 \quad , \quad \boldsymbol{j} = \frac{\hbar}{2im} \left(\psi^* \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \psi^* \right) . \tag{2.214}$$

are the particle density and the particle current, respectively.

2.9 Constraints

A mechanical system of N point particles in d dimensions possesses $n = dN$ degrees of freedom[6]. To specify these degrees of freedom, we can choose any independent set of generalized coordinates $\{q_1, \ldots, q_n\}$. Oftentimes, however, not all n coordinates are independent.

Consider, for example, the situation in Fig. 2.12, where a cylinder of radius a rolls over a half-cylinder of radius R. If there is no slippage, then the angles θ_1 and θ_2 are not independent, and they obey the *equation of constraint*,

$$R\,\theta_1 = a\,(\theta_2 - \theta_1) \; . \tag{2.215}$$

In this case, we can easily solve the constraint equation and substitute $\theta_2 = \big(1 + \frac{R}{a}\big)\,\theta_1$. In other cases, though, the equation of constraint might not be so easily solved (*e.g.* it may be nonlinear). How then do we proceed?

[6]For N rigid bodies, the number of degrees of freedom is $n' = \frac{1}{2}d(d+1)N$, corresponding to d center-of-mass coordinates and $\frac{1}{2}d(d-1)$ angles of orientation for each particle. The dimension of the group of rotations in d dimensions is $\frac{1}{2}d(d-1)$, corresponding to the number of parameters in a general rank-d orthogonal matrix (*i.e.* an element of the group $O(d)$).

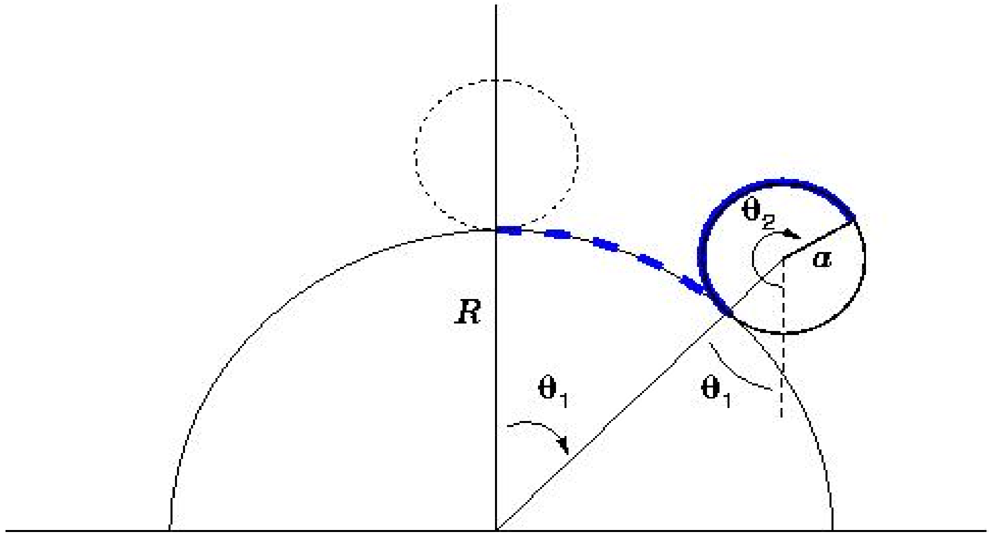

Figure 2.12: A cylinder of radius a rolls along a half-cylinder of radius R. When there is no slippage, the angles θ_1 and θ_2 obey the constraint equation $R\theta_1 = a(\theta_2 - \theta_1)$.

2.9.1 Constraints and Variational Calculus

Before addressing the subject of constrained dynamical systems, let's consider the issue of constraints in the broader context of variational calculus. Suppose we have a functional

$$F[y(x)] = \int\limits_{x_a}^{x_b} dx\, L(y, y', x) \ , \tag{2.216}$$

which we want to extremize subject to some constraints. Here y may stand for a set of functions $\{y_\sigma(x)\}$. There are two classes of constraints we will consider:

1. *Integral constraints:* These are of the form

$$\int\limits_{x_a}^{x_b} dx\, N_l(y, y', x) = C_l \ , \tag{2.217}$$

   where k labels the constraint.

2. *Holonomic constraints:* These are of the form

$$G_k(y, x) = 0 \ . \tag{2.218}$$

The cylinders system in Fig. 2.12 provides an example of a holonomic constraint. There, $G(\theta, t) = R\,\theta_1 - a\,(\theta_2 - \theta_1) = 0$. As an example of a problem with an integral constraint, suppose we want to know the shape of a hanging rope of fixed length C. This means we minimize the rope's potential energy,

$$U[y(x)] = \lambda g \int\limits_{x_a}^{x_b} ds\, y(x) = \lambda g \int\limits_{x_a}^{x_b} dx\, y \sqrt{1 + y'^2} \ , \tag{2.219}$$

where λ is the linear mass density of the rope, subject to the fixed-length constraint

$$C = \int\limits_{x_a}^{x_b} ds = \int\limits_{x_a}^{x_b} dx\, \sqrt{1 + y'^2} \ . \tag{2.220}$$

Note $ds = \sqrt{dx^2 + dy^2}$ is the differential element of arc length along the rope. To solve problems like these, we turn to Lagrange's method of *undetermined multipliers*.

2.9.2 Constrained Extremization of Functions

Given $F(x_1, \ldots, x_n)$ to be extremized subject to k constraints of the form $G_j(x_1, \ldots, x_n) = 0$ where $j = 1, \ldots, k$, construct

$$F^*\big(x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_k\big) \equiv F(x_1, \ldots, x_n) + \sum_{j=1}^{k} \lambda_j \, G_j(x_1, \ldots, x_n) \tag{2.221}$$

which is a function of the $(n+k)$ variables $\big\{x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_k\big\}$. Now freely extremize the extended function F^*:

$$\begin{aligned} dF^* &= \sum_{\sigma=1}^{n} \frac{\partial F^*}{\partial x_\sigma} \, dx_\sigma + \sum_{j=1}^{k} \frac{\partial F^*}{\partial \lambda_j} \, d\lambda_j & (2.222) \\ &= \sum_{\sigma=1}^{n} \left(\frac{\partial F}{\partial x_\sigma} + \sum_{j=1}^{k} \lambda_j \, \frac{\partial G_j}{\partial x_\sigma} \right) dx_\sigma + \sum_{j=1}^{k} G_j \, d\lambda_j = 0 & (2.223) \end{aligned}$$

This results in the $(n+k)$ equations

$$\begin{aligned} \frac{\partial F}{\partial x_\sigma} + \sum_{j=1}^{k} \lambda_j \, \frac{\partial G_j}{\partial x_\sigma} &= 0 \qquad (\sigma = 1, \ldots, n) & (2.224) \\ G_j &= 0 \qquad (j = 1, \ldots, k) \ . & (2.225) \end{aligned}$$

The interpretation of all this is as follows. The n equations in 2.224 can be written in vector form as

$$\boldsymbol{\nabla} F + \sum_{j=1}^{k} \lambda_j \, \boldsymbol{\nabla} G_j = 0 \ . \tag{2.226}$$

This says that the (n-component) vector $\boldsymbol{\nabla} F$ is linearly dependent upon the k vectors $\boldsymbol{\nabla} G_j$. Thus, any movement in the direction of $\boldsymbol{\nabla} F$ must necessarily entail movement along one or more of the directions $\boldsymbol{\nabla} G_j$. This would require violating the constraints, since movement along $\boldsymbol{\nabla} G_j$ takes us off the level set $G_j = 0$. Were $\boldsymbol{\nabla} F$ linearly *independent* of the set $\{\boldsymbol{\nabla} G_j\}$, this would mean that we could find a differential displacement $d\boldsymbol{x}$ which has finite overlap with $\boldsymbol{\nabla} F$ but zero overlap with each $\boldsymbol{\nabla} G_j$. Thus $\boldsymbol{x} + d\boldsymbol{x}$ would still satisfy $G_j(\boldsymbol{x} + d\boldsymbol{x}) = 0$, but F would change by the finite amount $dF = \boldsymbol{\nabla} F(\boldsymbol{x}) \cdot d\boldsymbol{x}$.

2.9.3 Extremization of Functionals : Integral Constraints

Given a functional

$$F\big[\{y_\sigma(x)\}\big] = \int\limits_{x_a}^{x_b} \!dx\, L\big(\{y_\sigma\}, \{y'_\sigma\}, x\big) \qquad (\sigma = 1, \ldots, n) \tag{2.227}$$

subject to boundary conditions $\delta y_\sigma(x_a) = \delta y_\sigma(x_b) = 0$ and k constraints of the form

$$\int\limits_{x_a}^{x_b} \!dx\, N_l\big(\{y_\sigma\}, \{y'_\sigma\}, x\big) = C_l \qquad (l = 1, \ldots, k)\ , \tag{2.228}$$

construct the extended functional

$$F^*\big[\{y_\sigma(x)\}; \{\lambda_j\}\big] \equiv \int\limits_{x_a}^{x_b} \!dx\, \bigg\{ L\big(\{y_\sigma\}, \{y'_\sigma\}, x\big) + \sum_{l=1}^k \lambda_l\, N_l\big(\{y_\sigma\}, \{y'_\sigma\}, x\big) \bigg\} - \sum_{l=1}^k \lambda_l\, C_l \tag{2.229}$$

and freely extremize over $\{y_1, \ldots, y_n; \lambda_1, \ldots, \lambda_k\}$. This results in $(n + k)$ equations

$$\frac{\partial L}{\partial y_\sigma} - \frac{d}{dx}\bigg(\frac{\partial L}{\partial y'_\sigma}\bigg) + \sum_{l=1}^k \lambda_l \left\{ \frac{\partial N_l}{\partial y_\sigma} - \frac{d}{dx}\bigg(\frac{\partial N_l}{\partial y'_\sigma}\bigg)\right\} = 0 \qquad (\sigma = 1, \ldots, n) \tag{2.230}$$

$$\int\limits_{x_a}^{x_b} \!dx\, N_l\big(\{y_\sigma\}, \{y'_\sigma\}, x\big) = C_l \qquad (l = 1, \ldots, k)\ . \tag{2.231}$$

2.9.4 Extremization of Functionals : Holonomic Constraints

Given a functional

$$F\big[\{y_\sigma(x)\}\big] = \int\limits_{x_a}^{x_b} \!dx\, L\big(\{y_\sigma\}, \{y'_\sigma\}, x\big) \qquad (\sigma = 1, \ldots, n) \tag{2.232}$$

subject to boundary conditions $\delta y_\sigma(x_a) = \delta y_\sigma(x_b) = 0$ and k constraints of the form

$$G_j\big(\{y_\sigma(x)\}, x\big) = 0 \qquad (j = 1, \ldots, k) \ , \tag{2.233}$$

construct the extended functional

$$F^*\big[\{y_\sigma(x)\}; \{\lambda_j(x)\}\big] \equiv \int\limits_{x_a}^{x_b} \!dx \left\{ L\big(\{y_\sigma\}, \{y'_\sigma\}, x\big) + \sum_{j=1}^{k} \lambda_j \, G_j\big(\{y_\sigma\}\big) \right\} \tag{2.234}$$

and freely extremize over $\big\{y_1, \ldots, y_n; \lambda_1, \ldots, \lambda_k\big\}$:

$$\delta F^* = \int\limits_{x_a}^{x_b} \!dx \left\{ \sum_{\sigma=1}^{n} \left(\frac{\partial L}{\partial y_\sigma} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'_\sigma}\right) + \sum_{j=1}^{k} \lambda_j \, \frac{\partial G_j}{\partial y_\sigma} \right) \delta y_\sigma + \sum_{j=1}^{k} G_j \, \delta\lambda_j \right\} = 0 \ , \tag{2.235}$$

resulting in the $(n+k)$ equations

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'_\sigma}\right) - \frac{\partial L}{\partial y_\sigma} = \sum_{j=1}^{k} \lambda_j \, \frac{\partial G_j}{\partial y_\sigma} \qquad (\sigma = 1, \ldots, n) \tag{2.236}$$

$$G_j\big(\{y_\sigma\}, x\big) = 0 \qquad (j = 1, \ldots, k) \ . \tag{2.237}$$

2.9.5 Examples of Extremization with Constraints

Volume of a cylinder : As a warm-up problem, let's maximize the volume $V = \pi a^2 h$ of a cylinder of radius a and height h, subject to the constraint

$$G(a, h) = 2\pi a + \frac{h^2}{b} - \ell = 0 \ . \tag{2.238}$$

We therefore define

$$V^*(a, h, \lambda) \equiv V(a, h) + \lambda \, G(a, h) \ , \tag{2.239}$$

and set

$$\frac{\partial V^*}{\partial a} = 2\pi a h + 2\pi\lambda = 0 \tag{2.240}$$

$$\frac{\partial V^*}{\partial h} = \pi a^2 + 2\lambda \, \frac{h}{b} = 0 \tag{2.241}$$

$$\frac{\partial V^*}{\partial \lambda} = 2\pi a + \frac{h^2}{b} - \ell = 0 \ . \tag{2.242}$$

Solving these three equations simultaneously gives

$$a = \frac{2\ell}{5\pi} \quad , \quad h = \sqrt{\frac{b\ell}{5}} \quad , \quad \lambda = \frac{2\pi}{5^{3/2}} \, b^{1/2} \, \ell^{3/2} \quad , \quad V = \frac{4}{5^{5/2}\, \pi} \, \ell^{5/2} \, b^{1/2} \ . \tag{2.243}$$

Hanging rope : We minimize the energy functional

$$E\big[y(x)\big] = \mu g \int\limits_{x_1}^{x_2} dx \, y\sqrt{1+y'^2} \ , \tag{2.244}$$

where μ is the linear mass density, subject to the constraint of fixed total length,

$$C\big[y(x)\big] = \int\limits_{x_1}^{x_2} dx \, \sqrt{1+y'^2} \ . \tag{2.245}$$

Thus,

$$E^*\big[y(x), \lambda\big] = E\big[y(x)\big] + \lambda C\big[y(x)\big] = \int\limits_{x_1}^{x_2} dx \, L^*(y, y', x) \ , \tag{2.246}$$

with

$$L^*(y, y', x) = (\mu g y + \lambda)\, \sqrt{1+y'^2} \ . \tag{2.247}$$

Since $\frac{\partial L^*}{\partial x} = 0$ we have that

$$\mathcal{J} = y' \, \frac{\partial L^*}{\partial y'} - L^* = -\frac{\mu g y + \lambda}{\sqrt{1+y'^2}} \tag{2.248}$$

is constant. Thus,

$$\frac{dy}{dx} = \pm \mathcal{J}^{-1} \sqrt{(\mu g y + \lambda)^2 - \mathcal{J}^2} \ , \tag{2.249}$$

with solution

$$y(x) = -\frac{\lambda}{\mu g} + \frac{\mathcal{J}}{\mu g} \, \cosh\Big(\frac{\mu g}{\mathcal{J}}\,(x-a)\Big) \ . \tag{2.250}$$

Here, $\mathcal{J}$, a, and λ are constants to be determined by demanding $y(x_i) = y_i$ $(i = 1, 2)$, and that the total length of the rope is C.

Geodesic on a curved surface : Consider next the problem of a geodesic on a curved surface. Let the equation for the surface be

$$G(x, y, z) = 0 \ . \tag{2.251}$$

We wish to extremize the distance,

$$D = \int\limits_a^b ds = \int\limits_a^b \sqrt{dx^2 + dy^2 + dz^2} \ . \tag{2.252}$$

We introduce a parameter t defined on the unit interval: $t \in [0, 1]$, such that $x(0) = x_a$, $x(1) = x_b$, *etc.* Then D may be regarded as a functional, *viz.*

$$D\big[x(t), y(t), z(t)\big] = \int\limits_0^1 dt \, \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \ . \tag{2.253}$$

We impose the constraint by forming the extended functional, D^*:

$$D^*\big[x(t), y(t), z(t), \lambda(t)\big] \equiv \int\limits_0^1 dt \left\{ \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} + \lambda\, G(x, y, z) \right\} , \tag{2.254}$$

and we demand that the first functional derivatives of D^* vanish:

$$\frac{\delta D^*}{\delta x(t)} = -\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\right) + \lambda\, \frac{\partial G}{\partial x} = 0 \tag{2.255}$$

$$\frac{\delta D^*}{\delta y(t)} = -\frac{d}{dt}\left(\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\right) + \lambda\, \frac{\partial G}{\partial y} = 0 \tag{2.256}$$

$$\frac{\delta D^*}{\delta z(t)} = -\frac{d}{dt}\left(\frac{\dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\right) + \lambda\, \frac{\partial G}{\partial z} = 0 \tag{2.257}$$

$$\frac{\delta D^*}{\delta \lambda(t)} = G(x, y, z) = 0\ . \tag{2.258}$$

Thus,

$$\lambda(t) = \frac{v\ddot{x} - \dot{x}\dot{v}}{v^2\, \partial_x G} = \frac{v\ddot{y} - \dot{y}\dot{v}}{v^2\, \partial_y G} = \frac{v\ddot{z} - \dot{z}\dot{v}}{v^2\, \partial_z G}\ , \tag{2.259}$$

with $v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$ and $\partial_x \equiv \frac{\partial}{\partial x}$, *etc.* These three equations are supplemented by $G(x, y, z) = 0$, which is the fourth.

2.9.6 Application to Mechanics

Let us write our system of constraints in the differential form

$$\sum_{\sigma=1}^{n} g_{j\sigma}(q, t)\, dq_\sigma + h_j(q, t) dt = 0 \qquad (j = 1, \ldots, k)\ . \tag{2.260}$$

If the partial derivatives satisfy

$$\frac{\partial g_{j\sigma}}{\partial q_{\sigma'}} = \frac{\partial g_{j\sigma'}}{\partial q_\sigma} \quad , \quad \frac{\partial g_{j\sigma}}{\partial t} = \frac{\partial h_j}{\partial q_\sigma}\ , \tag{2.261}$$

then the differential can be integrated to give $dG(q, t) = 0$, where

$$g_{j\sigma} = \frac{\partial G_j}{\partial q_\sigma} \quad , \quad h_j = \frac{\partial G_j}{\partial t}\ . \tag{2.262}$$

The action functional is

$$S[\{q_\sigma(t)\}] = \int\limits_{t_a}^{t_b} dt\, L\big(\{q_\sigma\}, \{\dot{q}_\sigma\}, t\big) \qquad (\sigma = 1, \ldots, n)\ , \tag{2.263}$$

subject to boundary conditions $\delta q_\sigma(t_a) = \delta q_\sigma(t_b) = 0$. The first variation of S is given by

$$\delta S = \int\limits_{t_a}^{t_b} dt \sum_{\sigma=1}^{n} \left\{ \frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) \right\} \delta q_\sigma \ . \tag{2.264}$$

Since the $\{q_\sigma(t)\}$ are no longer independent, we cannot infer that the term in brackets vanishes for each σ. What are the constraints on the variations $\delta q_\sigma(t)$? The constraints are expressed in terms of *virtual displacements* which take no time: $\delta t = 0$. Thus,

$$\sum_{\sigma=1}^{n} g_{j\sigma}(q,t)\, \delta q_\sigma(t) = 0 \ . \tag{2.265}$$

We may now relax the constraint by introducing k undetermined functions $\lambda_j(t)$, by adding integrals of the above equations with undetermined coefficient functions to δS:

$$\sum_{\sigma=1}^{n} \left\{ \frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) + \sum_{j=1}^{k} \lambda_j(t)\, g_{j\sigma}(q,t) \right\} \delta q_\sigma(t) = 0 \ . \tag{2.266}$$

Now we can demand that the term in brackets vanish for all σ. Thus, we obtain a set of $(n+k)$ equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{k} \lambda_j(t)\, g_{j\sigma}(q,t) \equiv Q_\sigma \tag{2.267}$$

$$g_{j\sigma}(q,t)\, \dot{q}_\sigma + h_j(q,t) = 0 \ , \tag{2.268}$$

in $(n+k)$ unknowns $\big\{q_1, \ldots, q_n, \lambda_1, \ldots, \lambda_k\big\}$. Here, Q_σ is the *force of constraint conjugate to the generalized coordinate* q_σ. Thus, with

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} \quad , \quad F_\sigma = \frac{\partial L}{\partial q_\sigma} \quad , \quad Q_\sigma = \sum_{j=1}^{k} \lambda_j\, g_{j\sigma} \ , \tag{2.269}$$

we write Newton's second law as

$$\dot{p}_\sigma = F_\sigma + Q_\sigma \ . \tag{2.270}$$

Note that we can write

$$\frac{\delta S}{\delta \boldsymbol{q}(t)} = \frac{\partial L}{\partial \boldsymbol{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) \tag{2.271}$$

and that the *instantaneous* constraints may be written

$$\boldsymbol{g}_j \cdot \delta \boldsymbol{q} = 0 \qquad (j = 1, \ldots, k) \ . \tag{2.272}$$

Thus, by demanding

$$\frac{\delta S}{\delta \boldsymbol{q}(t)} + \sum_{j=1}^{k} \lambda_j\, \boldsymbol{g}_j = 0 \tag{2.273}$$

we require that the functional derivative be linearly dependent on the k vectors $\boldsymbol{g}_j$.

2.9.7 One cylinder rolling atop another

As an example of the constraint formalism, consider the system in Fig. 2.12, where a cylinder of radius a rolls atop a cylinder of radius R. We have two constraints:

$$r = R + a \qquad \text{(cylinders in contact)} \tag{2.274}$$

$$R\,\theta_1 = a\,(\theta_2 - \theta_1) \qquad \text{(no slipping)}\ , \tag{2.275}$$

from which we obtain the $g_{j\sigma}$:

$$g_{j\sigma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R+a & -a \end{pmatrix}\ , \tag{2.276}$$

which is to say

$$\frac{\partial G_1}{\partial r} = 1 \qquad \frac{\partial G_1}{\partial \theta_1} = 0 \qquad \frac{\partial G_1}{\partial \theta_2} = 0 \tag{2.277}$$

$$\frac{\partial G_2}{\partial r} = 0 \qquad \frac{\partial G_2}{\partial \theta_1} = R + a \qquad \frac{\partial G_2}{\partial \theta_2} = -a\ . \tag{2.278}$$

The Lagrangian is

$$L = T - U = \tfrac{1}{2} M \big(\dot r^2 + r^2\,\dot\theta_1^2\big) + \tfrac{1}{2} I\,\dot\theta_2^2 - Mgr\,\cos\theta_1\ , \tag{2.279}$$

where M and I are the mass and rotational inertia of the rolling cylinder, respectively. Note that the kinetic energy is a sum of center-of-mass translation $T_{\rm tr} = \frac{1}{2}M\big(\dot r^2 + r^2\,\dot\theta_1^2\big)$ and rotation about the center-of-mass, $T_{\rm rot} = \frac{1}{2} I\,\dot\theta_2^2$. The equations of motion are

$$\frac{d}{dt}\bigg(\frac{\partial L}{\partial r}\bigg) - \frac{\partial L}{\partial r} = M\ddot r - Mr\,\dot\theta_1^2 + Mg\cos\theta_1 = \lambda_1 \equiv Q_r \tag{2.280}$$

$$\frac{d}{dt}\bigg(\frac{\partial L}{\partial \theta_1}\bigg) - \frac{\partial L}{\partial \theta_1} = Mr^2\ddot\theta_1 + 2Mr\dot r\,\dot\theta_1 - Mgr\sin\theta_1 = (R+a)\,\lambda_2 \equiv Q_{\theta_1} \tag{2.281}$$

$$\frac{d}{dt}\bigg(\frac{\partial L}{\partial \theta_2}\bigg) - \frac{\partial L}{\partial \theta_2} = I\ddot\theta_2 = -a\,\lambda_2 \equiv Q_{\theta_2}\ . \tag{2.282}$$

To these three equations we add the two constraints, resulting in five equations in the five unknowns $\big\{r, \theta_1, \theta_2, \lambda_1, \lambda_2\big\}$.

We solve by first implementing the constraints, which give $r = (R+a)$ a constant (*i.e.* $\dot r = 0$), and $\dot\theta_2 = \big(1 + \frac{R}{a}\big)\,\dot\theta_1$. Substituting these into the above equations gives

$$-M(R+a)\,\dot\theta_1^2 + Mg\cos\theta_1 = \lambda_1 \tag{2.283}$$

$$M(R+a)^2\ddot\theta_1 - Mg(R+a)\sin\theta_1 = (R+a)\,\lambda_2 \tag{2.284}$$

$$I\bigg(\frac{R+a}{a}\bigg)\ddot\theta_1 = -a\lambda_2\ . \tag{2.285}$$

From eqn. 2.285 we obtain

$$\lambda_2 = -\frac{I}{a}\,\ddot\theta_2 = -\frac{R+a}{a^2}\,I\,\ddot\theta_1\ , \tag{2.286}$$

which we substitute into eqn. 2.284 to obtain

$$\bigg(M + \frac{I}{a^2}\bigg)(R+a)^2\ddot\theta_1 - Mg(R+a)\sin\theta_1 = 0\ . \tag{2.287}$$

Multiplying by $\dot\theta_1$, we obtain an exact differential, which may be integrated to yield

$$\tfrac{1}{2}M\bigg(1 + \frac{I}{Ma^2}\bigg)\dot\theta_1^2 + \frac{Mg}{R+a}\,\cos\theta_1 = \frac{Mg}{R+a}\cos\theta_1^\circ\ . \tag{2.288}$$

Here, we have assumed that $\dot\theta_1 = 0$ when $\theta_1 = \theta_1^\circ$, *i.e.* the rolling cylinder is released from rest at $\theta_1 = \theta_1^\circ$. Finally, inserting this result into eqn. 2.283, we obtain the radial force of constraint,

$$Q_r = \frac{Mg}{1+\alpha}\Big\{(3+\alpha)\,\cos\theta_1 - 2\,\cos\theta_1^\circ\Big\}\ , \tag{2.289}$$

where $\alpha = I/Ma^2$ is a dimensionless parameter $(0 \le \alpha \le 1)$. This is the radial component of the normal force between the two cylinders. When Q_r vanishes, the cylinders lose contact – the rolling cylinder flies off. Clearly this occurs at an angle $\theta_1 = \theta_1^*$, where

$$\theta_1^* = \cos^{-1}\bigg(\frac{2\cos\theta_1^\circ}{3+\alpha}\bigg)\ . \tag{2.290}$$

The detachment angle θ_1^* is an increasing function of α, which means that larger I delays detachment. This makes good sense, since when I is larger the gain in kinetic energy is split between translational and rotational motion of the rolling cylinder.

2.9.8 Frictionless Motion along a Curve

Consider the situation in Fig. 2.13 where a skier moves frictionlessly under the influence of gravity along a general curve $y = h(x)$. The Lagrangian for this problem is

$$L = \tfrac{1}{2}m(\dot x^2 + \dot y^2) - mgy \tag{2.291}$$

and the (holonomic) constraint is

$$G(x,y) = y - h(x) = 0\ . \tag{2.292}$$

Accordingly, the Euler-Lagrange equations are

$$\frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot q_\sigma}\bigg) - \frac{\partial L}{\partial q_\sigma} = \lambda\,\frac{\partial G}{\partial q_\sigma}\ , \tag{2.293}$$

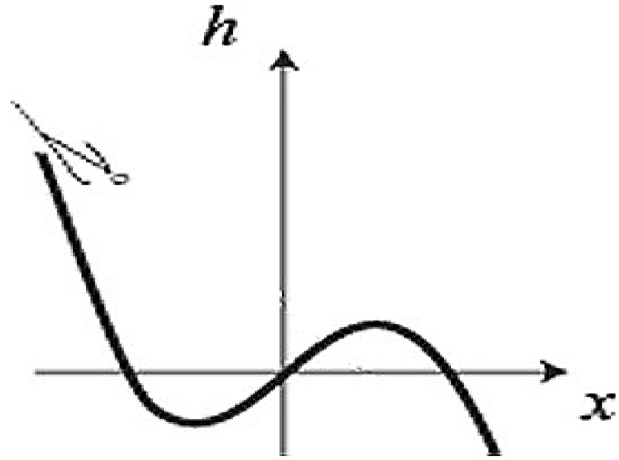

Figure 2.13: Frictionless motion under gravity along a curved surface. The skier flies off the surface when the normal force vanishes.

where $q_1 = x$ and $q_2 = y$. Thus, we obtain

$$m\ddot{x} = -\lambda\, h'(x) = Q_x \tag{2.294}$$

$$m\ddot{y} + mg = \lambda = Q_y \ . \tag{2.295}$$

We eliminate y in favor of x by invoking the constraint. Since we need $\ddot{y}$, we must differentiate the constraint, which gives

$$\dot{y} = h'(x)\,\dot{x} \qquad , \qquad \ddot{y} = h'(x)\,\ddot{x} + h''(x)\,\dot{x}^2 \ . \tag{2.296}$$

Using the second Euler-Lagrange equation, we then obtain

$$\frac{\lambda}{m} = g + h'(x)\,\ddot{x} + h''(x)\,\dot{x}^2 \ . \tag{2.297}$$

Finally, we substitute this into the first E-L equation to obtain an equation for x alone:

$$\Big(1 + \big[h'(x)\big]^2\Big)\,\ddot{x} + h'(x)\,h''(x)\,\dot{x}^2 + g\,h'(x) = 0 \ . \tag{2.298}$$

Had we started by eliminating $y = h(x)$ at the outset, writing

$$L(x, \dot{x}) = \tfrac{1}{2}m\Big(1 + \big[h'(x)\big]^2\Big)\,\dot{x}^2 - mg\,h(x) \ , \tag{2.299}$$

we would also have obtained this equation of motion.

The skier flies off the curve when the vertical force of constraint $Q_y = \lambda$ starts to become negative, because the curve can only supply a positive normal force. Suppose the skier starts from rest at a height y_0. We may then determine the point x at which the skier detaches from the curve by setting $\lambda(x) = 0$. To do so, we must eliminate $\dot{x}$ and $\ddot{x}$ in terms of x. For $\ddot{x}$, we may use the equation of motion to write

$$\ddot{x} = -\bigg(\frac{gh' + h'\,h''\,\dot{x}^2}{1 + h'^2}\bigg) \ , \tag{2.300}$$

which allows us to write

$$\lambda = m\bigg(\frac{g + h''\,\dot x^2}{1 + h'^2}\bigg) \ . \tag{2.301}$$

To eliminate $\dot x$, we use conservation of energy,

$$E = mgy_0 = \tfrac{1}{2}m\big(1 + h'^2\big)\,\dot x^2 + mgh \ , \tag{2.302}$$

which fixes

$$\dot x^2 = 2g\bigg(\frac{y_0 - h}{1 + h'^2}\bigg) \ . \tag{2.303}$$

Putting it all together, we have

$$\lambda(x) = \frac{mg}{\big(1 + h'^2\big)}\Big\{1 + h'^2 + 2(y_0 - h)\,h''\Big\} \ . \tag{2.304}$$

The skier detaches from the curve when $\lambda(x) = 0$, *i.e.* when

$$1 + h'^2 + 2(y_0 - h)\,h'' = 0 \ . \tag{2.305}$$

There is a somewhat easier way of arriving at the same answer. This is to note that the skier must fly off when the local centripetal force equals the gravitational force normal to the curve, *i.e.*

$$\frac{m\,v^2(x)}{R(x)} = mg\,\cos\theta(x) \ , \tag{2.306}$$

where $R(x)$ is the local radius of curvature. Now $\tan\theta = h'$, so $\cos\theta = \big(1 + h'^2\big)^{-1/2}$. The square of the velocity is $v^2 = \dot x^2 + \dot y^2 = \big(1 + h'^2\big)\,\dot x^2$. What is the local radius of curvature $R(x)$? This can be determined from the following argument, and from the sketch in Fig. 2.14. Writing $x = x^* + \epsilon$, we have

$$y = h(x^*) + h'(x^*)\,\epsilon + \tfrac{1}{2}h''(x^*)\,\epsilon^2 + \ldots \ . \tag{2.307}$$

We now drop a perpendicular segment of length z from the point (x, y) to the line which is tangent to the curve at $\big(x^*, h(x^*)\big)$. According to Fig. 2.14, this means

$$\begin{pmatrix} \epsilon \\ y \end{pmatrix} = \eta \cdot \tfrac{1}{\sqrt{1+h'^2}}\begin{pmatrix} 1 \\ h' \end{pmatrix} - z \cdot \tfrac{1}{\sqrt{1+h'^2}}\begin{pmatrix} -h' \\ 1 \end{pmatrix} \ . \tag{2.308}$$

Thus, we have

$$\begin{aligned} y &= h'\,\epsilon + \tfrac{1}{2}h''\,\epsilon^2 \\ &= h'\left(\frac{\eta + z\,h'}{\sqrt{1 + h'^2}}\right) + \tfrac{1}{2}h''\bigg(\frac{\eta + z\,h'}{\sqrt{1 + h'^2}}\bigg)^2 \\ &= \frac{\eta\,h' + z\,h'^2}{\sqrt{1 + h'^2}} + \frac{h''\,\eta^2}{2\big(1 + h'^2\big)} + \mathcal{O}(\eta z) \\ &= \frac{\eta\,h' - z}{\sqrt{1 + h'^2}} \ , \end{aligned} \tag{2.309}$$

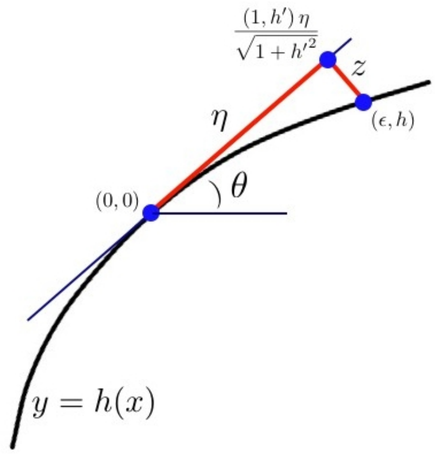

Figure 2.14: Finding the local radius of curvature: $z = \eta^2/2R$.

from which we obtain

$$z = -\frac{h''\,\eta^2}{2\big(1+h'^2\big)^{3/2}} + \mathcal{O}(\eta^3) \tag{2.310}$$

and therefore

$$R(x) = -\frac{1}{h''(x)} \cdot \Big(1 + \big[h'(x)\big]^2\Big)^{3/2} \ . \tag{2.311}$$

Thus, the detachment condition,

$$\frac{mv^2}{R} = -\frac{m\,h''\,\dot{x}^2}{\sqrt{1+h'^2}} = \frac{mg}{\sqrt{1+h'^2}} = mg\,\cos\theta \tag{2.312}$$

reproduces the result from eqn. 2.301.

2.9.9 Constraints and conservation laws

We have seen how invariance of the Lagrangian with respect to a one-parameter family of coordinate transformations results in an associated conserved quantity $\varLambda$, and how a lack of explicit time dependence in L results in the conservation of the Hamiltonian H. In deriving both these results, however, we used the equations of motion $\dot{p}_\sigma = F_\sigma$. What happens when we have constraints, in which case $\dot{p}_\sigma = F_\sigma + Q_\sigma$?

Let's begin with the Hamiltonian. We have $H = \dot{q}_\sigma\, p_\sigma - L$, hence

$$\begin{aligned}\frac{dH}{dt} &= \bigg(p_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma}\bigg)\ddot{q}_\sigma + \bigg(\dot{p}_\sigma - \frac{\partial L}{\partial q_\sigma}\bigg)\dot{q}_\sigma - \frac{\partial L}{\partial t} \\ &= Q_\sigma\,\dot{q}_\sigma - \frac{\partial L}{\partial t} \ . \end{aligned} \tag{2.313}$$

We now use

$$Q_\sigma \, \dot q_\sigma = \lambda_j \, g_{j\sigma} \, \dot q_\sigma = -\lambda_j \, h_j \tag{2.314}$$

to obtain

$$\frac{dH}{dt} = -\lambda_j \, h_j - \frac{\partial L}{\partial t} \ . \tag{2.315}$$

We therefore conclude that *in a system with constraints of the form* $g_{j\sigma} \, \dot q_\sigma + h_j = 0$, *the Hamiltonian is conserved if each* $h_j = 0$ *and if* L *is not explicitly dependent on time.* In the case of holonomic constraints, $h_j = \frac{\partial G_j}{\partial t}$, so H is conserved if neither L nor any of the constraints G_j is explicitly time-dependent.

Next, let us rederive Noether's theorem when constraints are present. We assume a one-parameter family of transformations $q_\sigma \to \tilde q_\sigma(\zeta)$ leaves L invariant. Then

$$\begin{aligned} 0 = \frac{dL}{d\zeta} &= \frac{\partial L}{\partial \tilde q_\sigma} \, \frac{\partial \tilde q_\sigma}{\partial \zeta} + \frac{\partial L}{\partial \dot{\tilde q}_\sigma} \frac{\partial \dot{\tilde q}_\sigma}{\partial \zeta} \\ &= \big(\dot{\tilde p}_\sigma - \tilde Q_\sigma\big) \, \frac{\partial \tilde q_\sigma}{\partial \zeta} + \tilde p_\sigma \, \frac{d}{dt}\bigg(\frac{\partial \tilde q_\sigma}{\partial \zeta}\bigg) \\ &= \frac{d}{dt}\bigg(\tilde p_\sigma \, \frac{\partial \tilde q_\sigma}{\partial \zeta}\bigg) - \lambda_j \, \tilde g_{j\sigma} \, \frac{\partial \tilde q_\sigma}{\partial \zeta} \ . \end{aligned} \tag{2.316}$$

Now let us write the constraints in differential form as

$$\tilde g_{j\sigma} \, d\tilde q_\sigma + \tilde h_j \, dt + \tilde k_j \, d\zeta = 0 \ . \tag{2.317}$$

We now have

$$\varLambda \equiv \tilde p_\sigma \, \frac{\partial \tilde q_\sigma}{\partial \zeta} \qquad \Rightarrow \qquad \frac{d\varLambda}{dt} = \lambda_j \, \tilde k_j \ , \tag{2.318}$$

which says that *if the constraints are independent of* ζ *then* $\varLambda$ *is conserved.* For holonomic constraints, this means that

$$G_j\big(\tilde q(\zeta), t\big) = 0 \quad \Rightarrow \quad \tilde k_j = \frac{\partial G_j}{\partial \zeta} = 0 \ , \tag{2.319}$$

i.e. $G_j(\tilde q, t)$ has no explicit ζ dependence. Again, there is a conserved Noether charge associated with each continuous one-parameter family which leaves L and the constraints invariant:

$$\varLambda_a = \frac{\partial L}{\partial \dot{\tilde q}_\sigma} \, \frac{\partial \tilde q_\sigma}{\partial \zeta_a}\bigg|_{\zeta=0} \ . \tag{2.320}$$

2.10 Central Forces and Orbital Mechanics

Consider two particles interacting via a potential $U(\boldsymbol{r}_1, \boldsymbol{r}_2) = U\big(|\boldsymbol{r}_1 - \boldsymbol{r}_2|\big)$. Such a potential, which depends only on the relative distance between the particles, is called a *central* potential. The Lagrangian of this system is then

$$L = T - U = \tfrac{1}{2} m_1 \dot{\boldsymbol{r}}_1^2 + \tfrac{1}{2} m_2 \dot{\boldsymbol{r}}_2^2 - U\big(|\boldsymbol{r}_1 - \boldsymbol{r}_2|\big) \ . \tag{2.321}$$

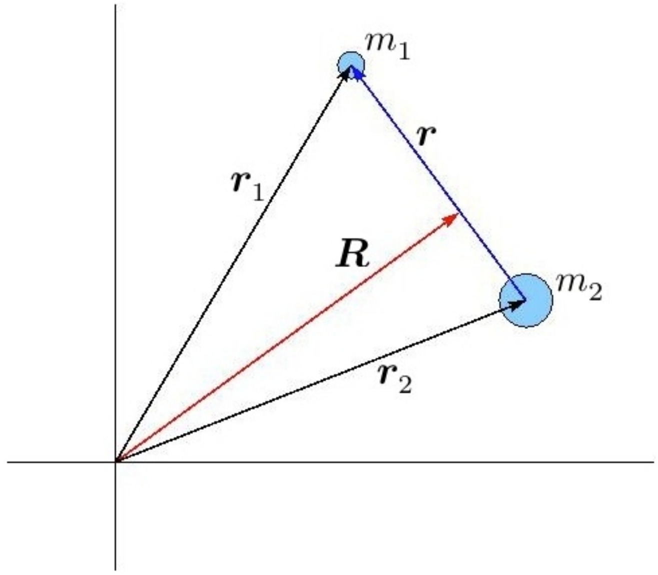

Figure 2.15: Center-of-mass ($\boldsymbol{R}$) and relative ($\boldsymbol{r}$) coordinates.

2.10.1 Center-of-Mass (CM) and Relative Coordinates

The two-body central force problem may always be reduced to two independent one-body problems, by transforming to center-of-mass ($\boldsymbol{R}$) and relative ($\boldsymbol{r}$) coordinates (see Fig. 2.15), *viz.*

$$\boldsymbol{R} = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{m_1 + m_2} \qquad\qquad \boldsymbol{r}_1 = \boldsymbol{R} + \frac{m_2}{m_1 + m_2}\, \boldsymbol{r} \tag{2.322}$$

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 \qquad\qquad \boldsymbol{r}_2 = \boldsymbol{R} - \frac{m_1}{m_1 + m_2}\, \boldsymbol{r} \tag{2.323}$$

We then have

$$L = \tfrac{1}{2} m_1 \dot{\boldsymbol{r}}_1^{\,2} + \tfrac{1}{2} m_2 \dot{\boldsymbol{r}}_2^{\,2} - U\big(|\boldsymbol{r}_1 - \boldsymbol{r}_2|\big) \tag{2.324}$$

$$= \tfrac{1}{2} M \dot{\boldsymbol{R}}^2 + \tfrac{1}{2} \mu \dot{\boldsymbol{r}}^2 - U(r) \ . \tag{2.325}$$

where

$$M = m_1 + m_2 \qquad \text{(total mass)} \tag{2.326}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \qquad \text{(reduced mass)} \ . \tag{2.327}$$

2.10.2 Solution to the CM problem

We have $\partial L/\partial \boldsymbol{R} = 0$, which gives $\ddot{\boldsymbol{R}} = 0$ and hence

$$\boldsymbol{R}(t) = \boldsymbol{R}(0) + \dot{\boldsymbol{R}}(0)\, t \ . \tag{2.328}$$

Thus, the CM problem is trivial. The center-of-mass moves at constant velocity.

2.10.3 Solution to the Relative Coordinate Problem

`Angular momentum conservation:` We have that $\boldsymbol{\ell} = \boldsymbol{r} \times \boldsymbol{p} = \mu \boldsymbol{r} \times \dot{\boldsymbol{r}}$ is a constant of the motion. This means that the motion $\boldsymbol{r}(t)$ is confined to a plane perpendicular to $\boldsymbol{\ell}$. It is convenient to adopt two-dimensional polar coordinates (r, ϕ). The magnitude of $\boldsymbol{\ell}$ is

$$\ell = \mu r^2 \dot{\phi} = 2\mu \dot{\mathcal{A}} \tag{2.329}$$

where $d\mathcal{A} = \frac{1}{2} r^2 d\phi$ is the differential element of area subtended relative to the force center. *The relative coordinate vector for a central force problem subtends equal areas in equal times.* This is known as *Kepler's Second Law.*

`Energy conservation:` The equation of motion for the relative coordinate is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{r}}}\right) = \frac{\partial L}{\partial \boldsymbol{r}} \quad \Rightarrow \quad \mu \ddot{\boldsymbol{r}} = -\frac{\partial U}{\partial \boldsymbol{r}} \ . \tag{2.330}$$

Taking the dot product with $\dot{\boldsymbol{r}}$, we have

$$\begin{aligned} 0 &= \mu \ddot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} + \frac{\partial U}{\partial \boldsymbol{r}} \cdot \dot{\boldsymbol{r}} \\ &= \frac{d}{dt}\Big\{ \tfrac{1}{2} \mu \dot{\boldsymbol{r}}^2 + U(r) \Big\} = \frac{dE}{dt} \ . \end{aligned} \tag{2.331}$$

Thus, the relative coordinate contribution to the total energy is itself conserved. The total energy is of course $E_{\text{tot}} = E + \frac{1}{2} M \dot{\boldsymbol{R}}^2$.

Since $\boldsymbol{\ell}$ is conserved, and since $\boldsymbol{r} \cdot \boldsymbol{\ell} = 0$, all motion is confined to a plane perpendicular to $\boldsymbol{\ell}$. Choosing coordinates such that $\hat{\boldsymbol{z}} = \hat{\boldsymbol{\ell}}$, we have

$$\begin{aligned} E &= \tfrac{1}{2} \mu \dot{\boldsymbol{r}}^2 + U(r) = \tfrac{1}{2} \mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) \\ &= \tfrac{1}{2} \mu \dot{r}^2 + U_{\text{eff}}(r) \end{aligned} \tag{2.332}$$

$$U_{\text{eff}}(r) = \frac{\ell^2}{2\mu r^2} + U(r) \ . \tag{2.333}$$

`Integration of the Equations of Motion, Step I:` The second order equation for $r(t)$ is

$$\frac{dE}{dt} = 0 \quad \Rightarrow \quad \mu \ddot{r} = \frac{\ell^2}{\mu r^3} - \frac{dU(r)}{dr} = -\frac{dU_{\text{eff}}(r)}{dr} \ . \tag{2.334}$$

However, conservation of energy reduces this to a first order equation, via

$$\dot{r} = \pm\sqrt{\frac{2}{\mu}\Big(E - U_{\text{eff}}(r)\Big)} \quad\Rightarrow\quad dt = \pm\frac{\sqrt{\frac{\mu}{2}}\,dr}{\sqrt{E - \frac{\ell^2}{2\mu r^2} - U(r)}} \ . \tag{2.335}$$

This gives $t(r)$, which must be inverted to obtain $r(t)$. In principle this is possible. Note that a constant of integration also appears at this stage – call it $r_0 = r(t=0)$.

`Integration of the Equations of Motion, Step II:` After finding $r(t)$ one can integrate to find $\phi(t)$ using the conservation of ℓ:

$$\dot{\phi} = \frac{\ell}{\mu r^2} \quad\Rightarrow\quad d\phi = \frac{\ell}{\mu r^2(t)}\,dt \ . \tag{2.336}$$

This gives $\phi(t)$, and introduces another constant of integration – call it $\phi_0 = \phi(t=0)$.

`Pause to Reflect on the Number of Constants:` Confined to the plane perpendicular to $\boldsymbol{\ell}$, the relative coordinate vector has two degrees of freedom. The equations of motion are second order in time, leading to *four* constants of integration. Our four constants are E, ℓ, r_0, and ϕ_0. Life is good!

`Geometric Equation of the Orbit:` From $\ell = \mu r^2 \dot{\phi}$, we have

$$\frac{d}{dt} = \frac{\ell}{\mu r^2}\,\frac{d}{d\phi} \ , \tag{2.337}$$

leading to

$$\frac{d^2 r}{d\phi^2} - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 = \frac{\mu r^4}{\ell^2}\,F(r) + r \tag{2.338}$$

where $F(r) = -dU(r)/dr$ is the magnitude of the central force. This second order equation may be reduced to a first order one using energy conservation:

$$\begin{aligned} E &= \tfrac{1}{2}\mu\dot{r}^2 + U_{\text{eff}}(r) \\ &= \frac{\ell^2}{2\mu r^4}\left(\frac{dr}{d\phi}\right)^2 + U_{\text{eff}}(r) \ . \end{aligned} \tag{2.339}$$

Thus,

$$d\phi = \pm\frac{\ell}{\sqrt{2\mu}} \cdot \frac{dr}{r^2\sqrt{E - U_{\text{eff}}(r)}} \ , \tag{2.340}$$

which can be integrated to yield $\phi(r)$, and then inverted to yield $r(\phi)$. Note that only one integration need be performed to obtain the geometric shape of the orbit, while two integrations – one for $r(t)$ and one for $\phi(t)$ – must be performed to obtain the full motion of the system.

It is sometimes convenient to rewrite this equation in terms of the variable $s = 1/r$:

$$\frac{d^2 s}{d\phi^2} + s = -\frac{\mu}{\ell^2 s^2} F\big(s^{-1}\big) \ . \tag{2.341}$$

As an example, suppose the geometric orbit is $r(\phi) = k\, e^{\alpha\phi}$, known as a logarithmic spiral. What is the force? We invoke (2.338), with $s''(\phi) = \alpha^2\, s$, yielding

$$F\big(s^{-1}\big) = -(1+\alpha^2)\,\frac{\ell^2}{\mu}\, s^3 \quad \Rightarrow \quad F(r) = -\frac{C}{r^3} \tag{2.342}$$

with

$$\alpha^2 = \frac{\mu C}{\ell^2} - 1 \ . \tag{2.343}$$

The general solution for $s(\phi)$ for this force law is

$$s(\phi) = \begin{cases} A\cosh(\alpha\phi) + B\sinh(-\alpha\phi) & \text{if } \ell^2 > \mu C \\ \\ A'\,\cos\big(|\alpha|\phi\big) + B'\,\sin\big(|\alpha|\phi\big) & \text{if } \ell^2 < \mu C \ . \end{cases} \tag{2.344}$$

The logarithmic spiral shape is a special case of the first kind of orbit.

2.10.4 Precession

`Almost Circular Orbits:` A circular orbit with $r(t) = r_0$ satisfies $\ddot r = 0$, which means that $U'_{\rm eff}(r_0) = 0$, which says that $F(r_0) = -\ell^2/\mu r_0^3$. This is negative, indicating that a circular orbit is possible only if the force is attractive over some range of distances. Since $\dot r = 0$ as well, we must also have $E = U_{\rm eff}(r_0)$. An almost circular orbit has $r(t) = r_0 + \eta(t)$, where $|\eta/r_0| \ll 1$. To lowest order in η, one derives the equations

$$\frac{d^2\eta}{dt^2} = -\omega^2\,\eta \qquad , \qquad \omega^2 = \frac{1}{\mu}\, U''_{\rm eff}(r_0) \ . \tag{2.345}$$

If $\omega^2 > 0$, the circular orbit is *stable* and the perturbation oscillates harmonically. If $\omega^2 < 0$, the circular orbit is *unstable* and the perturbation grows exponentially. For the geometric shape of the perturbed orbit, we write $r = r_0 + \eta$, and from (2.338) we obtain

$$\frac{d^2\eta}{d\phi^2} = \left(\frac{\mu r_0^4}{\ell^2}\, F'(r_0) - 3\right)\eta = -\beta^2\,\eta \ , \tag{2.346}$$

with

$$\beta^2 = 3 + \frac{d\ln F(r)}{d\ln r}\bigg|_{r_0} = \left(\frac{\mu\omega r_0}{\ell}\right)^2 \ . \tag{2.347}$$

The solution here is

$$\eta(\phi) = \eta_0\,\cos\beta(\phi - \delta_0) \ , \tag{2.348}$$

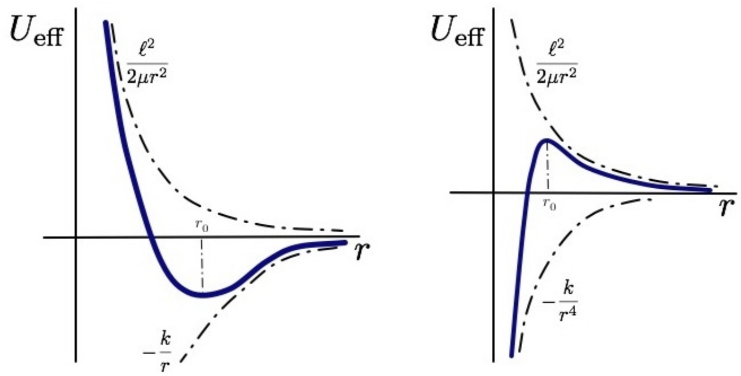

Figure 2.16: Stable and unstable circular orbits. Left panel: $U(r) = -k/r$ produces a stable circular orbit. Right panel: $U(r) = -k/r^4$ produces an unstable circular orbit.

where η_0 and δ_0 are initial conditions. Setting $\eta = \eta_0$, we obtain the sequence of ϕ values

$$\phi_n = \delta_0 + \frac{2\pi n}{\beta} \ , \tag{2.349}$$

at which $\eta(\phi)$ is a local maximum, *i.e.* at *apoapsis*, where $r = r_0 + \eta_0$. Setting $r = r_0 - \eta_0$ is the condition for closest approach, *i.e. periapsis.* This yields the identical set if angles, just shifted by π. The difference,

$$\Delta\phi = \phi_{n+1} - \phi_n - 2\pi = 2\pi\big(\beta^{-1} - 1\big) \ , \tag{2.350}$$

is the amount by which the apsides (*i.e.* periapsis and apoapsis) *precess* during each cycle. If $\beta > 1$, the apsides advance, *i.e.* it takes less than a complete revolution $\Delta\phi = 2\pi$ between successive periapses. If $\beta < 1$, the apsides retreat, and it takes longer than a complete revolution between successive periapses. The situation is depicted in Fig. 2.17 for the case $\beta = 1.1$. Below, we will exhibit a soluble model in which the precessing orbit may be determined exactly. Finally, note that if $\beta = p/q$ is a rational number, then the orbit is *closed*, *i.e.* it eventually retraces itself, after every q revolutions.

As an example, let $U(r) = kr^{-\alpha}$. Solving for a circular orbit, we write

$$U'_{\rm eff}(r) = \frac{\alpha k}{r^{\alpha+1}} - \frac{\ell^2}{\mu r^3} = 0 \ , \tag{2.351}$$

which has a solution only for $\alpha k > 0$, corresponding to an attractive potential. We then find

$$r_0 = \left(\frac{\ell^2}{\alpha k \mu}\right)^{1/(2-\alpha)} \ . \tag{2.352}$$

The force law is $F(r) = -\alpha k\, r^{-(1+\alpha)}$, yielding $\beta^2 = 2 - \alpha$. The shape of the perturbed orbits follows from $\eta'' = -\beta^2\, \eta$. Thus, while circular orbits exist whenever $\alpha k > 0$, small perturbations about these orbits are stable only for $\beta^2 > 0$, *i.e.* for $\alpha < 2$. One then has

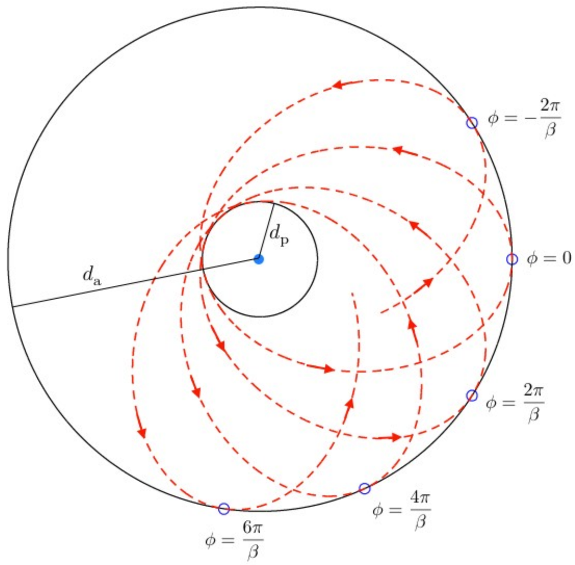

Figure 2.17: Precession in a soluble model, with geometric orbit $r(\phi) = r_0/(1 - \varepsilon \cos \beta\phi)$, shown here with $\beta = 1.1$. Periapsis and apoapsis advance by $\Delta\phi = 2\pi(1 - \beta^{-1})$ per cycle.

$\eta(\phi) = A \cos \beta(\phi - \phi_0)$. The perturbed orbits are closed, at least to lowest order in η, for $\alpha = 2 - (p/q)^2$, *i.e.* for $\beta = p/q$. The situation is depicted in Fig. 2.16, for the potentials $U(r) = -k/r$ ($\alpha = 1$) and $U(r) = -k/r^4$ ($\alpha = 4$).

Precession in a Soluble Model: Let's start with the answer and work backwards. Consider the geometrical orbit,

$$r(\phi) = \frac{r_0}{1 - \epsilon \cos \beta\phi} \ . \tag{2.353}$$

Our interest is in bound orbits, for which $0 \le \epsilon < 1$ (see Fig. 2.17). What sort of potential gives rise to this orbit? Writing $s = 1/r$ as before, we have

$$s(\phi) = s_0 \, (1 - \varepsilon \cos \beta\phi) \ . \tag{2.354}$$

Substituting into (2.341), we have

$$\begin{aligned} -\frac{\mu}{\ell^2 s^2} \, F\big(s^{-1}\big) &= \frac{d^2 s}{d\phi^2} + s \\ &= \beta^2 \, s_0 \, \epsilon \cos \beta\phi + s \\ &= (1 - \beta^2) \, s + \beta^2 \, s_0 \ , \end{aligned} \tag{2.355}$$

from which we conclude

$$F(r) = -\frac{k}{r^2} + \frac{C}{r^3} \ , \tag{2.356}$$

with

$$k = \beta^2 s_0 \, \frac{\ell^2}{\mu} \quad , \quad C = (\beta^2 - 1) \, \frac{\ell^2}{\mu} \ . \tag{2.357}$$

The corresponding potential is

$$U(r) = -\frac{k}{r} + \frac{C}{2r^2} + U_\infty \ , \tag{2.358}$$

where U_∞ is an arbitrary constant, conveniently set to zero. If μ and C are given, we have

$$r_0 = \frac{\ell^2}{\mu k} + \frac{C}{k} \quad , \quad \beta = \sqrt{1 + \frac{\mu C}{\ell^2}} \ . \tag{2.359}$$

When $C = 0$, these expressions recapitulate those from the Kepler problem. Note that when $\ell^2 + \mu C < 0$ that the effective potential is monotonically increasing as a function of r. In this case, the angular momentum barrier is overwhelmed by the (attractive, $C < 0$) inverse square part of the potential, and $U_{\text{eff}}(r)$ is monotonically increasing. The orbit then passes through the force center. It is a useful exercise to derive the total energy for the orbit,

$$E = (\varepsilon^2 - 1) \, \frac{\mu k^2}{2(\ell^2 + \mu C)} \qquad \Longleftrightarrow \qquad \varepsilon^2 = 1 + \frac{2E(\ell^2 + \mu C)}{\mu k^2} \ . \tag{2.360}$$

2.10.5 The Kepler Problem: $U(r) = -k\, r^{-1}$

Geometric Shape of Orbits: The force is $F(r) = -kr^{-2}$, hence the equation for the geometric shape of the orbit is

$$\frac{d^2 s}{d\phi^2} + s = -\frac{\mu}{\ell^2 s^2} \, F(s^{-1}) = \frac{\mu k}{\ell^2} \ , \tag{2.361}$$

with $s = 1/r$. Thus, the most general solution is

$$s(\phi) = s_0 - C \cos(\phi - \phi_0) \ , \tag{2.362}$$

where C and ϕ_0 are constants. Thus,

$$r(\phi) = \frac{r_0}{1 - \varepsilon \cos(\phi - \phi_0)} \ , \tag{2.363}$$

where $r_0 = \ell^2/\mu k$ and where we have defined a new constant $\varepsilon \equiv C r_0$.

Laplace-Runge-Lenz vector: Consider the vector

$$\boldsymbol{A} = \boldsymbol{p} \times \boldsymbol{\ell} - \mu k \, \hat{\boldsymbol{r}} \ , \tag{2.364}$$

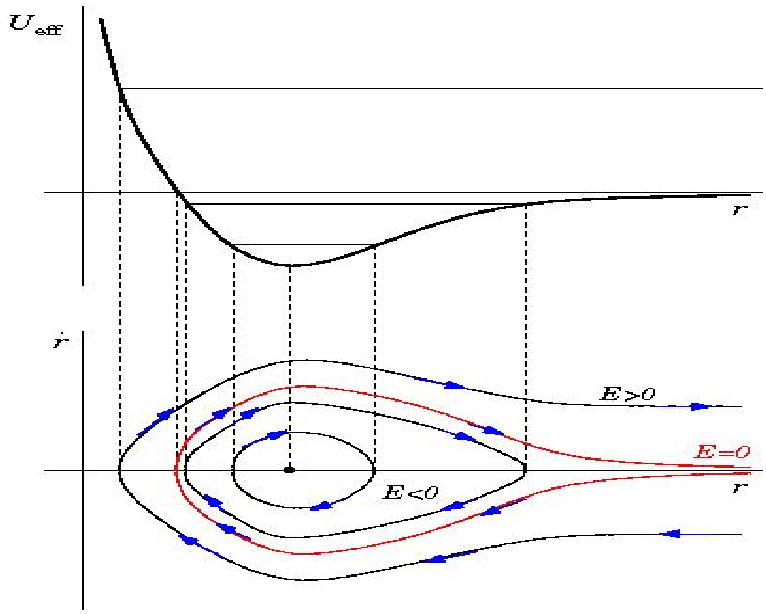

Figure 2.18: The effective potential for the Kepler problem, and associated phase curves. The orbits are geometrically described as conic sections: hyperbolae ($E > 0$), parabolae ($E = 0$), ellipses ($E_{\rm min} < E < 0$), and circles ($E = E_{\rm min}$).

where $\hat{\boldsymbol{r}} = \boldsymbol{r}/|\boldsymbol{r}|$ is the unit vector pointing in the direction of $\boldsymbol{r}$. We may now show that $\boldsymbol{A}$ is conserved:

$$\begin{aligned}\frac{d\boldsymbol{A}}{dt} &= \frac{d}{dt}\Big\{\boldsymbol{p}\times\boldsymbol{\ell} - \mu k\frac{\boldsymbol{r}}{r}\Big\}\\ &= \dot{\boldsymbol{p}}\times\boldsymbol{\ell} + \boldsymbol{p}\times\dot{\boldsymbol{\ell}} - \mu k\,\frac{r\dot{\boldsymbol{r}} - \boldsymbol{r}\dot r}{r^2}\\ &= -\frac{k\boldsymbol{r}}{r^3}\times(\mu\boldsymbol{r}\times\dot{\boldsymbol{r}}) - \mu k\,\frac{\dot{\boldsymbol{r}}}{r} + \mu k\,\frac{\dot r\boldsymbol{r}}{r^2}\\ &= -\mu k\,\frac{\boldsymbol{r}(\boldsymbol{r}\cdot\dot{\boldsymbol{r}})}{r^3} + \mu k\,\frac{\dot{\boldsymbol{r}}(\boldsymbol{r}\cdot\boldsymbol{r})}{r^3} - \mu k\,\frac{\dot{\boldsymbol{r}}}{r} + \mu k\,\frac{\dot r\boldsymbol{r}}{r^2} = 0\ .\end{aligned} \tag{2.365}$$

So $\boldsymbol{A}$ is a conserved vector which clearly lies in the plane of the motion. $\boldsymbol{A}$ points toward periapsis, *i.e.* toward the point of closest approach to the force center.

Let's assume *apo*apsis occurs at $\phi = \phi_0$. Then

$$\boldsymbol{A}\cdot\boldsymbol{r} = -Ar\cos(\phi-\phi_0) = \ell^2 - \mu k r \tag{2.366}$$

giving

$$r(\phi) = \frac{\ell^2}{\mu k - A\cos(\phi-\phi_0)} = \frac{a(1-\varepsilon^2)}{1-\varepsilon\cos(\phi-\phi_0)}\ , \tag{2.367}$$

where

$$\varepsilon = \frac{A}{\mu k} \qquad , \qquad a(1-\varepsilon^2) = \frac{\ell^2}{\mu k}\ . \tag{2.368}$$

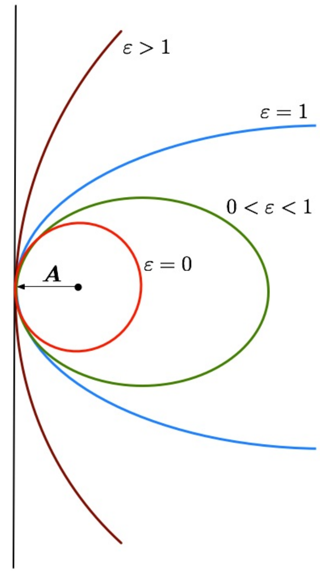

Figure 2.19: Keplerian orbits are conic sections, classified according to eccentricity: hyperbola ($\epsilon > 1$), parabola ($\epsilon = 1$), ellipse ($0 < \epsilon < 1$), and circle ($\epsilon = 0$). The Laplace-Runge-Lenz vector, $\boldsymbol{A}$, points toward periapsis.

The orbit is a *conic section* with eccentricity ε. Squaring $\boldsymbol{A}$, onefinds

$$\begin{aligned}\boldsymbol{A}^2 &= (\boldsymbol{p}\times\boldsymbol{\ell})^2 - 2\mu k\hat{\boldsymbol{r}}\cdot\boldsymbol{p}\times\boldsymbol{\ell} + \mu^2 k^2 \\ &= p^2\ell^2 - 2\mu\ell^2\,\frac{k}{r} + \mu^2 k^2 \\ &= 2\mu\ell^2\left(\frac{p^2}{2\mu} - \frac{k}{r} + \frac{\mu k^2}{2\ell^2}\right) = 2\mu\ell^2\left(E + \frac{\mu k^2}{2\ell^2}\right) \end{aligned} \tag{2.369}$$

and thus

$$a = -\frac{k}{2E} \quad , \quad \varepsilon^2 = 1 + \frac{2E\ell^2}{\mu k^2} \ . \tag{2.370}$$

There are four classes of conic sections:

- *Circle:* $\varepsilon = 0$, $E = -\mu k^2/2\ell^2$, radius $a = \ell^2/\mu k$. The force center lies at the center of circle.
- *Ellipse:* $0 < \varepsilon < 1$, $-\mu k^2/2\ell^2 < E < 0$, semimajor axis $a = -k/2E$, semiminor axis $b = a\sqrt{1-\varepsilon^2}$. The force center is at one of the foci.
- *Parabola:* $\varepsilon = 1$, $E = 0$, force center is the focus.
- *Hyperbola:* $\varepsilon > 1$, $E > 0$, force center is closest focus (attractive) or farthest focus (repulsive).

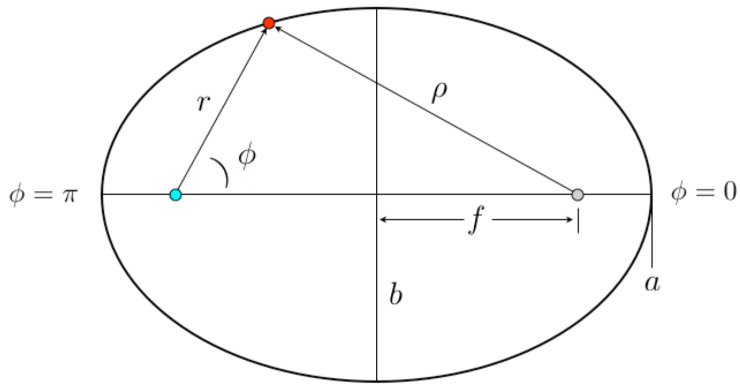

Figure 2.20: The Keplerian ellipse, with the force center at the left focus. The focal distance is $f = \varepsilon a$, where a is the semimajor axis length. The length of the semiminor axis is $b = \sqrt{1-\varepsilon^2}\, a$.

To see that the Keplerian orbits are indeed conic sections, consider the ellipse of Fig. 2.20. The law of cosines gives

$$\rho^2 = r^2 + 4f^2 - 4rf\cos\phi \ , \tag{2.371}$$

where $f = \varepsilon a$ is the focal distance. Now for any point on an ellipse, the sum of the distances to the left and right foci is a constant, and taking $\phi = 0$ we see that this constant is $2a$. Thus, $\rho = 2a - r$, and we have

$$\begin{aligned} (2a-r)^2 = 4a^2 - 4ar + r^2 &= r^2 + 4\varepsilon^2 a^2 - 4\varepsilon r\cos\phi \\ \Rightarrow \quad r(1-\varepsilon\cos\phi) &= a(1-\varepsilon^2) \ . \end{aligned} \tag{2.372}$$

Thus, we obtain

$$r(\phi) = \frac{a\,(1-\varepsilon^2)}{1-\varepsilon\cos\phi} \ , \tag{2.373}$$

and we therefore conclude that

$$r_0 = \frac{\ell^2}{\mu k} = a\,(1-\varepsilon^2) \ . \tag{2.374}$$

Next let us examine the energy,

$$\begin{aligned} E &= \tfrac{1}{2}\mu\dot{r}^2 + U_{\rm eff}(r) \\ &= \tfrac{1}{2}\mu\bigg(\frac{\ell}{\mu r^2}\,\frac{dr}{d\phi}\bigg)^2 + \frac{\ell^2}{2\mu r^2} - \frac{k}{r} \\ &= \frac{\ell^2}{2\mu}\left(\frac{ds}{d\phi}\right)^2 + \frac{\ell^2}{2\mu}\,s^2 - ks \ , \end{aligned} \tag{2.375}$$

with

$$s = \frac{1}{r} = \frac{\mu k}{\ell^2}\Big(1-\varepsilon\cos\phi\Big) \ . \tag{2.376}$$

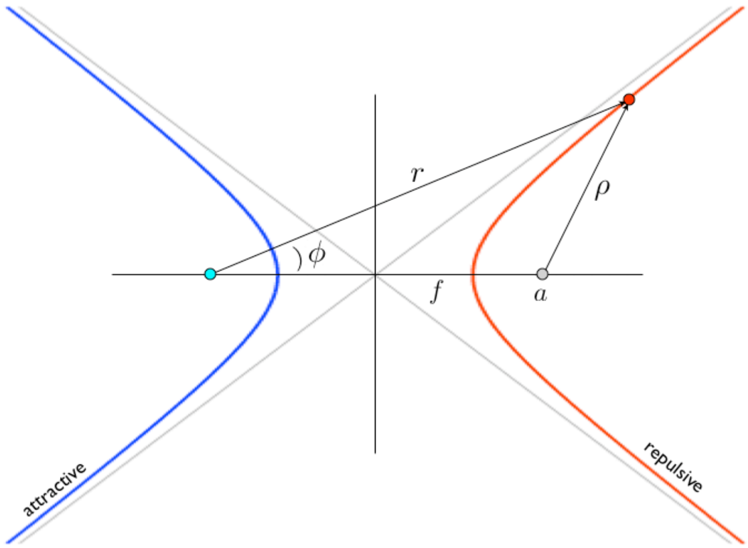

Figure 2.21: The Keplerian hyperbolae, with the force center at the left focus. The left (blue) branch corresponds to an attractive potential, while the right (red) branch corresponds to a repulsive potential. The equations of these branches are $r = \rho = \mp 2a$, where the top sign corresponds to the left branch and the bottom sign to the right branch.

Thus,

$$\frac{ds}{d\phi} = \frac{\mu k}{\ell^2}\,\varepsilon\,\sin\phi\ , \tag{2.377}$$

and

$$\begin{aligned}\left(\frac{ds}{d\phi}\right)^2 &= \frac{\mu^2 k^2}{\ell^4}\,\varepsilon^2\,\sin^2\!\phi \\ &= \frac{\mu^2 k^2 \varepsilon^2}{\ell^4} - \left(\frac{\mu k}{\ell^2} - s\right)^2 \\ &= -s^2 + \frac{2\mu k}{\ell^2}\,s + \frac{\mu^2 k^2}{\ell^4}\,\big(\varepsilon^2 - 1\big)\ .\end{aligned} \tag{2.378}$$

Substituting this into eqn. 2.375, we obtain

$$E = \frac{\mu k^2}{2\ell^2}\,\big(\varepsilon^2 - 1\big)\ . \tag{2.379}$$

For the hyperbolic orbit, depicted in Fig. 2.21, we have $r - \rho = \mp 2a$, depending on whether we are on the attractive or repulsive branch, respectively. We then have

$$\begin{aligned}(r \pm 2a)^2 = 4a^2 \pm 4ar + r^2 &= r^2 + 4\varepsilon^2 a^2 - 4\varepsilon r\cos\phi \\ \Rightarrow \quad r(\pm 1 + \varepsilon\cos\phi) &= a(\varepsilon^2 - 1)\ .\end{aligned} \tag{2.380}$$

This yields

$$r(\phi) = \frac{a\,(\varepsilon^2 - 1)}{\pm 1 + \varepsilon\cos\phi}\ . \tag{2.381}$$

Period of Bound Kepler Orbits: From $\ell = \mu r^2 \dot{\phi} = 2\mu \dot{\mathcal{A}}$, the period is $\tau = 2\mu \mathcal{A}/\ell$, where $\mathcal{A} = \pi a^2 \sqrt{1-\varepsilon^2}$ is the area enclosed by the orbit. This gives

$$\tau = 2\pi \left(\frac{\mu a^3}{k}\right)^{1/2} = 2\pi \left(\frac{a^3}{GM}\right)^{1/2} \tag{2.382}$$

as well as

$$\frac{a^3}{\tau^2} = \frac{GM}{4\pi^2} , \tag{2.383}$$

where $k = G m_1 m_2$ and $M = m_1 + m_2$ is the total mass. For planetary orbits, $m_1 = M_\odot$ is the solar mass and $m_2 = m_{\mathrm{p}}$ is the planetary mass. We then have

$$\frac{a^3}{\tau^2} = \left(1 + \frac{m_{\mathrm{p}}}{M_\odot}\right) \frac{GM_\odot}{4\pi^2} \approx \frac{GM_\odot}{4\pi^2} , \tag{2.384}$$

which is to an excellent approximation independent of the planetary mass. (Note that $m_{\mathrm{p}}/M_\odot \approx 10^{-3}$ even for Jupiter.) This analysis also holds, *mutatis mutandis,* for the case of satellites orbiting the earth, and indeed in any case where the masses are grossly disproportionate in magnitude.

Escape Velocity: The threshold for escape from a gravitational potential occurs at $E = 0$. Since $E = T + U$ is conserved, we determine the *escape velocity* for a body a distance r from the force center by setting

$$E = 0 = \tfrac{1}{2}\mu v_{\mathrm{esc}}^2(r) - \frac{GMm}{r} \qquad \Rightarrow \qquad v_{\mathrm{esc}}(r) = \sqrt{\frac{2G(M+m)}{r}} . \tag{2.385}$$

When $M \gg m$, $v_{\mathrm{esc}}(r) = \sqrt{2GM/r}$. Thus, for an object at the surface of the earth, $v_{\mathrm{esc}} = \sqrt{2gR_{\mathrm{E}}} = 11.2\,\mathrm{km/s}$.

Satellites and Spacecraft: A satellite in a circular orbit a distance h above the earth's surface has an orbital period

$$\tau = \frac{2\pi}{\sqrt{GM_{\mathrm{E}}}} \left(R_{\mathrm{E}} + h\right)^{3/2} , \tag{2.386}$$

where we take $m_{\mathrm{satellite}} \ll M_{\mathrm{E}}$. For low earth orbit (LEO), $h \ll R_{\mathrm{E}} = 6.37 \times 10^6\,\mathrm{m}$, in which case $\tau_{\mathrm{LEO}} = 2\pi\sqrt{R_{\mathrm{E}}/g} = 1.4\,\mathrm{hr}$.

Consider a weather satellite in an elliptical orbit whose closest approach to the earth (perigee) is 200 km above the earth's surface and whose farthest distance (apogee) is 7200 km above the earth's surface. What is the satellite's orbital period? From Fig. 2.20, we see that

$$\begin{aligned} d_{\mathrm{apogee}} &= R_{\mathrm{E}} + 7200\,\mathrm{km} = 13571\,\mathrm{km} \\ d_{\mathrm{perigee}} &= R_{\mathrm{E}} + 200\,\mathrm{km} = 6971\,\mathrm{km} \\ a &= \tfrac{1}{2}(d_{\mathrm{apogee}} + d_{\mathrm{perigee}}) = 10071\,\mathrm{km} . \end{aligned} \tag{2.387}$$

We then have

$$\tau = \left(\frac{a}{R_{\text{E}}}\right)^{3/2} \cdot \tau_{\text{LEO}} \approx 2.65\,\text{hr} \ . \tag{2.388}$$

What happens if a spacecraft in orbit about the earth fires its rockets? Clearly the energy and angular momentum of the orbit will change, and this means the shape will change. If the rockets are fired (in the direction of motion) at perigee, then perigee itself is unchanged, because $\boldsymbol{v} \cdot \boldsymbol{r} = 0$ is left unchanged at this point. However, E is increased, hence the eccentricity $\varepsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}$ increases. This is the most efficient way of boosting a satellite into an orbit with higher eccentricity. Conversely, and somewhat paradoxically, when a satellite in LEO loses energy due to frictional drag of the atmosphere, the energy E decreases. Initially, because the drag is weak and the atmosphere is isotropic, the orbit remains circular. Since E decreases, $\langle T \rangle = -E$ must *increase*, which means that the frictional forces cause the satellite to speed up!

2.10.6 Two Examples of Orbital Mechanics

Problem #1: At perigee of an elliptical Keplerian orbit, a satellite receives an impulse $\Delta\boldsymbol{p} = p_0 \hat{\boldsymbol{r}}$. Describe the resulting orbit.

Solution #1: Since the impulse is radial, the angular momentum $\boldsymbol{\ell} = \boldsymbol{r} \times \boldsymbol{p}$ is unchanged. The energy, however, does change, with $\Delta E = p_0^2/2\mu$. Thus,

$$\varepsilon_{\text{f}}^2 = 1 + \frac{2E_{\text{f}}\ell^2}{\mu k^2} = \varepsilon_{\text{i}}^2 + \left(\frac{\ell p_0}{\mu k}\right)^2 \ . \tag{2.389}$$

The new semimajor axis length is

$$\begin{aligned} a_{\text{f}} &= \frac{\ell^2/\mu k}{1 - \varepsilon_{\text{f}}^2} = a_{\text{i}} \cdot \frac{1 - \varepsilon_{\text{i}}^2}{1 - \varepsilon_{\text{f}}^2} \\ &= \frac{a_{\text{i}}}{1 - (a_{\text{i}} p_0^2/\mu k)} \ . \end{aligned} \tag{2.390}$$

The shape of the final orbit must also be a Keplerian ellipse, described by

$$r_{\text{f}}(\phi) = \frac{\ell^2}{\mu k} \cdot \frac{1}{1 - \varepsilon_{\text{f}} \cos(\phi + \delta)} \ , \tag{2.391}$$

where the phase shift δ is determined by setting

$$r_{\text{i}}(\pi) = r_{\text{f}}(\pi) = \frac{\ell^2}{\mu k} \cdot \frac{1}{1 + \varepsilon_{\text{i}}} \ . \tag{2.392}$$

Solving for δ, we obtain

$$\delta = \cos^{-1}\left(\varepsilon_{\text{i}}/\varepsilon_{\text{f}}\right) \ . \tag{2.393}$$

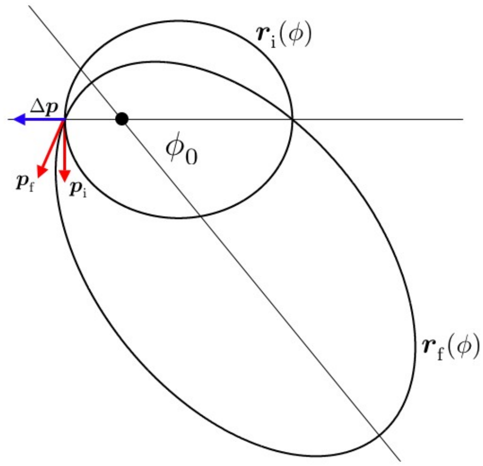

Figure 2.22: At perigee of an elliptical orbit $r_{\mathrm{i}}(\phi)$, a radial impulse $\Delta\boldsymbol{p}$ is applied. The shape of the resulting orbit $r_{\mathrm{f}}(\phi)$ is shown.

The situation is depicted in Fig. 2.22.

Problem #2: Which is more energy efficient – to send nuclear waste outside the solar system, or to send it into the Sun?

Solution #2: Escape velocity for the solar system is $v_{\mathrm{esc},\odot}(r) = \sqrt{2GM_\odot/r}$. At a distance a_{E}, we then have $v_{\mathrm{esc},\odot}(a_{\mathrm{E}}) = \sqrt{2}\,v_{\mathrm{E}}$, where $v_{\mathrm{E}} = \sqrt{GM_\odot/a_{\mathrm{E}}} = 2\pi a_{\mathrm{E}}/\tau_{\mathrm{E}} = 29.9\,\mathrm{km/s}$ is the velocity of the earth in its orbit. The satellite is launched from earth, and clearly the most energy efficient launch will be one in the direction of the earth's motion, in which case the velocity after escape from earth must be $u = \big(\sqrt{2}-1\big)v_{\mathrm{E}} = 12.4\,\mathrm{km/s}$. The speed just above the earth's atmosphere must then be $\tilde{u}$, where

$$\tfrac{1}{2}m\tilde{u}^2 - \frac{GM_{\mathrm{E}}m}{R_{\mathrm{E}}} = \tfrac{1}{2}mu^2 \ , \tag{2.394}$$

or, in other words,

$$\tilde{u}^2 = u^2 + v_{\mathrm{esc,E}}^2 \ . \tag{2.395}$$

We compute $\tilde{u} = 16.7\,\mathrm{km/s}$.

The second method is to place the trash ship in an elliptical orbit whose perihelion is the Sun's radius, $R_\odot = 6.98\times10^8\,\mathrm{m}$, and whose aphelion is a_{E}. Using the general equation

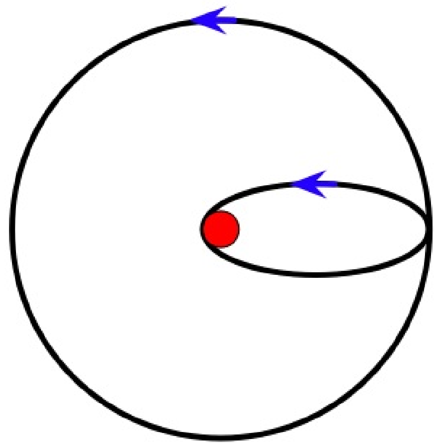

Figure 2.23: The larger circular orbit represents the orbit of the earth. The elliptical orbit represents that for an object orbiting the Sun with distance at perihelion equal to the Sun's radius.

$r(\phi) = (\ell^2/\mu k)/(1 - \varepsilon\cos\phi)$ for a Keplerian ellipse, we therefore solve the two equations

$$r(\phi = \pi) = R_\odot = \frac{1}{1+\varepsilon} \cdot \frac{\ell^2}{\mu k} \tag{2.396}$$

$$r(\phi = 0) = a_{\mathrm{E}} = \frac{1}{1-\varepsilon} \cdot \frac{\ell^2}{\mu k} \ . \tag{2.397}$$

We thereby obtain

$$\varepsilon = \frac{a_{\mathrm{E}} - R_\odot}{a_{\mathrm{E}} + R_\odot} = 0.991 \ , \tag{2.398}$$

which is a very eccentric ellipse, and

$$\begin{aligned} \frac{\ell^2}{\mu k} &= \frac{a_{\mathrm{E}}^2\, v^2}{G(M_\odot + m)} \approx a_{\mathrm{E}} \cdot \frac{v^2}{v_{\mathrm{E}}^2} \\ &= (1-\varepsilon)\, a_{\mathrm{E}} = \frac{2a_{\mathrm{E}} R_\odot}{a_{\mathrm{E}} + R_\odot} \ . \end{aligned} \tag{2.399}$$

Hence,

$$v^2 = \frac{2R_\odot}{a_{\mathrm{E}} + R_\odot}\, v_{\mathrm{E}}^2 \ , \tag{2.400}$$

and the necessary velocity relative to earth is

$$u = \left(\sqrt{\frac{2R_\odot}{a_{\mathrm{E}} + R_\odot}} - 1\right) v_{\mathrm{E}} \approx -0.904\, v_{\mathrm{E}} \ , \tag{2.401}$$

i.e. $u = -27.0\,\mathrm{km/s}$. Launch is in the opposite direction from the earth's orbital motion, and from $\tilde{u}^2 = u^2 + v_{\mathrm{esc,E}}^2$ we find $\tilde{u} = -29.2\,\mathrm{km/s}$, which is larger (in magnitude) than in the first scenario. Thus, it is cheaper to ship the trash out of the solar system than to send it crashing into the Sun, by a factor $\tilde{u}_{\mathrm{I}}^2/\tilde{u}_{\mathrm{II}}^2 = 0.327$.

2.11 Mission to Neptune

Four earth-launched spacecraft have escaped the solar system: *Pioneer 10* (launch 3/3/72), *Pioneer 11* (launch 4/6/73), *Voyager 1* (launch 9/5/77), and *Voyager 2* (launch 8/20/77).[7] The latter two are still functioning, and each are moving away from the Sun at a velocity of roughly 3.5 AU/yr.

As the first objects of earthly origin to leave our solar system, both *Pioneer* spacecraft featured a graphic message in the form of a 6" x 9" gold anodized plaque affixed to the spacecrafts' frame. This plaque was designed in part by the late astronomer and popular science writer Carl Sagan. The humorist Dave Barry, in an essay entitled *Bring Back Carl's Plaque*, remarks,

> But the really bad part is what they put on the plaque. I mean, if we're going to have a plaque, it ought to at least show the aliens what we're really like, right? Maybe a picture of people eating cheeseburgers and watching "The Dukes of Hazzard." Then if aliens found it, they'd say, "Ah. Just plain folks."
>
> But no. Carl came up with this incredible science-fair-wimp plaque that features drawings of – you are not going to believe this – a hydrogen atom and naked people. To represent the entire Earth! This is crazy! Walk the streets of any town on this planet, and the two things you will almost never see are hydrogen atoms and naked people.

During August, 1989, *Voyager 2* investigated the planet Neptune. A direct trip to Neptune along a Keplerian ellipse with $r_{\rm p} = a_{\rm E} = 1\,{\rm AU}$ and $r_{\rm a} = a_{\rm N} = 30.06\,{\rm AU}$ would take 30.6 years. To see this, note that $r_{\rm p} = a\,(1-\varepsilon)$ and $r_{\rm a} = a\,(1+\varepsilon)$ yield

$$a = \tfrac{1}{2}\big(a_{\rm E} + a_{\rm N}\big) = 15.53\,{\rm AU} \quad , \quad \varepsilon = \frac{a_{\rm N} - a_{\rm E}}{a_{\rm N} + a_{\rm E}} = 0.9356 \ . \tag{2.402}$$

Thus,

$$\tau = \tfrac{1}{2}\,\tau_{\rm E} \cdot \left(\frac{a}{a_{\rm E}}\right)^{3/2} = 30.6\,{\rm yr} \ . \tag{2.403}$$

The energy cost per kilogram of such a mission is computed as follows. Let the speed of the probe after its escape from earth be $v_{\rm p} = \lambda v_{\rm E}$, and the speed just above the atmosphere (*i.e.* neglecting atmospheric friction) is v_0. For the most efficient launch possible, the probe is shot in the direction of earth's instantaneous motion about the Sun. Then we must have

$$\tfrac{1}{2} m\, v_0^2 - \frac{G M_{\rm E} m}{R_{\rm E}} = \tfrac{1}{2} m\,(\lambda - 1)^2\, v_{\rm E}^2 \ , \tag{2.404}$$

[7] There is a very nice discussion in the Barger and Olsson book on 'Grand Tours of the Outer Planets'. Here I reconstruct and extend their discussion.

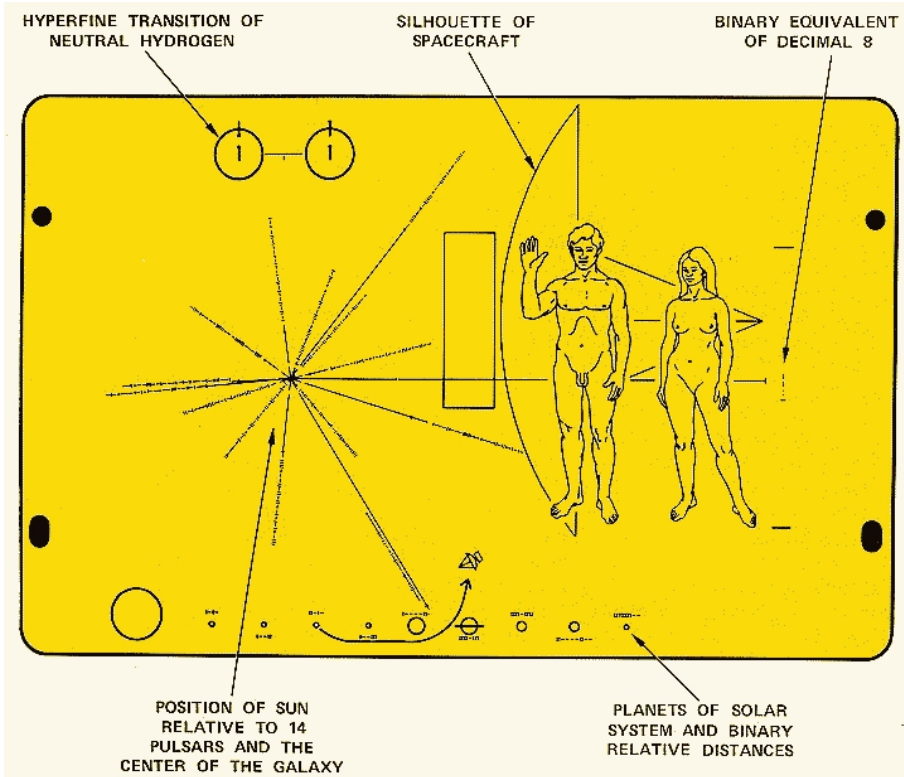

Figure 2.24: The unforgivably dorky *Pioneer 10* and *Pioneer 11* plaque.

since the speed of the probe in the frame of the earth is $v_{\rm p} - v_{\rm E} = (\lambda - 1)\, v_{\rm E}$. Thus,

$$\frac{E}{m} = \tfrac{1}{2} v_0^2 = \Big[\tfrac{1}{2}(\lambda-1)^2 + h\Big] v_{\rm E}^2 \qquad (2.405)$$

$$v_{\rm E}^2 = \frac{GM_\odot}{a_{\rm E}} = 6.24\times 10^7\ {\rm J/kg}\ ,$$

where

$$h \equiv \frac{M_{\rm E}}{M_\odot}\cdot\frac{a_{\rm E}}{R_{\rm E}} = 7.050\times 10^{-2}\ . \qquad (2.406)$$

Therefore, a convenient dimensionless measure of the energy is

$$\eta \equiv \frac{2E}{mv_{\rm E}^2} = \frac{v_0^2}{v_{\rm E}^2} = (\lambda-1)^2 + 2h\ . \qquad (2.407)$$

As we shall derive below, a direct mission to Neptune requires

$$\lambda \ge \sqrt{\frac{2a_{\rm N}}{a_{\rm N} + a_{\rm E}}} = 1.3913\ , \qquad (2.408)$$

which is close to the criterion for escape from the solar system, $\lambda_{\rm esc} = \sqrt{2}$. Note that about 52% of the energy is expended after the probe escapes the Earth's pull, and 48% is expended in liberating the probe from Earth itself.

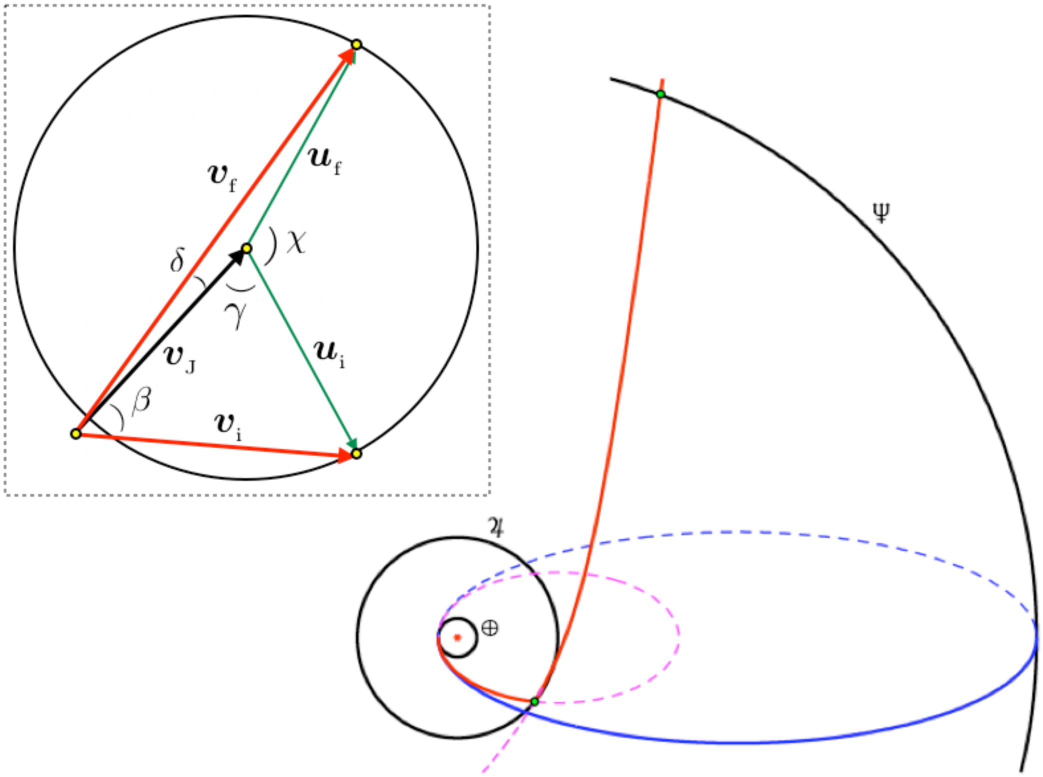

Figure 2.25: Mission to Neptune. The figure at the lower right shows the orbits of Earth, Jupiter, and Neptune in black. The cheapest (in terms of energy) direct flight to Neptune, shown in blue, would take 30.6 years. By swinging past the planet Jupiter, the satellite can pick up great speed and with even less energy the mission time can be cut to 8.5 years (red curve). The inset in the upper left shows the scattering event with Jupiter.

This mission can be done much more economically by taking advantage of a Jupiter flyby, as shown in Fig. 2.25. The idea of a flyby is to steal some of Jupiter's momentum and then fly away very fast before Jupiter realizes and gets angry. The CM frame of the probe-Jupiter system is of course the rest frame of Jupiter, and in this frame conservation of energy means that the final velocity $\boldsymbol{u}_{\mathrm{f}}$ is of the same magnitude as the initial velocity $\boldsymbol{u}_{\mathrm{i}}$. However, in the frame of the Sun, the initial and final velocities are $\boldsymbol{v}_{\mathrm{J}} + \boldsymbol{u}_{\mathrm{i}}$ and $\boldsymbol{v}_{\mathrm{J}} + \boldsymbol{u}_{\mathrm{f}}$, respectively, where $\boldsymbol{v}_{\mathrm{J}}$ is the velocity of Jupiter in the rest frame of the Sun. If, as shown in the inset to Fig. 2.25, $\boldsymbol{u}_{\mathrm{f}}$ is roughly parallel to $\boldsymbol{v}_{\mathrm{J}}$, the probe's velocity in the Sun's frame will be enhanced. Thus, the motion of the probe is broken up into three segments:

$$
\begin{aligned}
\mathrm{I}: &\quad \text{Earth to Jupiter} \\
\mathrm{II}: &\quad \text{Scatter off Jupiter's gravitational pull} \\
\mathrm{III}: &\quad \text{Jupiter to Neptune}
\end{aligned}
$$

We now analyze each of these segments in detail. In so doing, it is useful to recall that the general form of a Keplerian orbit is

$$r(\phi) = \frac{d}{1 - \varepsilon \cos\phi} \quad , \quad d = \frac{\ell^2}{\mu k} = \big|\varepsilon^2 - 1\big|\, a \ . \tag{2.409}$$

The energy is

$$E = (\varepsilon^2 - 1)\, \frac{\mu k^2}{2\ell^2} \ , \tag{2.410}$$

with $k = GMm$, where M is the mass of either the Sun or a planet. In either case, M dominates, and $\mu = Mm/(M+m) \simeq m$ to extremely high accuracy. The time for the trajectory to pass from $\phi = \phi_1$ to $\phi = \phi_2$ is

$$T = \int\! dt = \int\limits_{\phi_1}^{\phi_2} \frac{d\phi}{\dot\phi} = \frac{\mu}{\ell} \int\limits_{\phi_1}^{\phi_2} \! d\phi \, r^2(\phi) = \frac{\ell^3}{\mu k^2} \int\limits_{\phi_1}^{\phi_2} \frac{d\phi}{\big[1 - \varepsilon \cos\phi\big]^2} \ . \tag{2.411}$$

For reference,

$$\begin{array}{lll} a_{\rm E} = 1\,{\rm AU} & a_{\rm J} = 5.20\,{\rm AU} & a_{\rm N} = 30.06\,{\rm AU} \\ M_{\rm E} = 5.972 \times 10^{24}\,{\rm kg} & M_{\rm J} = 1.900 \times 10^{27}\,{\rm kg} & M_{\odot} = 1.989 \times 10^{30}\,{\rm kg} \end{array}$$

with $1\,{\rm AU} = 1.496 \times 10^8\,{\rm km}$. Here $a_{\rm E,J,N}$ and $M_{\rm E,J,\odot}$ are the orbital radii and masses of Earth, Jupiter, and Neptune, and the Sun. The last thing we need to know is the radius of Jupiter,

$$R_{\rm J} = 9.558 \times 10^{-4}\,{\rm AU} \ .$$

We need $R_{\rm J}$ because the distance of closest approach to Jupiter, or *perijove*, must be $R_{\rm J}$ or greater. Otherwise the probe crashes into Jupiter!

2.11.1 I. Earth to Jupiter

The probe's velocity at perihelion is $v_{\rm p} = \lambda v_{\rm E}$. The angular momentum is $\ell = \mu a_{\rm E} \cdot \lambda v_{\rm E}$, whence

$$d = \frac{(a_{\rm E} \lambda v_{\rm E})^2}{GM_\odot} = \lambda^2\, a_{\rm E} \ . \tag{2.412}$$

From $r(\pi) = a_{\rm E}$, we obtain

$$\varepsilon = \lambda^2 - 1 \ . \tag{2.413}$$

This orbit will intersect the orbit of Jupiter if $r_{\rm a} \ge a_{\rm J}$, which means

$$\frac{d}{1-\varepsilon} \ge a_{\rm J} \quad \Rightarrow \quad \lambda \ge \sqrt{\frac{2a_{\rm J}}{a_{\rm J} + a_{\rm E}}} = 1.2952 \ . \tag{2.414}$$

If this inequality holds, then intersection of Jupiter's orbit will occur for

$$\phi_{\rm J} = 2\pi - \cos^{-1}\!\left(\frac{a_{\rm J} - \lambda^2 a_{\rm E}}{(\lambda^2 - 1)\, a_{\rm J}}\right) \ . \tag{2.415}$$

Finally, the time for this portion of the trajectory is

$$\tau_{\rm EJ} = \tau_{\rm E} \cdot \lambda^3 \int\limits_{\pi}^{\phi_{\rm J}} \frac{d\phi}{2\pi} \, \frac{1}{\big[1 - (\lambda^2 - 1)\cos\phi\big]^2} \ . \tag{2.416}$$

2.11.2 II. Encounter with Jupiter

We are interested in the final speed $\boldsymbol{v}_{\rm f}$ of the probe after its encounter with Jupiter. We will determine the speed $v_{\rm f}$ and the angle δ which the probe makes with respect to Jupiter after its encounter. According to the geometry of Fig. 2.25,

$$v_{\rm f}^2 = v_{\rm J}^2 + u^2 - 2uv_{\rm J}\cos(\chi + \gamma) \tag{2.417}$$

$$\cos\delta = \frac{v_{\rm J}^2 + v_{\rm f}^2 - u^2}{2v_{\rm f}v_{\rm J}} \tag{2.418}$$

Note that

$$v_{\rm J}^2 = \frac{GM_\odot}{a_{\rm J}} = \frac{a_{\rm E}}{a_{\rm J}} \cdot v_{\rm E}^2 \ . \tag{2.419}$$

But what are u, χ, and γ?

To determine u, we invoke

$$u^2 = v_{\rm J}^2 + v_{\rm i}^2 - 2v_{\rm J}v_{\rm i}\cos\beta \ . \tag{2.420}$$

The initial velocity (in the frame of the Sun) when the probe crosses Jupiter's orbit is given by energy conservation:

$$\tfrac{1}{2}m(\lambda v_{\rm E})^2 - \frac{GM_\odot m}{a_{\rm E}} = \tfrac{1}{2}mv_{\rm i}^2 - \frac{GM_\odot m}{a_{\rm J}} \ , \tag{2.421}$$

which yields

$$v_i^2 = \bigg(\lambda^2 - 2 + \frac{2a_{\rm E}}{a_{\rm J}}\bigg)\, v_{\rm E}^2 \ . \tag{2.422}$$

As for β, we invoke conservation of angular momentum:

$$\mu(v_{\rm i}\cos\beta)a_{\rm J} = \mu(\lambda v_{\rm E})a_{\rm E} \quad \Rightarrow \quad v_{\rm i}\cos\beta = \lambda\,\frac{a_{\rm E}}{a_{\rm J}}\,v_{\rm E} \ . \tag{2.423}$$

The angle γ is determined from

$$v_{\rm J} = v_{\rm i}\cos\beta + u\cos\gamma \ . \tag{2.424}$$

Putting all this together, we obtain

$$v_{\rm i} = v_{\rm E}\sqrt{\lambda^2 - 2 + 2x} \tag{2.425}$$

$$u = v_{\rm E}\sqrt{\lambda^2 - 2 + 3x - 2\lambda x^{3/2}} \tag{2.426}$$

$$\cos\gamma = \frac{\sqrt{x} - \lambda x}{\sqrt{\lambda^2 - 2 + 3x - 2\lambda x^{3/2}}} \ , \tag{2.427}$$

where

$$x \equiv \frac{a_{\rm E}}{a_{\rm J}} = 0.1923 \ . \tag{2.428}$$

We next consider the scattering of the probe by the planet Jupiter. In the Jovian frame, we may write

$$r(\phi) = \frac{\kappa R_{\rm J}\,(1+\varepsilon_{\rm J})}{1+\varepsilon_{\rm J}\cos\phi} \ , \tag{2.429}$$

where perijove occurs at

$$r(0) = \kappa R_{\rm J} \ . \tag{2.430}$$

Here, κ is a dimensionless quantity, which is simply perijove in units of the Jovian radius. Clearly we require $\kappa > 1$ or else the probe crashes into Jupiter! The probe's energy in this frame is simply $E = \frac{1}{2}mu^2$, which means the probe enters into a hyperbolic orbit about Jupiter. Next, from

$$E = \frac{k}{2}\,\frac{\varepsilon^2 - 1}{\ell^2/\mu k} \tag{2.431}$$

$$\frac{\ell^2}{\mu k} = (1+\varepsilon)\,\kappa R_{\rm J} \tag{2.432}$$

we find

$$\varepsilon_{\rm J} = 1 + \kappa \bigg(\frac{R_{\rm J}}{a_{\rm E}}\bigg)\bigg(\frac{M_\odot}{M_{\rm J}}\bigg)\bigg(\frac{u}{v_{\rm E}}\bigg)^2 \ . \tag{2.433}$$

The opening angle of the Keplerian hyperbola is then $\phi_{\rm c} = \cos^{-1}\big(\varepsilon_{\rm J}^{-1}\big)$, and the angle χ is related to $\phi_{\rm c}$ through

$$\chi = \pi - 2\phi_{\rm c} = \pi - 2\cos^{-1}\bigg(\frac{1}{\varepsilon_{\rm J}}\bigg) \ . \tag{2.434}$$

Therefore, we may finally write

$$v_{\rm f} = \sqrt{x\,v_{\rm E}^2 + u^2 + 2\,u\,v_{\rm E}\sqrt{x}\cos(2\phi_{\rm c} - \gamma)} \tag{2.435}$$

$$\cos\delta = \frac{x\,v_{\rm E}^2 + v_{\rm f}^2 - u^2}{2\,v_{\rm f}\,v_{\rm E}\sqrt{x}} \ . \tag{2.436}$$

2.11.3 III. Jupiter to Neptune

Immediately after undergoing gravitational scattering off Jupiter, the energy and angular momentum of the probe are

$$E = \tfrac{1}{2}mv_{\rm f}^2 - \frac{GM_\odot m}{a_{\rm J}} \tag{2.437}$$

and

$$\ell = \mu\,v_{\rm f}\,a_{\rm J}\cos\delta \ . \tag{2.438}$$

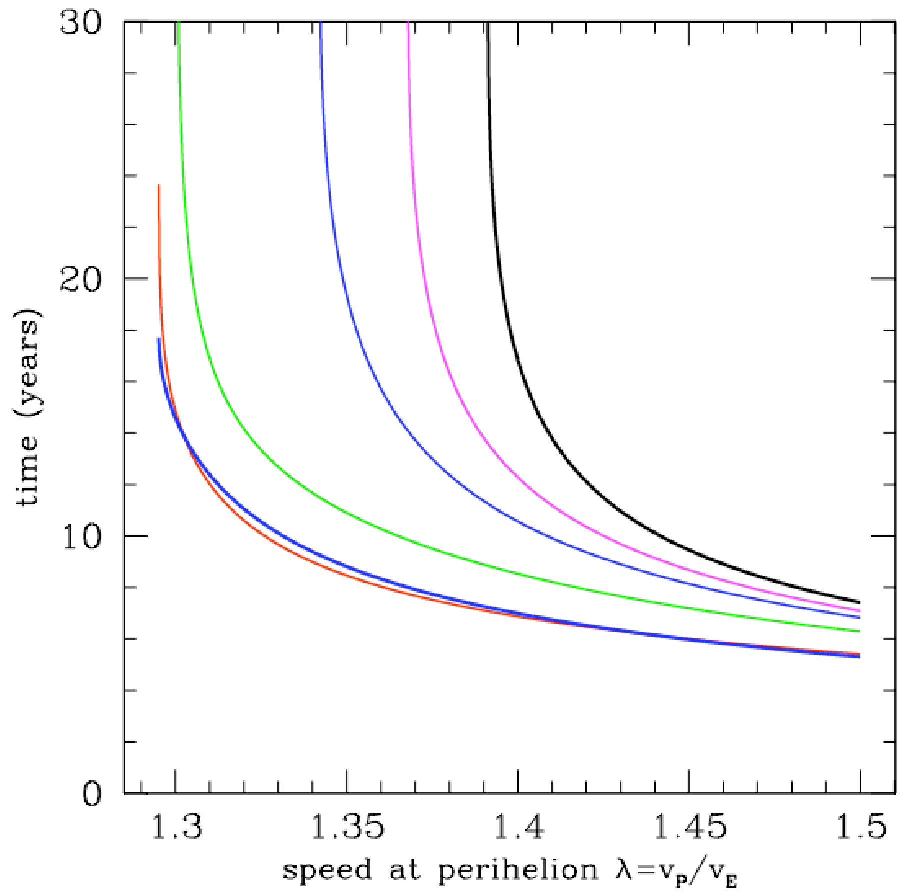

Figure 2.26: Total time for Earth-Neptune mission as a function of dimensionless velocity at perihelion, $\lambda = v_{\rm p}/v_{\rm E}$. Six different values of κ, the value of perijove in units of the Jovian radius, are shown: $\kappa = 1.0$ (thick blue), $\kappa = 5.0$ (red), $\kappa = 20$ (green), $\kappa = 50$ (blue), $\kappa = 100$ (magenta), and $\kappa = \infty$ (thick black).

We write the geometric equation for the probe's orbit as

$$r(\phi) = \frac{d}{1 + \varepsilon \cos(\phi - \phi_{\rm J} - \alpha)} \ , \tag{2.439}$$

where

$$d = \frac{\ell^2}{\mu k} = \left(\frac{v_{\rm f}\, a_{\rm J} \cos\delta}{v_{\rm E}\, a_{\rm E}} \right)^2 a_{\rm E} \ . \tag{2.440}$$

Setting $E = (\mu k^2/2\ell^2)(\varepsilon^2 - 1)$, we obtain the eccentricity

$$\varepsilon = \sqrt{1 + \left(\frac{v_{\rm f}^2}{v_{\rm E}^2} - \frac{2a_{\rm E}}{a_{\rm J}} \right) \frac{d}{a_{\rm E}}} \ . \tag{2.441}$$

Note that the orbit is hyperbolic – the probe will escape the Sun – if $v_{\mathrm{f}} > v_{\mathrm{E}} \cdot \sqrt{2x}$. The condition that this orbit intersect Jupiter at $\phi = \phi_{\mathrm{J}}$ yields

$$\cos\alpha = \frac{1}{\varepsilon}\left(\frac{d}{a_{\mathrm{J}}} - 1\right) , \tag{2.442}$$

which determines the angle α. Interception of Neptune occurs at

$$\frac{d}{1 + \varepsilon\cos(\phi_{\mathrm{N}} - \phi_{\mathrm{J}} - \alpha)} = a_{\mathrm{N}} \qquad \Rightarrow \qquad \phi_{\mathrm{N}} = \phi_{\mathrm{J}} + \alpha + \cos^{-1}\frac{1}{\varepsilon}\left(\frac{d}{a_{\mathrm{N}}} - 1\right) . \tag{2.443}$$

We then have

$$\tau_{\mathrm{JN}} = \tau_{\mathrm{E}} \cdot \left(\frac{d}{a_{\mathrm{E}}}\right)^3 \int\limits_{\phi_{\mathrm{J}}}^{\phi_{\mathrm{N}}} \frac{d\phi}{2\pi} \, \frac{1}{\big[1 + \varepsilon\cos(\phi - \phi_{\mathrm{J}} - \alpha)\big]^2} . \tag{2.444}$$

The total time to Neptune is then the sum,

$$\tau_{\mathrm{EN}} = \tau_{\mathrm{EJ}} + \tau_{\mathrm{JN}} . \tag{2.445}$$

2.12 Conservative Forces in Higher Dimensions

The Lagrangian is $L = \frac{1}{2} m\dot{\boldsymbol{r}}^2 - U(\boldsymbol{r})$, and the equations of motion are

$$m\ddot{\boldsymbol{r}} = -\boldsymbol{\nabla} U(\boldsymbol{r}) \equiv \boldsymbol{F}(\boldsymbol{r}) . \tag{2.446}$$

The work done *on* the particle in moving from $\boldsymbol{r}_a$ to $\boldsymbol{r}_b$ is

$$\begin{aligned} W_{ab} &= \int\limits_{\boldsymbol{r}_a}^{\boldsymbol{r}_b} d\boldsymbol{r} \cdot \boldsymbol{F}(\boldsymbol{r}) \\ &= U\big(\boldsymbol{r}_a\big) - U\big(\boldsymbol{r}_b\big) . \end{aligned} \tag{2.447}$$

Since the total energy $E = T + U$ is conserved,

$$W_{ab} = T_b - T_a , \tag{2.448}$$

and the work done on the system in going from a to b is the change in kinetic energy $\Delta T = T_b - T_a$. Clearly the work done depends only on the endpoints; this is also clear from application of Stokes' theorem to the integral of $\boldsymbol{F} \cdot d\boldsymbol{r}$ around a closed path,

$$\oint\limits_{\mathcal{C}} d\boldsymbol{r} \cdot \boldsymbol{F} = \int\limits_{\mathrm{int}(\mathcal{C})} \boldsymbol{\nabla} \times \boldsymbol{F} \cdot \hat{\boldsymbol{n}} \, d\Sigma = 0 , \tag{2.449}$$

since $\boldsymbol{\nabla} \times \boldsymbol{F} = -\boldsymbol{\nabla} \times \boldsymbol{\nabla} U = 0$. Thus, if γ_1 and γ_2 are two paths connecting $\boldsymbol{r}_a$ to $\boldsymbol{r}_b$, then applying the above result to the closed path $\mathcal{C} = \gamma_2^{-1} \circ \gamma_1$ establishes that the work done along these two paths is the same.

2.12.1 Polar Coordinates in $d = 2$ Dimensions

We have

$$\boldsymbol{r} = \cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}} \qquad \Longrightarrow \qquad d\hat{\boldsymbol{r}} = \hat{\boldsymbol{\phi}}\,d\phi \tag{2.450}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}} \qquad \Longrightarrow \qquad d\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{r}}\,d\phi \ , \tag{2.451}$$

from which we obtain

$$\dot{\boldsymbol{r}} = \frac{d}{dt}\,(r\hat{\boldsymbol{r}}) = \dot{r}\,\hat{\boldsymbol{r}} + r\,\dot{\phi}\,\hat{\boldsymbol{\phi}} \ , \tag{2.452}$$

and hence

$$\begin{aligned} \boldsymbol{a} = \frac{d^2}{dt^2}(r\hat{\boldsymbol{r}}) &= \ddot{r}\,\hat{\boldsymbol{r}} + \dot{r}\,\dot{\hat{\boldsymbol{r}}} + \dot{r}\,\dot{\phi}\,\hat{\boldsymbol{\phi}} + r\,\ddot{\phi}\,\hat{\boldsymbol{\phi}} + r\,\dot{\phi}\,\dot{\hat{\boldsymbol{\phi}}} \\ &= \big(\ddot{r} - r\dot{\phi}^2\big)\,\hat{\boldsymbol{r}} + \big(2\dot{r}\dot{\phi} + r\ddot{\phi}\big)\,\hat{\boldsymbol{\phi}} \\ &= \big(\ddot{r} - r\dot{\phi}^2\big)\,\hat{\boldsymbol{r}} + \frac{1}{r}\frac{d}{dt}\big(r^2\dot{\phi}\big)\hat{\boldsymbol{\phi}} \ . \end{aligned} \tag{2.453}$$

Newton's second law,

$$m\ddot{\boldsymbol{r}} = -\boldsymbol{\nabla} U = -\hat{\boldsymbol{r}}\,\frac{\partial U}{\partial r} - \frac{\hat{\boldsymbol{\phi}}}{r}\,\frac{\partial U}{\partial \phi} \ , \tag{2.454}$$

therefore gives

$$m\big(\ddot{r} - r\dot{\phi}^2\big) = -\frac{\partial U}{\partial r} \tag{2.455}$$

$$m\frac{d}{dt}\big(r^2\dot{\phi}\big) = -\frac{\partial U}{\partial \phi} \ . \tag{2.456}$$

If U is independent of ϕ, the angular momentum $\ell = mr^2\dot{\phi}$ is conserved. Of course, all this follows directly from writing the Lagrangian in polar coordinates,

$$L = \tfrac{1}{2}m\big(\dot{r}^2 + r^2\dot{\phi}^2\big) - U(r,\phi) \ , \tag{2.457}$$

and computing the Euler-Lagrange equations of motion.

Assuming $U(r)$ is a *central potential*, which is to say that it is dependent only on the magnitude r and not on ϕ, we may substitute $\dot{\phi} = \ell/mr^2$ into the equation for $\ddot{r}$, to obtain

$$m\ddot{r} = -\frac{\partial U}{\partial r} + \frac{\ell^2}{2mr^3} \equiv -\frac{\partial U_{\rm eff}}{\partial r} \ , \tag{2.458}$$

where the *effective potential* is defined as

$$U_{\rm eff}(r) = U(r) + \frac{\ell^2}{2mr^2} \ . \tag{2.459}$$

The second term, which diverges as $r \to 0$, is called the *angular momentum barrier*. Multiplying both sides of $m\ddot{r} = -U'_{\rm eff}(r)$ by $\dot{r}$, we have that

$$E = \tfrac{1}{2}m\dot{r}^2 + U_{\rm eff}(r) \tag{2.460}$$

is conserved. We will discuss this case in more detail shortly.

2.13 Systems of Particles

Consider a system of many particles, with

$$T = \tfrac{1}{2} \sum_a m_a \dot{\boldsymbol{r}}_a^2 \tag{2.461}$$

$$U = \sum_a V(\boldsymbol{r}_a) + \sum_{a<b} v\big(|\boldsymbol{r}_a - \boldsymbol{r}_b|\big) \ . \tag{2.462}$$

Here, $V(\boldsymbol{r})$ is the *external* (or one-body) potential, and $v(\boldsymbol{r}-\boldsymbol{r}')$ is the *interparticle* potential, which we assume to be central, depending only on the distance between any pair of particles. The equations of motion are

$$m_a \, \ddot{\boldsymbol{r}}_a = \boldsymbol{F}_a^{(\mathrm{ext})} + \boldsymbol{F}_a^{(\mathrm{int})} \ , \tag{2.463}$$

with

$$\boldsymbol{F}_a^{(\mathrm{ext})} = -\frac{\partial V(\boldsymbol{r}_a)}{\partial \boldsymbol{r}_a} \tag{2.464}$$

$$\boldsymbol{F}_a^{(\mathrm{int})} = -\sum_b \frac{\partial v\big(|\boldsymbol{r}_a - \boldsymbol{r}_b|\big)}{\boldsymbol{r}_a} \equiv \sum_b \boldsymbol{F}_{ab}^{(\mathrm{int})} \ . \tag{2.465}$$

Here, $\boldsymbol{F}_{ab}^{(\mathrm{int})}$ is the force exerted on particle a by particle b:

$$\boldsymbol{F}_{ab}^{(\mathrm{int})} = \frac{\partial v\big(|\boldsymbol{r}_a - \boldsymbol{r}_b|\big)}{\partial \boldsymbol{r}_a} = -\frac{\boldsymbol{r}_a - \boldsymbol{r}_b}{|\boldsymbol{r}_a - \boldsymbol{r}_b|} \, v'\big(|\boldsymbol{r}_a - \boldsymbol{r}_b|\big) \ . \tag{2.466}$$

Note that $\boldsymbol{F}_{ab}^{(\mathrm{int})} = -\boldsymbol{F}_{ba}^{(\mathrm{int})}$, otherwise known as Newton's Third Law. It is convenient to abbreviate $\boldsymbol{r}_{ab} \equiv \boldsymbol{r}_a - \boldsymbol{r}_b$, in which case we may write the interparticle force as $\boldsymbol{F}_{ab}^{(\mathrm{int})} = -\hat{\boldsymbol{r}}_{ab} \, v'\big(r_{ab}\big)$.

Consider now the total momentum of the system, $\boldsymbol{P} = \sum_a \boldsymbol{p}_a$. Its rate of change is

$$\frac{d\boldsymbol{P}}{dt} = \sum_a \dot{\boldsymbol{p}}_a = \sum_a \boldsymbol{F}_a^{(\mathrm{ext})} + \overbrace{\sum_{a\neq b} \boldsymbol{F}_{ab}^{(\mathrm{int})}}^{\boldsymbol{F}_{ab}^{(\mathrm{int})} + \boldsymbol{F}_{ba}^{(\mathrm{int})} = 0} = \boldsymbol{F}_{\mathrm{tot}}^{(\mathrm{ext})} \ , \tag{2.467}$$

since the sum over all internal forces cancels as a result of Newton's Third Law. We write

$$\boldsymbol{P} = \sum_a m_a \dot{\boldsymbol{r}}_a = M\dot{\boldsymbol{R}} \tag{2.468}$$

$$M = \sum_a m_a \quad \text{(total mass)} \tag{2.469}$$

$$\boldsymbol{R} = \frac{\sum_a m_a \boldsymbol{r}_a}{\sum_a m_a} \quad \text{(center-of-mass)} \ . \tag{2.470}$$

Next, consider the total angular momentum,

$$
\begin{aligned}
\boldsymbol{L} &= \sum_a \boldsymbol{r}_a \times \boldsymbol{p}_a = \sum_a m_a \boldsymbol{r}_a \times \dot{\boldsymbol{r}}_a \\
\frac{d\boldsymbol{L}}{dt} &= \sum_a \big\{ m_a \dot{\boldsymbol{r}}_a \times \dot{\boldsymbol{r}}_a + m_a \boldsymbol{r}_a \times \ddot{\boldsymbol{r}}_a \big\} \\
&= \sum_a \boldsymbol{r}_a \times \boldsymbol{F}_a^{(\text{ext})} + \sum_{a \neq b} \boldsymbol{r}_a \times \boldsymbol{F}_{ab}^{(\text{int})} \\
&= \sum_a \boldsymbol{r}_a \times \boldsymbol{F}_a^{(\text{ext})} + \overbrace{\tfrac{1}{2} \sum_{a \neq b} (\boldsymbol{r}_a - \boldsymbol{r}_b) \times \boldsymbol{F}_{ab}^{(\text{int})}}^{\boldsymbol{r}_{ab} \times \boldsymbol{F}_{ab}^{(\text{int})} = 0} \\
&= \boldsymbol{N}_{\text{tot}}^{(\text{ext})} \ .
\end{aligned} \tag{2.471}
$$

Finally, it is useful to establish the result

$$
T = \tfrac{1}{2} \sum_a m_a \dot{\boldsymbol{r}}_a^2 = \tfrac{1}{2} M \dot{\boldsymbol{R}}^2 + \tfrac{1}{2} \sum_a m_a \big(\dot{\boldsymbol{r}}_a - \dot{\boldsymbol{R}} \big)^2 \ , \tag{2.472}
$$

which says that the kinetic energy may be written as a sum of two terms, those being the kinetic energy of the center-of-mass motion, and the kinetic energy of the particles relative to the center-of-mass.

Recall the "work-energy theorem" for conservative systems,

$$
\begin{aligned}
0 &= \int\limits_{\text{initial}}^{\text{final}} dE = \int\limits_{\text{initial}}^{\text{final}} dT + \int\limits_{\text{initial}}^{\text{final}} dU \\
&= T_{\text{f}} - T_{\text{i}} - \sum_a \int\limits_{\boldsymbol{r}_{a,\text{i}}}^{\boldsymbol{r}_{a,\text{f}}} d\boldsymbol{r}_a \cdot \boldsymbol{F}_a
\end{aligned} \tag{2.473}
$$

$$
\Delta T = T_{\text{f}} - T_{\text{i}} = \sum_a \int\limits_{\boldsymbol{r}_{a,\text{i}}}^{\boldsymbol{r}_{a,\text{f}}} d\boldsymbol{r}_a \cdot \boldsymbol{F}_a \ . \tag{2.474}
$$

Note that for continuous systems, we replace

$$
\sum_a m_a \, G(\boldsymbol{r}_a) \longrightarrow \int d^3\!r \, \rho(\boldsymbol{r}) \, G(\boldsymbol{r}) \ , \tag{2.475}
$$

where $\rho(\boldsymbol{r})$ is the mass density, and $G(\boldsymbol{r})$ is any function.

2.14 Mechanical Similarity and Virial Theorem

2.14.1 Mechanical Similarity

In certain cases of interest, the potential is a homogeneous function of the coordinates. This means

$$U\big(\lambda\boldsymbol{r}_1,\ldots,\lambda\boldsymbol{r}_N\big)=\lambda^k\,U\big(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N\big)\ . \tag{2.476}$$

Here, k is the *degree of homogeneity* of U. Familiar examples include gravity,

$$U\big(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N\big)=-G\sum_{a<b}\frac{m_a\,m_b}{|\boldsymbol{r}_a-\boldsymbol{r}_b|}\quad;\quad k=-1\ , \tag{2.477}$$

and the harmonic oscillator,

$$U\big(q_1,\ldots,q_n\big)=\tfrac{1}{2}\sum_{\sigma,\sigma'}\mathrm{V}_{\sigma\sigma'}\,q_\sigma\,q_{\sigma'}\quad;\quad k=+2\ . \tag{2.478}$$

The sum of two homogeneous functions is itself homogeneous only if the component functions themselves are of the same degree of homogeneity.

Now suppose we rescale distances and times, defining

$$\boldsymbol{r}_a=\lambda\boldsymbol{r}'_a\quad\forall a\qquad,\qquad t=\beta t'\ . \tag{2.479}$$

Clearly

$$\frac{d\boldsymbol{r}'_a}{dt'}=\frac{\lambda}{\beta}\,\frac{d\boldsymbol{r}_a}{dt}\ , \tag{2.480}$$

and so

$$\begin{aligned} L&=\tfrac{1}{2}\sum_a m_a\bigg(\frac{d\boldsymbol{r}_a}{dt}\bigg)^2-U\big(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N\big)\\ &=\frac{\lambda^2}{\beta^2}\,\tfrac{1}{2}\sum_a m_a\bigg(\frac{d\boldsymbol{r}'_a}{dt'}\bigg)^2-\lambda^k\,U\big(\boldsymbol{r}'_1,\ldots,\boldsymbol{r}'_N\big)\ . \end{aligned} \tag{2.481}$$

We now demand

$$\frac{\lambda^2}{\beta^2}=\lambda^k\quad\Rightarrow\quad\beta=\lambda^{1-\frac{1}{2}k}\ , \tag{2.482}$$

which yields

$$L\Bigg(\{\boldsymbol{r}_a\},\bigg\{\frac{d\boldsymbol{r}_a}{dt}\bigg\},t\Bigg)=\lambda^k\,L\Bigg(\{\boldsymbol{r}'_a\},\bigg\{\frac{d\boldsymbol{r}'_a}{dt'}\bigg\},t'\Bigg)\ . \tag{2.483}$$

This means that if $r_a(t)$ is a solution to the motion, then so is $\boldsymbol{r}'_a(t')$, *i.e.*we may substitute

$$\boldsymbol{r}_a(t)\longrightarrow\boldsymbol{r}_a(t;\lambda)\equiv\lambda\,\boldsymbol{r}_a\big(\lambda^{\frac{1}{2}k-1}\,t\big) \tag{2.484}$$

to obtain a one-parameter family of solutions.

If $\boldsymbol{r}(t)$ is periodic with period T, the $\boldsymbol{r}_a(t;\lambda)$ is periodic with period $T' = \lambda^{1-\frac{1}{2}k}\,T$. Thus,

$$\left(\frac{T'}{T}\right) = \left(\frac{L'}{L}\right)^{1-\frac{1}{2}k} . \tag{2.485}$$

Here, $\lambda = L'/L$ is the ratio of length scales. Velocities, energies and angular momenta scale accordingly:

$$\big[v\big] = \frac{L}{T} \qquad \Rightarrow \qquad \frac{v'}{v} = \frac{L'}{L}\Big/\frac{T'}{T} = \lambda^{\frac{1}{2}k} \tag{2.486}$$

$$\big[E\big] = \frac{ML^2}{T^2} \qquad \Rightarrow \qquad \frac{E'}{E} = \left(\frac{L'}{L}\right)^2\Big/\left(\frac{T'}{T}\right)^2 = \lambda^k \tag{2.487}$$

$$\big[\boldsymbol{L}\big] = \frac{ML^2}{T} \qquad \Rightarrow \qquad \frac{|\boldsymbol{L}'|}{|\boldsymbol{L}|} = \left(\frac{L'}{L}\right)^2\Big/\frac{T'}{T} = \lambda^{(1+\frac{1}{2}k)} . \tag{2.488}$$

As examples, consider:

(i) *Harmonic Oscillator* : Here $k = 2$ and therefore

$$q_\sigma(t) \longrightarrow q_\sigma(t;\lambda) = \lambda\, q_\sigma(t) . \tag{2.489}$$

Thus, rescaling lengths alone gives another solution.

(ii) *Kepler Problem* : This is gravity, for which $k = -1$. Thus,

$$\boldsymbol{r}(t) \longrightarrow \boldsymbol{r}(t;\lambda) = \lambda\,\boldsymbol{r}\big(\lambda^{-3/2}t\big) . \tag{2.490}$$

Thus, $r^3 \propto t^2$, *i.e.*

$$\left(\frac{L'}{L}\right)^3 = \left(\frac{T'}{T}\right)^2 , \tag{2.491}$$

also known as Kepler's Third Law.

2.14.2 Virial Theorem

The virial theorem is a statement about the time-averaged motion of a mechanical system. Define the *virial*,

$$G(q,p) = \sum_\sigma p_\sigma\, q_\sigma . \tag{2.492}$$

Then

$$\begin{aligned} \frac{dG}{dt} &= \sum_\sigma \big(\dot p_\sigma\, q_\sigma + p_\sigma\, \dot q_\sigma\big) \\ &= \sum_\sigma q_\sigma\, F_\sigma + \sum_\sigma \dot q_\sigma\, \frac{\partial L}{\partial \dot q_\sigma} . \end{aligned} \tag{2.493}$$

Now suppose that $T = \frac{1}{2}\sum_{\sigma,\sigma'} \mathrm{T}_{\sigma\sigma'}\, \dot{q}_\sigma \dot{q}_{\sigma'}$ is homogeneous of degree $k = 2$ in $\dot{q}$, and that U is homogeneous of degree zero in $\dot{q}$. Then

$$\sum_\sigma \dot{q}_\sigma \, \frac{\partial L}{\partial \dot{q}_\sigma} = \sum_\sigma \dot{q}_\sigma \, \frac{\partial T}{\partial \dot{q}_\sigma} = 2\,T, \tag{2.494}$$

which follows from Euler's theorem on homogeneous functions:

$$f\big(\lambda\, x_1, \ldots, \lambda\, x_N\big) = \lambda^k \, f\big(x_1, \ldots, x_N\big) \tag{2.495}$$

$$\Rightarrow \qquad \sum_{i=1}^N x_i \, \frac{\partial}{\partial x_i} \, f\big(x_1, \ldots, x_N\big) = k \, f\big(x_1, \ldots, x_N\big) \; . \tag{2.496}$$

Now consider the time average of $\dot{G}$ over a period τ:

$$\begin{aligned} \left\langle \frac{dG}{dt} \right\rangle &= \frac{1}{\tau} \int\limits_0^\tau \! dt \; \frac{dG}{dt} \\ &= \frac{1}{\tau} \Big[G(\tau) - G(0) \Big] \; . \end{aligned} \tag{2.497}$$

If $G(t)$ is bounded, then in the limit $\tau \to \infty$ we must have $\langle \dot{G} \rangle = 0$. Any bounded motion, such as the orbit of the earth around the Sun, will result in $\langle \dot{G} \rangle_{\tau\to\infty} = 0$. But then

$$\left\langle \frac{dG}{dt} \right\rangle = 2 \, \langle T \rangle + \Big\langle \sum_\sigma q_\sigma \, F_\sigma \Big\rangle = 0 \; , \tag{2.498}$$

which implies

$$\begin{aligned} \langle T \rangle &= -\tfrac{1}{2} \Big\langle \sum_\sigma q_\sigma \, F_\sigma \Big\rangle = + \Big\langle \tfrac{1}{2} \sum_\sigma q_\sigma \, \frac{\partial U}{\partial q_\sigma} \Big\rangle \\ &= \Big\langle \tfrac{1}{2} \sum_a \boldsymbol{r}_a \cdot \boldsymbol{\nabla}_{\!a} U\big(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N\big) \Big\rangle && (2.499) \\ &= \tfrac{1}{2} k \, \langle U \rangle \; , && (2.500) \end{aligned}$$

where the last line pertains to homogeneous potentials of degree k. Finally, since $T + U = E$ is conserved, we have

$$\langle T \rangle = \frac{k\,E}{k+2} \quad , \quad \langle U \rangle = \frac{2\,E}{k+2} \; . \tag{2.501}$$

2.15 Elastic Collisions

A collision or 'scattering event' is said to be *elastic* if it results in no change in the internal state of any of the particles involved. Thus, no internal energy is liberated or captured in an

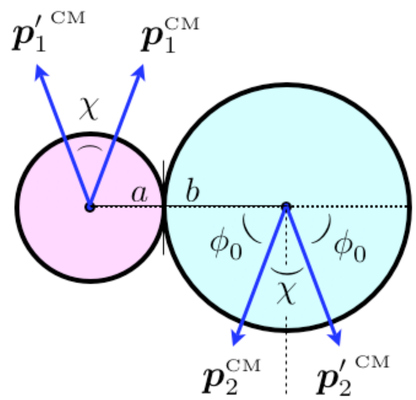

Figure 2.27: The scattering of two hard spheres of radii a and b The scattering angle is χ.

elastic process. Consider the elastic scattering of two particles. Recall the relation between laboratory coordinates $\{\boldsymbol{r}_1, \boldsymbol{r}_2\}$ and the CM and relative coordinates $\{\boldsymbol{R}, \boldsymbol{r}\}$:

$$\boldsymbol{R} = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{m_1 + m_2} \qquad\qquad \boldsymbol{r}_1 = \boldsymbol{R} + \frac{m_2}{m_1 + m_2}\,\boldsymbol{r} \tag{2.502}$$

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 \qquad\qquad \boldsymbol{r}_2 = \boldsymbol{R} - \frac{m_1}{m_1 + m_2}\,\boldsymbol{r} \tag{2.503}$$

If external forces are negligible, the CM momentum $\boldsymbol{P} = M\dot{\boldsymbol{R}}$ is constant, and therefore the frame of reference whose origin is tied to the CM position is an inertial frame of reference. In this frame,

$$\boldsymbol{v}_1^{\mathrm{CM}} = \frac{m_2\,\boldsymbol{v}}{m_1 + m_2} \quad , \quad \boldsymbol{v}_2^{\mathrm{CM}} = -\frac{m_1\,\boldsymbol{v}}{m_1 + m_2} \ , \tag{2.504}$$

where $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2 = \boldsymbol{v}_1^{\mathrm{CM}} - \boldsymbol{v}_2^{\mathrm{CM}}$ is the relative velocity, which is the same in both L and CM frames. Note that the CM momenta satisfy

$$\boldsymbol{p}_1^{\mathrm{CM}} = m_1 \boldsymbol{v}_1^{\mathrm{CM}} = \mu \boldsymbol{v} \tag{2.505}$$

$$\boldsymbol{p}_2^{\mathrm{CM}} = m_2 \boldsymbol{v}_2^{\mathrm{CM}} = -\mu \boldsymbol{v} \ , \tag{2.506}$$

where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass. Thus, $\boldsymbol{p}_1^{\mathrm{CM}} + \boldsymbol{p}_2^{\mathrm{CM}} = 0$ and the total momentum in the CM frame is zero. We may then write

$$\boldsymbol{p}_1^{\mathrm{CM}} \equiv p_0 \hat{\boldsymbol{n}} \quad , \quad \boldsymbol{p}_2^{\mathrm{CM}} \equiv -p_0 \hat{\boldsymbol{n}} \quad \Rightarrow \quad E^{\mathrm{CM}} = \frac{p_0^2}{2m_1} + \frac{p_0^2}{2m_2} = \frac{p_0^2}{2\mu} \ . \tag{2.507}$$

The energy is evaluated when the particles are asymptotically far from each other, in which case the potential energy is assumed to be negligible. After the collision, energy and momentum conservation require

$$\boldsymbol{p}_1'^{\,\mathrm{CM}} \equiv p_0 \hat{\boldsymbol{n}}' \quad , \quad \boldsymbol{p}_2'^{\,\mathrm{CM}} \equiv -p_0 \hat{\boldsymbol{n}}' \quad \Rightarrow \quad E'^{\,\mathrm{CM}} = E^{\mathrm{CM}} = \frac{p_0^2}{2\mu} \ . \tag{2.508}$$

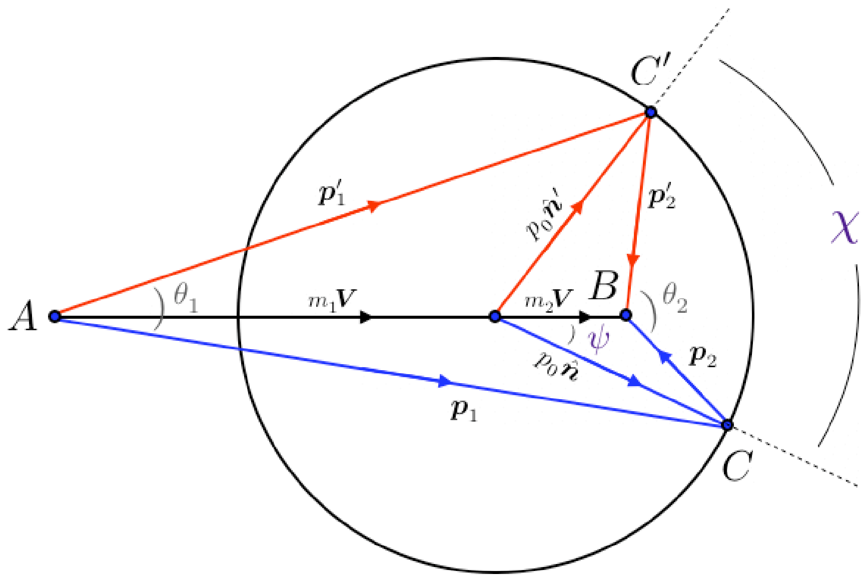

Figure 2.28: Scattering of two particles of masses m_1 and m_2. The scattering angle χ is the angle between $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}}'$.

The angle between $\boldsymbol{n}$ and $\boldsymbol{n}'$ is the *scattering angle* χ:

$$\boldsymbol{n} \cdot \boldsymbol{n}' \equiv \cos\chi \ . \tag{2.509}$$

The value of χ depends on the details of the scattering process, *i.e.* on the interaction potential $U(r)$. As an example, consider the scattering of two hard spheres, depicted in Fig. 2.27. The potential is

$$U(r) = \begin{cases} \infty & \text{if } r \le a+b \\ 0 & \text{if } r > a+b \ . \end{cases} \tag{2.510}$$

Clearly the scattering angle is $\chi = \pi - 2\eta$, where η is the angle between the initial momentum of either sphere and a line containing their two centers at the moment of contact.

There is a simple geometric interpretation of these results, depicted in Fig. 2.28. We have

$$\boldsymbol{p}_1 = m_1\boldsymbol{V} + p_0\hat{\boldsymbol{n}} \qquad\qquad \boldsymbol{p}_1' = m_1\boldsymbol{V} + p_0\hat{\boldsymbol{n}}' \tag{2.511}$$

$$\boldsymbol{p}_2 = m_2\boldsymbol{V} - p_0\hat{\boldsymbol{n}} \qquad\qquad \boldsymbol{p}_2' = m_2\boldsymbol{V} - p_0\hat{\boldsymbol{n}}' \ . \tag{2.512}$$

So draw a circle of radius p_0 whose center is the origin. The vectors $p_0\hat{\boldsymbol{n}}$ and $p_0\hat{\boldsymbol{n}}'$ must both lie along this circle. We define the angle ψ between $\boldsymbol{V}$ and $\boldsymbol{n}$:

$$\hat{\boldsymbol{V}} \cdot \boldsymbol{n} = \cos\psi \ . \tag{2.513}$$

It is now an exercise in geometry, using the law of cosines, to determine everything of

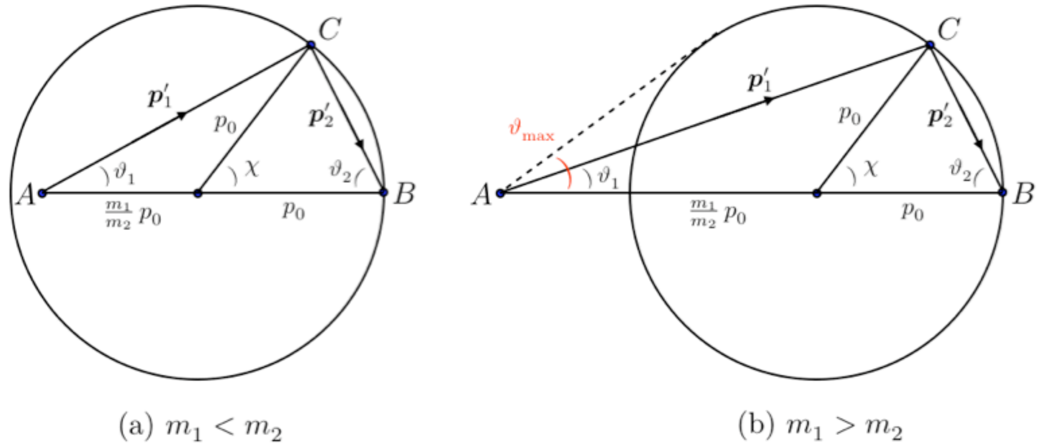

Figure 2.29: Scattering when particle 2 is initially at rest.

interest in terms of the quantities V, v, ψ, and χ. For example, the momenta are

$$p_1 = \sqrt{m_1^2\,V^2 + \mu^2 v^2 + 2 m_1 \mu V v \cos\psi} \tag{2.514}$$

$$p_1' = \sqrt{m_1^2\,V^2 + \mu^2 v^2 + 2 m_1 \mu V v \cos(\chi - \psi)} \tag{2.515}$$

$$p_2 = \sqrt{m_2^2\,V^2 + \mu^2 v^2 - 2 m_2 \mu V v \cos\psi} \tag{2.516}$$

$$p_2' = \sqrt{m_2^2\,V^2 + \mu^2 v^2 - 2 m_2 \mu V v \cos(\chi - \psi)}\ , \tag{2.517}$$

and the scattering angles are

$$\theta_1 = \tan^{-1}\!\left(\frac{\mu v \sin\psi}{\mu v \cos\psi + m_1 V}\right) + \tan^{-1}\!\left(\frac{\mu v \sin(\chi-\psi)}{\mu v \cos(\chi-\psi) + m_1 V}\right) \tag{2.518}$$

$$\theta_2 = \tan^{-1}\!\left(\frac{\mu v \sin\psi}{\mu v \cos\psi - m_2 V}\right) + \tan^{-1}\!\left(\frac{\mu v \sin(\chi-\psi)}{\mu v \cos(\chi-\psi) - m_2 V}\right)\ . \tag{2.519}$$

If particle 2, say, is initially at rest, the situation is somewhat simpler. In this case, $\boldsymbol{V} = m_1\boldsymbol{V}/(m_1+m_2)$ and $m_2\boldsymbol{V} = \mu\boldsymbol{v}$, which means the point B lies on the circle in Fig. 2.29 ($m_1 \neq m_2$) and Fig. 2.30 ($m_1 = m_2$). Let $\vartheta_{1,2}$ be the angles between the directions of motion after the collision and the direction $\boldsymbol{V}$ of impact. The scattering angle χ is the angle through which particle 1 turns in the CM frame. Clearly

$$\tan\vartheta_1 = \frac{\sin\chi}{\frac{m_1}{m_2} + \cos\chi} \qquad , \qquad \vartheta_2 = \tfrac{1}{2}(\pi - \chi)\ . \tag{2.520}$$

We can also find the speeds v_1' and v_2' in terms of v and χ, from

$$p_1'^{\,2} = p_0^2 + \left(\tfrac{m_1}{m_2}\,p_0\right)^2 - 2\,\tfrac{m_1}{m_2}\,p_0^2\,\cos(\pi-\chi) \tag{2.521}$$

and

$$p_2^2 = 2\,p_0^2\,(1-\cos\chi)\ . \tag{2.522}$$

These equations yield

$$v_1' = \frac{\sqrt{m_1^2+m_2^2+2m_1m_2\cos\chi}}{m_1+m_2}\,v \qquad , \qquad v_2' = \frac{2m_1v}{m_1+m_2}\,\sin(\tfrac{1}{2}\chi)\ . \tag{2.523}$$

The angle $\vartheta_{\rm max}$ from Fig. 2.29(b) is given by $\sin\vartheta_{\rm max} = \frac{m_2}{m_1}$. Note that when $m_1 = m_2$ we have $\vartheta_1 + \vartheta_2 = \pi$. A sketch of the orbits in the cases of both repulsive and attractive scattering, in both the laboratory and CM frames, in shown in Fig. 2.31.

2.15.1 Central Force Scattering

Consider a single particle of mass μ movng in a central potential $U(r)$, or a two body central force problem in which μ is the reduced mass. Recall that

$$\frac{dr}{dt} = \frac{d\phi}{dt}\cdot\frac{dr}{d\phi} = \frac{\ell}{\mu r^2}\cdot\frac{dr}{d\phi}\ , \tag{2.524}$$

and therefore

$$\begin{aligned} E &= \tfrac{1}{2}\mu\dot r^2 + \frac{\ell^2}{2\mu r^2} + U(r) \\ &= \frac{\ell^2}{2\mu r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{\ell^2}{2\mu r^2} + U(r)\ . \end{aligned} \tag{2.525}$$

Solving for $\frac{dr}{d\phi}$, we obtain

$$\frac{dr}{d\phi} = \pm\sqrt{\frac{2\mu r^4}{\ell^2}\big(E-U(r)\big) - r^2}\ , \tag{2.526}$$

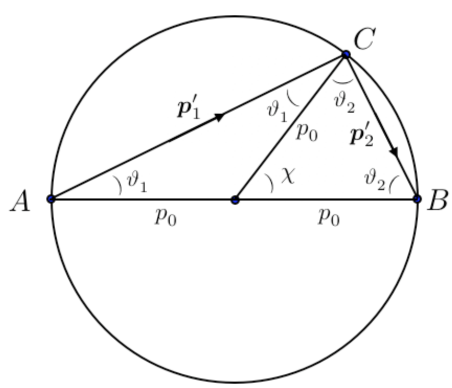

Figure 2.30: Scattering of identical mass particles when particle 2 is initially at rest.

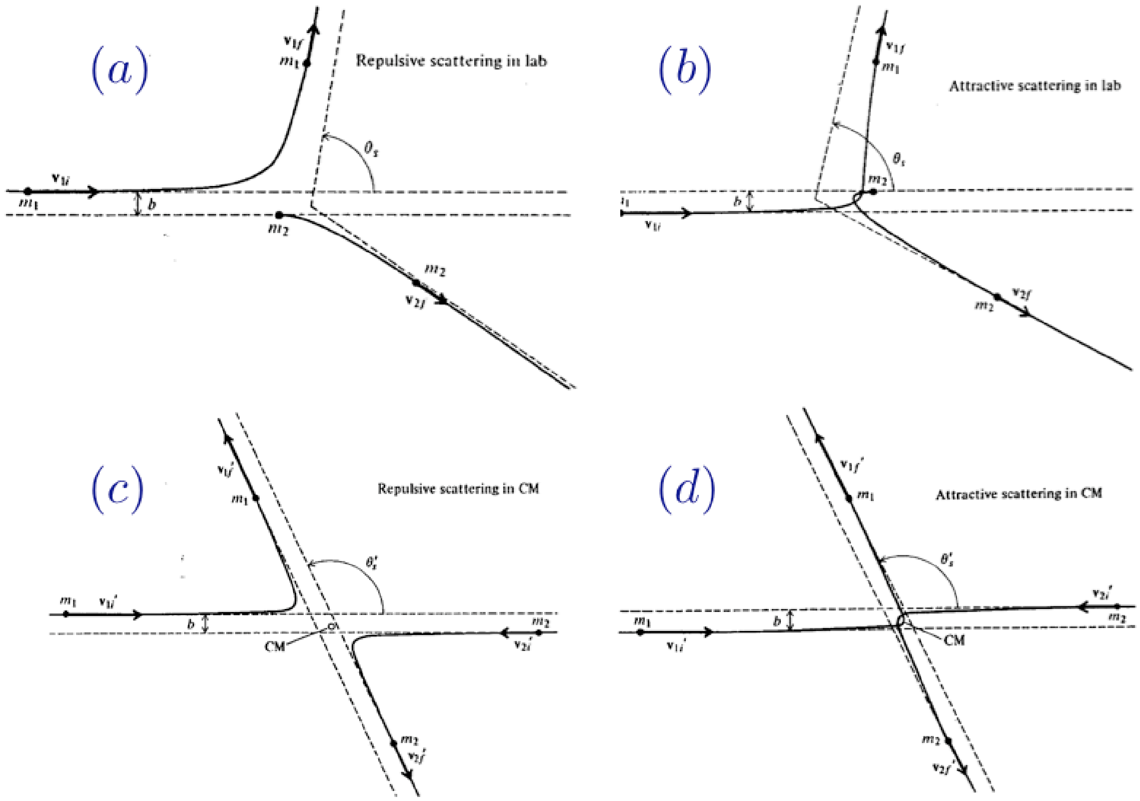

Figure 2.31: Repulsive (A,C) and attractive (B,D) scattering in the lab (A,B) and CM (C,D) frames, assuming particle 2 starts from rest in the lab frame.

Consulting Fig. 2.32, we have that

$$\phi_0 = \frac{\ell}{\sqrt{2\mu}} \int\limits_{r_{\rm p}}^{\infty} \frac{dr}{r^2\sqrt{E - U_{\rm eff}(r)}} \ , \tag{2.527}$$

where $r_{\rm p}$ is the radial distance at periapsis, and where

$$U_{\rm eff}(r) = \frac{\ell^2}{2\mu r^2} + U(r) \tag{2.528}$$

is the effective potential, as before. From Fig. 2.32, we conclude that the scattering angle is

$$\chi = \big|\pi - 2\phi_0\big| \ . \tag{2.529}$$

It is convenient to define the *impact parameter* b as the distance of the asymptotic trajectory from a parallel line containing the force center. The geometry is shown again in Fig. 2.32. Note that the energy and angular momentum, which are conserved, can be evaluated at infinity using the impact parameter:

$$E = \tfrac{1}{2}\mu v_\infty^2 \qquad , \qquad \ell = \mu v_\infty b \ . \tag{2.530}$$

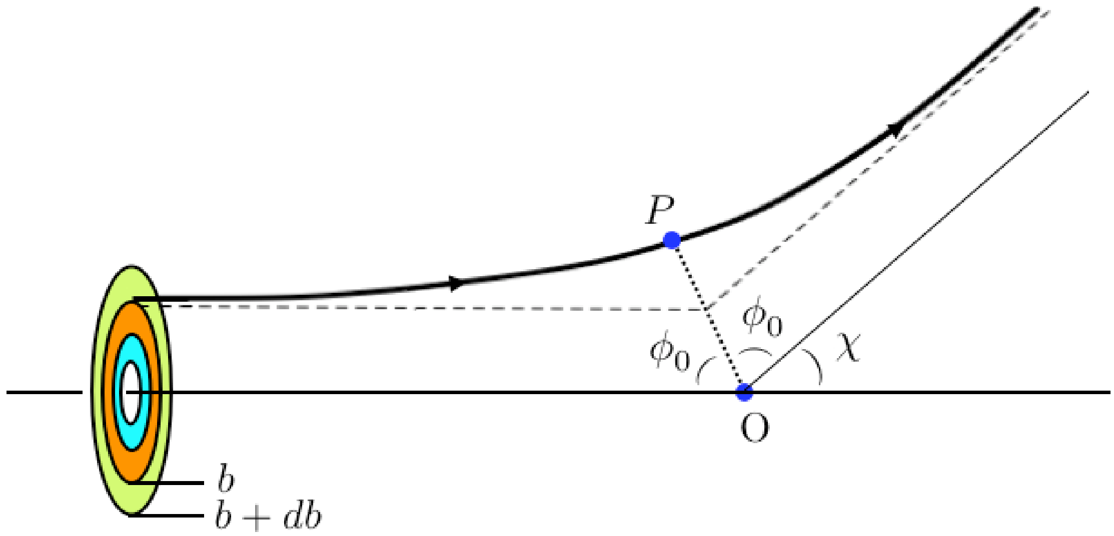

Figure 2.32: Scattering in the CM frame. O is the force center and P is the point of periapsis. The impact parameter is b, and χ is the scattering angle. ϕ_0 is the angle through which the relative coordinate moves between periapsis and infinity.

Substituting for $\ell(b)$, we have

$$\phi_0(E,b) = \int\limits_{r_{\rm p}}^{\infty} \frac{dr}{r^2} \, \frac{b}{\sqrt{1 - \frac{b^2}{r^2} - \frac{U(r)}{E}}} \ , \tag{2.531}$$

In physical applications, we are often interested in the deflection of a beam of incident particles by a scattering center. We define the *differential scattering cross section* $d\sigma$ by

$$d\sigma = \frac{\#\text{ of particles scattered into solid angle } d\Omega \text{ per unit time}}{\text{incident flux}} \ . \tag{2.532}$$

Now for particles of a given energy E there is a unique relationship between the scattering angle χ and the impact parameter b, as we have just derived in eqn. 2.531. The differential solid angle is given by $d\Omega = 2\pi \sin\chi \, d\chi$, hence

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\!\chi} \left| \frac{db}{d\chi} \right| = \left| \frac{d\,(\frac{1}{2}b^2)}{d\cos\chi} \right| \ . \tag{2.533}$$

Note that $\frac{d\sigma}{d\Omega}$ has dimensions of area. The integral of $\frac{d\sigma}{d\Omega}$ over all solid angle is the *total scattering cross section*,

$$\sigma_{\rm T} = 2\pi \int\limits_0^{\pi} \! d\chi \ \sin\!\chi \, \frac{d\sigma}{d\Omega} \ . \tag{2.534}$$

Let's now work through some examples.

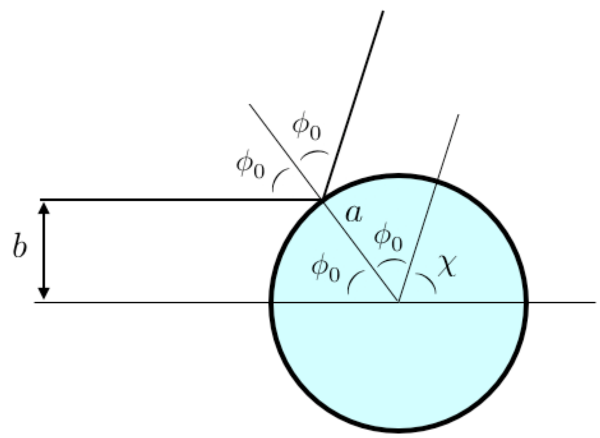

Figure 2.33: Geometry of hard sphere scattering.

Example #1 : Hard Sphere Scattering – Consider a point particle scattering off a hard sphere of radius a, or two hard spheres of radii a_1 and a_2 scattering off each other, with $a \equiv a_1 + a_2$. From the geometry of Fig. 2.33, we have $b = a \sin\phi_0$ and $\phi_0 = \frac{1}{2}(\pi - \chi)$, so

$$b^2 = a^2 \sin^2\left(\tfrac{1}{2}\pi - \tfrac{1}{2}\chi\right) = \tfrac{1}{2}a^2\,(1+\cos\chi)\ . \tag{2.535}$$

We therefore have

$$\frac{d\sigma}{d\Omega} = \frac{d\,(\frac{1}{2}b^2)}{d\cos\chi} = \tfrac{1}{4}\,a^2 \tag{2.536}$$

and $\sigma_{\rm T} = \pi a^2$. The total scattering cross section is simply the area of a sphere of radius a projected onto a plane perpendicular to the incident flux.

Example #2 : Rutherford Scattering – Consider scattering by the Kepler potential $U(r) = -\frac{k}{r}$. We assume that the orbits are unbound, *i.e.* they are Keplerian hyperbolae with $E > 0$, described by the equation

$$r(\phi) = \frac{a\,(\varepsilon^2-1)}{\pm 1 + \varepsilon\,\cos\phi} \quad\Rightarrow\quad \cos\phi_0 = \pm\,\frac{1}{\varepsilon}\ . \tag{2.537}$$

Recall that the eccentricity is given by

$$\varepsilon^2 = 1 + \frac{2E\ell^2}{\mu k^2} = 1 + \left(\frac{\mu b v_\infty}{k}\right)^2 . \tag{2.538}$$

We then have

$$\begin{aligned}\left(\frac{\mu b v_\infty}{k}\right)^2 &= \varepsilon^2 - 1\\ &= \sec^2\!\phi_0 - 1 = \tan^2\!\phi_0 = {\rm ctn}^2\big(\tfrac{1}{2}\chi\big)\ .\end{aligned} \tag{2.539}$$

Therefore

$$b(\chi) = \frac{k}{\mu v_\infty^2}\,{\rm ctn}\big(\tfrac{1}{2}\chi\big) \tag{2.540}$$

We finally obtain

$$\frac{d\sigma}{d\Omega} = \frac{d\left(\frac{1}{2}b^2\right)}{d\cos\chi} = \frac{1}{2}\left(\frac{k}{\mu v_\infty^2}\right)^2 \frac{d\,\mathrm{ctn}^2\left(\frac{1}{2}\chi\right)}{d\cos\chi}$$
$$= \frac{1}{2}\left(\frac{k}{\mu v_\infty^2}\right)^2 \frac{d}{d\cos\chi}\left(\frac{1+\cos\chi}{1-\cos\chi}\right)$$
$$= \left(\frac{k}{2\mu v_\infty^2}\right)^2 \csc^4\left(\tfrac{1}{2}\chi\right) , \tag{2.541}$$

which is the same as

$$\frac{d\sigma}{d\Omega} = \left(\frac{k}{4E}\right)^2 \csc^4\left(\tfrac{1}{2}\chi\right) . \tag{2.542}$$

Since $\frac{d\sigma}{d\Omega} \propto \chi^{-4}$ as $\chi \to 0$, the total cross section σ_{T} diverges! This is a consequence of the long-ranged nature of the Kepler/Coulomb potential. In electron-atom scattering, the Coulomb potential of the nucleus is *screened* by the electrons of the atom, and the $1/r$ behavior is cut off at large distances.

2.15.2 Transformation to Laboratory Coordinates

We previously derived the relation

$$\tan\vartheta = \frac{\sin\chi}{\gamma + \cos\chi} , \tag{2.543}$$

where $\vartheta \equiv \vartheta_1$ is the scattering angle for particle 1 in the laboratory frame, and $\gamma = \frac{m_1}{m_2}$ is the ratio of the masses. We now derive the differential scattering cross section in the laboratory frame. To do so, we note that particle conservation requires

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{L}} \cdot\, 2\pi \sin\vartheta\, d\vartheta = \left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{CM}} \cdot\, 2\pi\sin\chi\, d\chi , \tag{2.544}$$

which says

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{L}} = \left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{CM}} \cdot\, \frac{d\cos\chi}{d\cos\vartheta} . \tag{2.545}$$

From

$$\cos\vartheta = \frac{1}{\sqrt{1+\tan^2\!\vartheta}}$$
$$= \frac{\gamma+\cos\chi}{\sqrt{1+\gamma^2+2\gamma\cos\chi}} , \tag{2.546}$$

we derive

$$\frac{d\cos\vartheta}{d\cos\chi} = \frac{1+\gamma\cos\chi}{\left(1+\gamma^2+2\gamma\cos\chi\right)^{3/2}} \tag{2.547}$$

and, accordingly,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{L}} = \frac{\left(1+\gamma^2+2\gamma\cos\chi\right)^{3/2}}{1+\gamma\cos\chi} \cdot \left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{CM}} . \tag{2.548}$$

2.15.3 Small Angle Scattering

If the potential $U(r)$ is weak, the angle of deflection is small. Working in the laboratory frame, we then have

$$\vartheta_1 \approx \frac{\Delta p_y}{m_1 v_\infty} \tag{2.549}$$

with

$$\Delta p_y = \int\limits_{-\infty}^{\infty} dt\, F_y \approx -\int\limits_{-\infty}^{\infty} \frac{dx}{v_\infty}\, \frac{b}{r}\, \frac{dU}{dr} \ , \tag{2.550}$$

valid to lowest order in U. Thus,

$$\vartheta_1 = \frac{-2b}{m_1 v_\infty^2} \int\limits_b^\infty dr\, \frac{U'(r)}{\sqrt{r^2 - b^2}} \ , \tag{2.551}$$

which may be inverted to yield $\vartheta_1(b)$, thus giving

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm L} = \left| \frac{b(\vartheta_1)}{\vartheta_1} \cdot \frac{db}{d\vartheta_1} \right| \ . \tag{2.552}$$

2.16 Accelerated Coordinate Systems

A reference frame which is fixed with respect to a rotating rigid body is not inertial. The parade example of this is an observer fixed on the surface of the earth. Due to the rotation of the earth, such an observer is in a noninertial frame, and there are corresponding corrections to Newton's laws of motion which must be accounted for in order to correctly describe mechanical motion in the observer's frame. As is well known, these corrections involve fictitious centrifugal and Coriolis forces.

Consider an inertial frame with a fixed set of coordinate axes $\hat{\mathbf{e}}_\mu$, where μ runs from 1 to d, the dimension of space. Any vector $\boldsymbol{A}$ may be written in either basis:

$$\boldsymbol{A} = \sum_\mu A_\mu\, \hat{\mathbf{e}}_\mu = \sum_\mu A'_\mu\, \hat{\mathbf{e}}'_\mu \ , \tag{2.553}$$

where $A_\mu = \boldsymbol{A} \cdot \hat{\mathbf{e}}_\mu$ and $A'_\mu = \boldsymbol{A} \cdot \hat{\mathbf{e}}'_\mu$ are projections onto the different coordinate axes. We may now write

$$\begin{aligned} \left(\frac{d\boldsymbol{A}}{dt}\right)_{\rm inertial} &= \sum_\mu \frac{dA_\mu}{dt}\, \hat{\mathbf{e}}_\mu \\ &= \sum_i \frac{dA'_\mu}{dt}\, \hat{\mathbf{e}}'_\mu + \sum_i A'_\mu\, \frac{d\hat{\mathbf{e}}'_\mu}{dt} \ . \end{aligned} \tag{2.554}$$

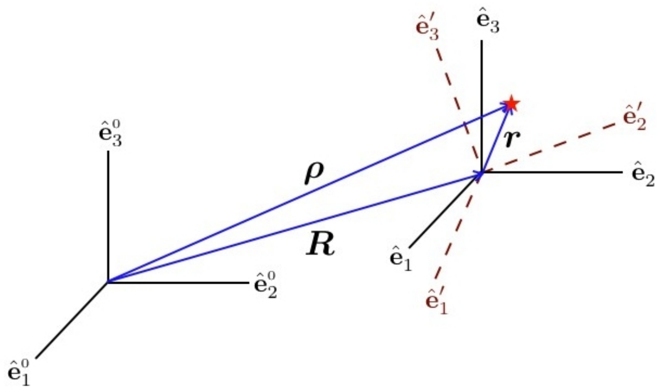

Figure 2.34: Reference frames related by both translation and rotation.

The first term on the RHS is $(d\boldsymbol{A}/dt)_{\text{body}}$, the time derivative of $\boldsymbol{A}$ along body-fixed axes, *i.e.* as seen by an observer rotating with the body. But what is $d\hat{\mathbf{e}}_i'/dt$? Well, we can always expand it in the $\{\hat{\mathbf{e}}_i'\}$ basis:

$$d\hat{\mathbf{e}}_\mu' = \sum_j d\Omega_{\mu\nu}\, \hat{\mathbf{e}}_\nu' \quad \Longleftrightarrow \quad d\Omega_{\mu\nu} \equiv d\hat{\mathbf{e}}_i' \cdot \hat{\mathbf{e}}_\nu' \ . \tag{2.555}$$

Note that $d\Omega_{\mu\nu} = -d\Omega_{\nu\mu}$ is antisymmetric, because

$$0 = d\big(\hat{\mathbf{e}}_\mu' \cdot \hat{\mathbf{e}}_\nu'\big) = d\Omega_{\nu\mu} + d\Omega_{\mu\nu} \ , \tag{2.556}$$

because $\hat{\mathbf{e}}_\mu' \cdot \hat{\mathbf{e}}_\nu' = \delta_{\mu\nu}$ is a constant. Now we may define $d\Omega_{12} \equiv d\Omega_3$, *et cyc.*, so that

$$d\Omega_{\mu\nu} = \epsilon_{\mu\nu\sigma}\, d\Omega_\sigma \quad , \quad \omega_\sigma \equiv \frac{d\Omega_\sigma}{dt} \ , \tag{2.557}$$

which yields

$$\frac{d\hat{\mathbf{e}}_\mu'}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}_\mu' \ . \tag{2.558}$$

Finally, we obtain the important result

$$\boxed{\left(\frac{d\boldsymbol{A}}{dt}\right)_{\text{inertial}} = \left(\frac{d\boldsymbol{A}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \boldsymbol{A}} \tag{2.559}$$

which is valid for any vector $\boldsymbol{A}$.

Applying this result to the position vector $\boldsymbol{r}$, we have

$$\left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{inertial}} = \left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \boldsymbol{r} \ . \tag{2.560}$$

Applying it twice,

$$\begin{aligned}\left(\frac{d^2\boldsymbol{r}}{dt^2}\right)_{\text{inertial}} &= \left(\frac{d}{dt}\bigg|_{\text{body}} + \boldsymbol{\omega}\times\right)\left(\frac{d}{dt}\bigg|_{\text{body}} + \boldsymbol{\omega}\times\right)\boldsymbol{r} \\ &= \left(\frac{d^2\boldsymbol{r}}{dt^2}\right)_{\text{body}} + \frac{d\boldsymbol{\omega}}{dt}\times\boldsymbol{r} + 2\,\boldsymbol{\omega}\times\left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{body}} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})\ .\end{aligned} \tag{2.561}$$

Note that $d\boldsymbol{\omega}/dt$ appears with no "inertial" or "body" label. This is because, upon invoking eq. 2.559,

$$\left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{inertial}} = \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{body}} + \boldsymbol{\omega}\times\boldsymbol{\omega}\ , \tag{2.562}$$

and since $\boldsymbol{\omega}\times\boldsymbol{\omega} = 0$, inertial and body-fixed observers will agree on the value of $\dot{\boldsymbol{\omega}}_{\text{inertial}} = \dot{\boldsymbol{\omega}}_{\text{body}} \equiv \dot{\boldsymbol{\omega}}$.

2.16.1 Translations

Suppose that frame K moves with respect to an inertial frame K^0, such that the origin of K lies at $\boldsymbol{R}(t)$. Suppose further that frame K' rotates with respect to K, but shares the same origin (see Fig. 2.34). Consider the motion of an object lying at position $\boldsymbol{\rho}$ relative to the origin of K^0, and $\boldsymbol{r}$ relative to the origin of K/K'. Thus,

$$\boldsymbol{\rho} = \boldsymbol{R} + \boldsymbol{r}\ , \tag{2.563}$$

and

$$\left(\frac{d\boldsymbol{\rho}}{dt}\right)_{\text{inertial}} = \left(\frac{d\boldsymbol{R}}{dt}\right)_{\text{inertial}} + \left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{body}} + \boldsymbol{\omega}\times\boldsymbol{r} \tag{2.564}$$

$$\begin{aligned}\left(\frac{d^2\boldsymbol{\rho}}{dt^2}\right)_{\text{inertial}} &= \left(\frac{d^2\boldsymbol{R}}{dt^2}\right)_{\text{inertial}} + \left(\frac{d^2\boldsymbol{r}}{dt^2}\right)_{\text{body}} + \frac{d\boldsymbol{\omega}}{dt}\times\boldsymbol{r} \\ &\qquad + 2\,\boldsymbol{\omega}\times\left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{body}} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})\ .\end{aligned} \tag{2.565}$$

Here, $\boldsymbol{\omega}$ is the angular velocity in the frame K or K'.

2.16.2 Motion on the Surface of the earth

The earth both rotates about its axis and orbits the Sun. If we add the infinitesimal effects of the two rotations,

$$\begin{aligned} d\boldsymbol{r}_1 &= \boldsymbol{\omega}_1\times\boldsymbol{r}\,dt \\ d\boldsymbol{r}_2 &= \boldsymbol{\omega}_2\times(\boldsymbol{r} + d\boldsymbol{r}_1)\,dt \\ d\boldsymbol{r} &= d\boldsymbol{r}_1 + d\boldsymbol{r}_2 \\ &= (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2)\,dt\times\boldsymbol{r} + \mathcal{O}\big((dt)^2\big)\ . \end{aligned} \tag{2.566}$$

Thus, *infinitesimal rotations add.* Dividing by dt, this means that

$$\boldsymbol{\omega} = \sum_i \boldsymbol{\omega}_i \ , \tag{2.567}$$

where the sum is over all the rotations. For the earth, $\boldsymbol{\omega} = \boldsymbol{\omega}_{\rm rot} + \boldsymbol{\omega}_{\rm orb}$.

- The rotation about earth's axis, $\boldsymbol{\omega}_{\rm rot}$ has magnitude $\omega_{\rm rot} = 2\pi/(1\,{\rm day}) = 7.29 \times 10^{-5}\,{\rm s}^{-1}$. The radius of the earth is $R_{\rm e} = 6.37 \times 10^3\,{\rm km}$.
- The orbital rotation about the Sun, $\boldsymbol{\omega}_{\rm orb}$ has magnitude $\omega_{\rm rot} = 2\pi/(1\,{\rm yr}) = 1.99 \times 10^{-7}\,{\rm s}^{-1}$. The radius of the earth is $a_{\rm e} = 1.50 \times 10^8\,{\rm km}$.

Thus, $\omega_{\rm orb}/\omega_{\rm rot} = 1/365.25 = 2.74 \times 10^{-3}$. There is also a very slow precession of the earth's axis of rotation, the period of which is about 25,000 years, which we will ignore. Note $\dot{\boldsymbol{\omega}} = 0$ for the earth. Thus, applying Newton's second law and then invoking eq. 2.566, we arrive at

$$m\left(\frac{d^2\boldsymbol{r}}{dt^2}\right)_{\!\rm earth} = \boldsymbol{F}^{(\rm tot)} - m\left(\frac{d^2\boldsymbol{R}}{dt^2}\right)_{\!\rm Sun} - \ \ 2m\,\boldsymbol{\omega}\times\left(\frac{d\boldsymbol{r}}{dt}\right)_{\!\rm earth} - \ \ m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) \ , \tag{2.568}$$

where $\boldsymbol{\omega} = \boldsymbol{\omega}_{\rm rot} + \boldsymbol{\omega}_{\rm orb}$, and where $\ddot{\boldsymbol{R}}_{\rm Sun}$ is the acceleration of the center of the earth around the Sun, assuming the Sun-fixed frame to be inertial. The force $\boldsymbol{F}_{(\rm tot)}$ is the total force on the object, and arises from three parts: (i) gravitational pull of the Sun, (ii) gravitational pull of the earth, and (iii) other earthly forces, such as springs, rods, surfaces, electric fields, *etc.*

On the earth's surface, the ratio of the Sun's gravity to the earth's is

$$\frac{F_\odot}{F_{\rm e}} = \frac{GM_\odot m}{a_{\rm e}^2}\bigg/\frac{GM_{\rm e} m}{R_{\rm e}^2} = \frac{M_\odot}{M_{\rm e}}\left(\frac{R_{\rm e}}{a_{\rm e}}\right)^2 \approx 6.02 \times 10^{-4} \ . \tag{2.569}$$

In fact, it is clear that the Sun's field precisely cancels with the term $m\,\ddot{\boldsymbol{R}}_{\rm Sun}$ at the earth's center, leaving only gradient contributions of even lower order, *i.e.* multiplied by $R_{\rm e}/a_{\rm e} \approx 4.25 \times 10^{-5}$. Thus, to an excellent approximation, we may neglect the Sun entirely and write

$$\boxed{\frac{d^2\boldsymbol{r}}{dt^2} = \frac{\boldsymbol{F}'}{m} + \boldsymbol{g} - 2\,\boldsymbol{\omega}\times\frac{d\boldsymbol{r}}{dt} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})} \tag{2.570}$$

Note that we've dropped the 'earth' label here and henceforth. We define $\boldsymbol{g} = -GM_{\rm e}\,\hat{\boldsymbol{r}}/r^2$, the acceleration due to gravity; $\boldsymbol{F}'$ is the sum of all earthly forces other than the earth's gravity. The last two terms on the RHS are corrections to $m\ddot{\boldsymbol{r}} = \boldsymbol{F}$ due to the noninertial frame of the earth, and are recognized as the Coriolis and centrifugal acceleration terms, respectively.

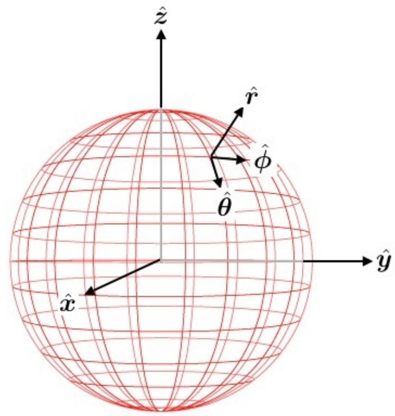

Figure 2.35: The locally orthonormal triad $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$.

2.16.3 Spherical Polar Coordinates

The locally orthonormal triad $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ varies with position. In terms of the body-fixed triad $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$, we have

$$\begin{aligned} \hat{\boldsymbol{r}} &= \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}} && (2.571)\\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\boldsymbol{x}} + \cos\theta\sin\phi\,\hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}} && (2.572)\\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}}\ . && (2.573) \end{aligned}$$

Inverting the relation between the triads $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ and $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$, we obtain

$$\begin{aligned} \hat{\boldsymbol{x}} &= \sin\theta\,\cos\phi\,\hat{\boldsymbol{r}} + \cos\theta\,\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} && (2.574)\\ \hat{\boldsymbol{y}} &= \sin\theta\,\sin\phi\,\hat{\boldsymbol{x}} + \cos\theta\,\sin\phi\,\hat{\boldsymbol{y}} + \cos\phi\,\hat{\boldsymbol{\phi}} && (2.575)\\ \hat{\boldsymbol{z}} &= \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\ . && (2.576) \end{aligned}$$

The differentials of these unit vectors are

$$\begin{aligned} d\hat{\boldsymbol{r}} &= \hat{\boldsymbol{\theta}}\,d\theta + \sin\theta\,\hat{\boldsymbol{\phi}}\,d\phi && (2.577)\\ d\hat{\boldsymbol{\theta}} &= -\hat{\boldsymbol{r}}\,d\theta + \cos\theta\,\hat{\boldsymbol{\phi}}\,d\phi && (2.578)\\ d\hat{\boldsymbol{\phi}} &= -\sin\theta\,\hat{\boldsymbol{r}}\,d\phi - \cos\theta\,\hat{\boldsymbol{\theta}}\,d\phi\ . && (2.579) \end{aligned}$$

Thus,

$$\begin{aligned} \dot{\boldsymbol{r}} &= \frac{d}{dt}\big(r\,\hat{\boldsymbol{r}}\big) = \dot{r}\,\hat{\boldsymbol{r}} + r\,\dot{\hat{\boldsymbol{r}}}\\ &= \dot{r}\,\hat{\boldsymbol{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\dot{\phi}\,\hat{\boldsymbol{\phi}}\ . && (2.580) \end{aligned}$$

If we differentiate a second time, we find, after some tedious accounting,

$$\begin{aligned} \ddot{\boldsymbol{r}} = \big(\ddot{r} - r\,\dot{\theta}^2 - r\,\sin^2\!\theta\,\dot{\phi}^2\big)\,\hat{\boldsymbol{r}} &+ \big(2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta} - r\,\sin\theta\cos\theta\,\dot{\phi}^2\big)\,\hat{\boldsymbol{\theta}}\\ &+ \big(2\,\dot{r}\,\dot{\phi}\,\sin\theta + 2\,r\,\dot{\theta}\,\dot{\phi}\,\cos\theta + r\,\sin\theta\,\ddot{\phi}\big)\,\hat{\boldsymbol{\phi}}\ . && (2.581) \end{aligned}$$

2.16.4 Centrifugal Force

One major distinction between the Coriolis and centrifugal forces is that the Coriolis force acts only on moving particles, whereas the centrifugal force is present even when $\dot{\boldsymbol{r}} = 0$. Thus, the equation for stationary equilibrium on the earth's surface is

$$m\boldsymbol{g} + \boldsymbol{F}' - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) = 0 \ , \tag{2.582}$$

involves the centrifugal term. We can write this as $\boldsymbol{F}' + m\widetilde{\boldsymbol{g}} = 0$, where

$$\widetilde{\boldsymbol{g}} = -\frac{GM_\mathrm{e}\,\hat{\boldsymbol{r}}}{r^2} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \tag{2.583}$$

$$= -\big(g_0 - \omega^2 R_\mathrm{e}\,\sin^2\theta\big)\,\hat{\boldsymbol{r}} + \omega^2\ R_\mathrm{e}\,\sin\theta\cos\theta\,\hat{\boldsymbol{\theta}} \ , \tag{2.584}$$

where $g_0 = GM_\mathrm{e}/R_\mathrm{e}^2 = 980\,\mathrm{cm}^2/\mathrm{s}$. Thus, on the equator, $\boldsymbol{g} = -\big(g_0 - \omega^2 R_\mathrm{e}\big)\,\hat{\boldsymbol{r}}$, with $\omega^2 R_\mathrm{e} \approx 3.39\,\mathrm{cm}^2/\mathrm{s}$, a small but significant correction. Thus, you weigh less on the equator. Note also the term in $\widetilde{\boldsymbol{g}}$ along $\hat{\boldsymbol{\theta}}$. This means that a plumb bob suspended from a general point above the earth's surface won't point exactly toward the earth's center. Moreover, if the earth were replaced by an equivalent mass of fluid, the fluid would rearrange itself so as to make its surface locally perpendicular to $\widetilde{\boldsymbol{g}}$. Indeed, the earth (and Sun) do exhibit quadrupolar distortions in their mass distributions – both are oblate spheroids. In fact, the observed difference $\tilde{g}(\theta = \frac{\pi}{2}) - \tilde{g}(\theta = 0) \approx 5.2\,\mathrm{cm/s}$, which is 53% greater than the naïvely expected value of $3.39\,\mathrm{cm/s}$. The earth's oblateness enhances the effect.

2.16.5 The Coriolis Force

The Coriolis force is given by $\boldsymbol{F}_\mathrm{Cor} = -2m\,\boldsymbol{\omega} \times \dot{\boldsymbol{r}}$. According to (2.570), the acceleration of a free particle ($\boldsymbol{F}' = 0$) is not along $\widetilde{\boldsymbol{g}}$ – an orthogonal component is generated by the Coriolis force. To actually solve the coupled equations of motion is difficult because the unit vectors $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ change with position, and hence with time. The following standard problem highlights some of the effects of the Coriolis and centrifugal forces.

PROBLEM: A cannonball is dropped from the top of a tower of height h located at a northerly latitude of λ. Assuming the cannonball is initially at rest with respect to the tower, and neglecting air resistance, calculate its deflection (magnitude and direction) due to (a) centrifugal and (b) Coriolis forces by the time it hits the ground. Evaluate for the case $h = 100$ m, $\lambda = 45°$. The radius of the earth is $R_\mathrm{e} = 6.4 \times 10^6\,\mathrm{m}$.

SOLUTION: The equation of motion for a particle near the earth's surface is

$$\ddot{\boldsymbol{r}} = -2\,\boldsymbol{\omega} \times \dot{\boldsymbol{r}} - g_0\,\hat{\boldsymbol{r}} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \ , \tag{2.585}$$

where $\omega = \omega\,\hat{\boldsymbol{z}}$, with $\omega = 2\pi/(24\,\mathrm{hrs}) = 7.3 \times 10^{-5}\,\mathrm{rad/s}$. Here, $g_0 = GM_\mathrm{e}/R_\mathrm{e}^2 = 980\,\mathrm{cm/s}^2$. We use a locally orthonormal coordinate system $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ and write

$$\boldsymbol{r} = x\,\hat{\boldsymbol{\theta}} + y\,\hat{\boldsymbol{\phi}} + (R_\mathrm{e} + z)\,\hat{\boldsymbol{r}} \ , \tag{2.586}$$

where $R_e = 6.4 \times 10^6$ m is the radius of the earth. Expressing $\hat{\boldsymbol{z}}$ in terms of our chosen orthonormal triad,

$$\hat{\boldsymbol{z}} = \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \ , \tag{2.587}$$

where $\theta = \frac{\pi}{2} - \lambda$ is the polar angle, or 'colatitude'. Since the height of the tower and the deflections are all very small on the scale of R_e, we may regard the orthonormal triad as fixed and time-independent. (In general, these unit vectors change as a function of $\boldsymbol{r}$.) Thus, we have $\dot{\boldsymbol{r}} \simeq \dot{x}\,\hat{\boldsymbol{\theta}} + \dot{y}\,\hat{\boldsymbol{\phi}} + \dot{z}\,\hat{\boldsymbol{r}}$, and we find

$$\hat{\boldsymbol{z}} \times \dot{\boldsymbol{r}} = -\dot{y}\,\cos\theta\,\hat{\boldsymbol{\theta}} + (\dot{x}\,\cos\theta + \dot{z}\,\sin\theta)\,\hat{\boldsymbol{\phi}} - \dot{y}\sin\theta\,\hat{\boldsymbol{r}} \tag{2.588}$$

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) = -\omega^2 R_e \sin\theta\cos\theta\,\hat{\boldsymbol{\theta}} - \omega^2 R_e \sin^2\!\theta\,\hat{\boldsymbol{r}} \ , \tag{2.589}$$

where we neglect the $\mathcal{O}(z)$ term in the second equation, since $z \ll R_e$.

The equation of motion, written in components, is then

$$\ddot{x} = 2\omega\cos\theta\,\dot{y} + \omega^2 R_e \sin\theta\cos\theta \tag{2.590}$$

$$\ddot{y} = -2\omega\cos\theta\,\dot{x} - 2\omega\sin\theta\,\dot{z} \tag{2.591}$$

$$\ddot{z} = -g_0 + 2\omega\sin\theta\,\dot{y} + \omega^2 R_e \sin^2\theta \ . \tag{2.592}$$

While these (inhomogeneous) equations are linear, they also are coupled, so an exact analytical solution is not trivial to obtain (but see below). Fortunately, the deflections are small, so we can solve this perturbatively. We write $x = x^{(0)} + \delta x$, *etc.*, and solve to lowest order by including only the g_0 term on the RHS. This gives $z^{(0)}(t) = z_0 - \frac{1}{2}g_0\,t^2$, along with $x^{(0)}(t) = y^{(0)}(t) = 0$. We then substitute this solution on the RHS and solve for the deflections, obtaining

$$\delta x(t) = \tfrac{1}{2}\omega^2 R_e\,\sin\theta\cos\theta\,t^2 \tag{2.593}$$

$$\delta y(t) = \tfrac{1}{3}\omega g_0\,\sin\theta\,t^3 \tag{2.594}$$

$$\delta z(t) = \tfrac{1}{2}\omega^2 R_e\,\sin^2\!\theta\,t^2 \ . \tag{2.595}$$

The deflection along $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{r}}$ is due to the centrifugal term, while that along $\hat{\boldsymbol{\phi}}$ is due to the Coriolis term. (At higher order, the two terms interact and the deflection in any given direction can't uniquely be associated to a single fictitious force.) To find the deflection of an object dropped from a height h, solve $z^{(0)}(t^*) = 0$ to obtain $t^* = \sqrt{2h/g_0}$ for the drop time, and substitute. For $h = 100$ m and $\lambda = \frac{\pi}{2}$, find $\delta x(t^*) = 17$ cm south (centrifugal) and $\delta y(t^*) = 1.6$ cm east (Coriolis).

In fact, an exact solution to (2.592) is readily obtained, via the following analysis. The equations of motion may be written $\dot{\boldsymbol{v}} = 2i\omega\mathcal{J}\boldsymbol{v} + \boldsymbol{b}$, or

$$\begin{pmatrix} \dot{v}_x \\ \dot{v}_y \\ \dot{v}_x \end{pmatrix} = 2i\,\omega \overbrace{\begin{pmatrix} 0 & -i\cos\theta & 0 \\ i\cos\theta & 0 & i\sin\theta \\ 0 & -i\sin\theta & 0 \end{pmatrix}}^{\mathcal{J}} \begin{pmatrix} v_x \\ v_y \\ v_x \end{pmatrix} + \overbrace{\begin{pmatrix} g_1\sin\theta\cos\theta \\ 0 \\ -g_0 + g_1\sin^2\!\theta \end{pmatrix}}^{\boldsymbol{b}} \tag{2.596}$$

with $g_1 \equiv \omega^2 R_{\rm e}$. Note that $\mathcal{J}^\dagger = \mathcal{J}$, *i.e.* $\mathcal{J}$ is a Hermitian matrix. The formal solution is

$$\boldsymbol{v}(t) = e^{2i\omega\mathcal{J}t}\,\boldsymbol{v}(0) + \left(\frac{e^{2i\omega\mathcal{J}t}-1}{2i\omega}\right)\mathcal{J}^{-1}\,\boldsymbol{b}\ . \tag{2.597}$$

When working with matrices, it is convenient to work in an eigenbasis. The characteristic polynomial for $\mathcal{J}$ is $P(\lambda) = \det\left(\lambda\cdot 1 - \mathcal{J}\right) = \lambda\,(\lambda^2-1)$, hence the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = +1$, and $\lambda_3 = -1$. The corresponding eigenvectors are easily found to be

$$\boldsymbol{\psi}_1 = \begin{pmatrix} \sin\theta \\ 0 \\ -\cos\theta \end{pmatrix} \quad , \quad \boldsymbol{\psi}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta \\ i \\ \sin\theta \end{pmatrix} \quad , \quad \boldsymbol{\psi}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta \\ -i \\ \sin\theta \end{pmatrix}\ . \tag{2.598}$$

Note that $\boldsymbol{\psi}_a^\dagger \cdot \boldsymbol{\psi}_{a'} = \delta_{aa'}$.

Expanding $\boldsymbol{v}$ and $\boldsymbol{b}$ in this eigenbasis, we have $\dot{v}_a = 2i\omega\lambda_a v_a + b_a$, where $v_a = \psi^*_{ia}\, v_i$ and $b_a = \psi^*_{ia}\, b_i$. The solution is

$$v_a(t) = v_a(0)\, e^{2i\lambda_a\omega t} + \left(\frac{e^{2i\lambda_a\omega t}-1}{2i\lambda_a\omega}\right) b_a\ , \tag{2.599}$$

which entails

$$v_i(t) = \left(\sum_a \psi_{ia}\left(\frac{e^{2i\lambda_a\omega t}-1}{2i\lambda_a\omega}\right)\psi^*_{ja}\right) b_j\ , \tag{2.600}$$

where we have taken $\boldsymbol{v}(0) = 0$, *i.e.* the object is released from rest. Doing the requisite matrix multiplications,

$$\begin{pmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \end{pmatrix} = \begin{pmatrix} t\sin^2\!\theta + \frac{\sin 2\omega t}{2\omega}\,\cos^2\!\theta & \frac{\sin^2\!\omega t}{\omega}\,\cos\theta & -\frac{1}{2}t\sin 2\theta + \frac{\sin 2\omega t}{4\omega}\,\sin 2\theta \\ -\frac{\sin^2\!\omega t}{\omega}\,\cos\theta & \frac{\sin 2\omega t}{2\omega} & -\frac{\sin^2\!\omega t}{\omega}\,\sin\theta \\ -\frac{1}{2}t\sin 2\theta + \frac{\sin 2\omega t}{4\omega}\,\sin 2\theta & \frac{\sin^2\!\omega t}{\omega}\,\sin\theta & t\cos^2\!\theta + \frac{\sin 2\omega t}{2\omega}\,\sin^2\!\theta \end{pmatrix} \begin{pmatrix} g_1\sin\theta\cos\theta \\ 0 \\ -g_0 + g_1\sin^2\!\theta \end{pmatrix}, \tag{2.601}$$

which says

$$\begin{aligned} v_x(t) &= \left(\tfrac{1}{2}\sin 2\theta + \tfrac{\sin 2\omega t}{4\omega t}\,\sin 2\theta\right)\cdot g_0 t + \tfrac{\sin 2\omega t}{4\omega t}\,\sin 2\theta\cdot g_1 t \\ v_y(t) &= \tfrac{\sin^2\!\omega t}{\omega t}\,\cdot g_0 t - \tfrac{\sin^2\!\omega t}{\omega t}\,\sin\theta\cdot g_1 t \\ v_z(t) &= -\left(\cos^2\!\theta + \tfrac{\sin 2\omega t}{2\omega t}\,\sin^2\!\theta\right)\cdot g_0 t + \tfrac{\sin^2\!\omega t}{2\omega t}\,\cdot g_1 t\ . \end{aligned} \tag{2.602}$$

Why is the deflection always to the east? The earth rotates eastward, and an object starting from rest in the earth's frame has initial angular velocity equal to that of the earth. To conserve angular momentum, the object must speed up as it falls.

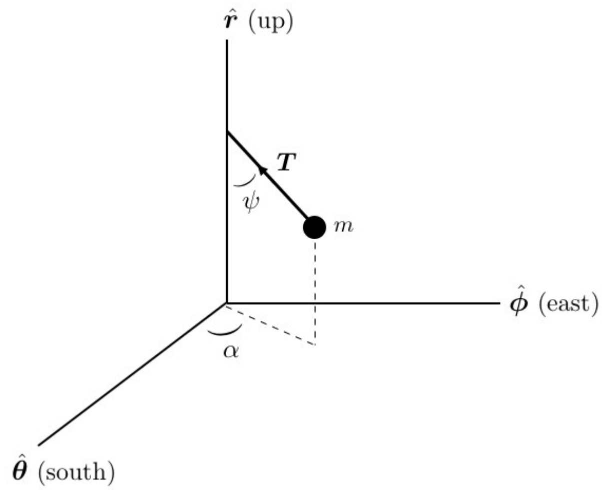

Figure 2.36: Foucault's pendulum.

2.16.6 Foucault's Pendulum

A pendulum swinging over one of the poles moves in a fixed inertial plane while the earth rotates underneath. Relative to the earth, the plane of motion of the pendulum makes one revolution every day. What happens at a general latitude? Assume the pendulum is located at colatitude θ and longitude ϕ. Assuming the length scale of the pendulum is small compared to $R_{\rm e}$, we can regard the local triad $\{\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{r}}\}$ as fixed. The situation is depicted in Fig. 2.36. We write

$$\boldsymbol{r} = x\,\hat{\boldsymbol{\theta}} + y\,\hat{\boldsymbol{\phi}} + z\,\hat{\boldsymbol{r}} \ , \tag{2.603}$$

with

$$x = \ell\,\sin\psi\,\cos\alpha \quad , \quad y = \ell\,\sin\psi\,\sin\alpha \quad , \quad z = \ell\,(1-\cos\psi) \ . \tag{2.604}$$

In our analysis we will ignore centrifugal effects, which are of higher order in ω, and we take $\boldsymbol{g} = -g\,\hat{\boldsymbol{r}}$. We also idealize the pendulum, and consider the suspension rod to be of negligible mass.

The total force on the mass m is due to gravity and tension:

$$\begin{aligned}\boldsymbol{F} &= m\boldsymbol{g} + \boldsymbol{T} \\ &= \big(-T\sin\psi\,\cos\alpha,\ -T\sin\psi\,\sin\alpha,\ T\cos\psi - mg\big) \\ &= \big(-Tx/\ell,\ -Ty/\ell,\ T - Mg - Tz/\ell\big) \ .\end{aligned} \tag{2.605}$$

The Coriolis term is

$$\boldsymbol{F}_{\rm Cor} = -2m\,\boldsymbol{\omega}\times\dot{\boldsymbol{r}} \tag{2.606}$$

$$\begin{aligned}&= -2m\,\omega\big(\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\big)\times\big(\dot x\,\hat{\boldsymbol{\theta}} + \dot y\,\hat{\boldsymbol{\phi}} + \dot z\,\hat{\boldsymbol{r}}\big) \\ &= 2m\omega\,\big(\dot y\cos\theta,\ -\dot x\cos\theta - \dot z\sin\theta,\ \dot y\sin\theta\big) \ .\end{aligned} \tag{2.607}$$

The equations of motion are $m\,\ddot{\boldsymbol{r}} = \boldsymbol{F} + \boldsymbol{F}_{\rm Cor}$:

$$m\ddot{x} = -Tx/\ell + 2m\omega\cos\theta\,\dot{y} \tag{2.608}$$

$$m\ddot{y} = -Ty/\ell - 2m\omega\cos\theta\,\dot{x} - 2m\omega\sin\theta\,\dot{z} \tag{2.609}$$

$$m\ddot{z} = T - mg - Tz/\ell + 2m\omega\sin\theta\,\dot{y}\ . \tag{2.610}$$

These three equations are to be solved for the three unknowns x, y, and T. Note that

$$x^2 + y^2 + (\ell - z)^2 = \ell^2\ , \tag{2.611}$$

so $z = z(x, y)$ is not an independent degree of freedom. This equation may be recast in the form $z = (x^2 + y^2 + z^2)/2\ell$ which shows that if x and y are both small, then z is at least of second order in smallness. Therefore, we will approximate $z \simeq 0$, in which case $\dot{z}$ may be neglected from the second equation of motion. Adding the first plus i times the second then gives the complexified equation

$$\ddot{\xi} = -\omega_0^2\,\xi - 2i\omega\cos\theta\,\dot{\xi}\ , \tag{2.612}$$

where $\xi \equiv x + iy$, and where $\omega_0 = \sqrt{g/\ell}$. The third equation is used to solve for T:

$$T \simeq mg - 2m\omega\sin\theta\,\dot{y}\ . \tag{2.613}$$

It is now a trivial matter to solve the homogeneous linear ODE of eq. 2.612. Writing

$$\xi = \xi_0\, e^{-i\Omega t} \tag{2.614}$$

and plugginh in to find Ω, we obtain

$$\Omega^2 - 2\omega_\perp\Omega - \omega_0^2 = 0\ , \tag{2.615}$$

with $\omega_\perp \equiv \omega\cos\theta$. The roots are

$$\Omega_\pm = \omega_\perp \pm \sqrt{\omega_0^2 + \omega_\perp^2}\ , \tag{2.616}$$

hence the most general solution is

$$\xi(t) = A_+\, e^{-i\Omega_+ t} + = A_-\, e^{-i\Omega_- t}\ . \tag{2.617}$$

Finally, if we take as initial conditions $x(0) = a$, $y(0) = 0$, $\dot{x}(0) = 0$, and $\dot{y}(0) = 0$, we obtain

$$x(t) = \left(\frac{a}{\nu}\right)\cdot\Big\{\omega_\perp\,\sin(\omega_\perp t)\,\sin(\nu t) + \nu\,\cos(\omega_\perp t)\,\cos(\nu t)\Big\} \tag{2.618}$$

$$y(t) = \left(\frac{a}{\nu}\right)\cdot\Big\{\omega_\perp\,\cos(\omega_\perp t)\,\sin(\nu t) - \nu\,\sin(\omega_\perp t)\,\cos(\nu t)\Big\}\ , \tag{2.619}$$

with $\nu = \sqrt{\omega_0^2 + \omega_\perp^2}$. Typically $\omega_0 \gg \omega_\perp$, since $\omega = 7.3 \times 10^{-5}\,\mathrm{s}^{-1}$. In the limit $\omega_\perp \ll \omega_0$, then, we have $\nu \approx \omega_0$ and

$$x(t) \simeq a\,\cos(\omega_\perp t)\,\cos(\omega_0 t) \quad , \quad y(t) \simeq -a\,\sin(\omega_\perp t)\,\cos(\omega_0 t)\ , \tag{2.620}$$

and the plane of motion rotates with angular frequency $-\omega_\perp$, *i.e.* the period is $|\sec\theta|$ days. Viewed from above, the rotation is clockwise in the northern hemisphere, where $\cos\theta > 0$ and counterclockwise in the southern hemisphere, where $\cos\theta < 0$.

2.17 Rigid Body Motion

A rigid body consists of a group of particles whose separations are all fixed in magnitude. Six independent coordinates are required to completely specify the position and orientation of a rigid body. For example, the location of the first particle is specified by three coordinates. A second particle requires only two coordinates since the distance to the first is fixed. Finally, a third particle requires only one coordinate, since its distance to the first two particles is fixed (think about the intersection of two spheres). The positions of all the remaining particles are then determined by their distances from the first three. Usually, one takes these six coordinates to be the center-of-mass position $\boldsymbol{R} = (X, Y, Z)$ and three angles specifying the orientation of the body (*e.g.* the Euler angles).

As derived in eqs. (2.467,2.471), the equations of motion are

$$\boldsymbol{P} = \sum_i m_i\,\dot{\boldsymbol{r}}_i \quad , \quad \dot{\boldsymbol{P}} = \boldsymbol{F}^{(\mathrm{ext})} \tag{2.621}$$

$$\boldsymbol{L} = \sum_i m_i\,\boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i \quad , \quad \dot{\boldsymbol{L}} = \boldsymbol{N}^{(\mathrm{ext})}\ . \tag{2.622}$$

These equations determine the motion of a rigid body.

`Examples of Rigid Bodies:` Our first example of a rigid body is of a wheel rolling with constant angular velocity $\dot{\phi} = \omega$, and without slipping, This is shown in Fig. 2.37. The no-slip condition is $dx = R\,d\phi$, so $\dot{x} = V_{\mathrm{CM}} = R\omega$. The velocity of a point within the wheel is

$$\boldsymbol{v} = \boldsymbol{V}_{\mathrm{CM}} + \boldsymbol{\omega} \times \boldsymbol{r}\ , \tag{2.623}$$

where $\boldsymbol{r}$ is measured from the center of the disk. The velocity of a point on the surface is then given by $\boldsymbol{v} = \omega R\big(\hat{\boldsymbol{x}} + \hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{r}}\big)$.

As a second example, consider a bicycle wheel of mass M and radius R affixed to a light, firm rod of length d, as shown in Fig. 2.38. Assuming $\boldsymbol{L}$ lies in the (x, y) plane, one computes the gravitational torque $\boldsymbol{N} = \boldsymbol{r} \times (M\boldsymbol{g}) = Mgd\,\hat{\boldsymbol{\phi}}$. The angular momentum vector then rotates with angular frequency $\dot{\phi}$. Thus,

$$d\phi = \frac{dL}{L} \quad \Longrightarrow \quad \dot{\phi} = \frac{Mgd}{L}\ . \tag{2.624}$$

But $L = MR^2\omega$, so the precession frequency is

$$\omega_{\mathrm{p}} = \dot{\phi} = \frac{gd}{\omega R^2} \ . \tag{2.625}$$

For $R = d = 30\,\mathrm{cm}$ and $\omega/2\pi = 200\,\mathrm{rpm}$, find $\omega_{\mathrm{p}}/2\pi \approx 15\,\mathrm{rpm}$. Note that we have here ignored the contribution to $\boldsymbol{L}$ from the precession itself, which lies along $\hat{\boldsymbol{z}}$, resulting in the *nutation* of the wheel. This is justified if $L_{\mathrm{p}}/L = (d^2/R^2) \cdot (\omega_{\mathrm{p}}/\omega) \ll 1$.

2.17.1 The Inertia Tensor

Suppose first that a point within the body itself is fixed. This eliminates the translational degrees of freedom from consideration. We now have

$$\bigg(\frac{d\boldsymbol{r}}{dt}\bigg)_{\mathrm{inertial}} = \boldsymbol{\omega} \times \boldsymbol{r} \ , \tag{2.626}$$

since $\dot{\boldsymbol{r}}_{\mathrm{body}} = 0$. The kinetic energy is then

$$\begin{aligned} T &= \tfrac{1}{2} \sum_i m_i \bigg(\frac{d\boldsymbol{r}_i}{dt}\bigg)^2_{\mathrm{inertial}} = \tfrac{1}{2} \sum_i m_i \, (\boldsymbol{\omega} \times \boldsymbol{r}_i) \cdot (\boldsymbol{\omega} \times \boldsymbol{r}_i) \\ &= \tfrac{1}{2} \sum_i m_i \Big[\omega^2 \, \boldsymbol{r}_i^2 - (\boldsymbol{\omega} \cdot \boldsymbol{r}_i)^2\Big] \equiv \tfrac{1}{2} I_{\alpha\beta} \, \omega_\alpha \, \omega_\beta \ , \end{aligned} \tag{2.627}$$

where ω_α is the component of $\boldsymbol{\omega}$ along the body-fixed axis $\boldsymbol{e}'_\alpha$. The quantity $I_{\alpha\beta}$ is the *inertia tensor,*

$$I_{\alpha\beta} = \sum_i m_i \Big(\boldsymbol{r}_i^2 \, \delta_{\alpha\beta} - r_{i,\alpha} \, r_{i,\beta}\Big) \tag{2.628}$$

$$= \int \! d^d r \, \varrho(\boldsymbol{r}) \, \Big(\boldsymbol{r}^2 \, \delta_{\alpha\beta} - r_\alpha \, r_\beta\Big) \quad \text{(continuous media)} \ . \tag{2.629}$$

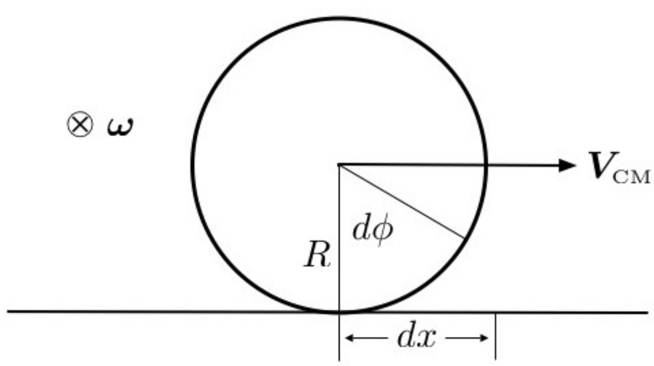

Figure 2.37: A wheel rolling to the right without slipping.

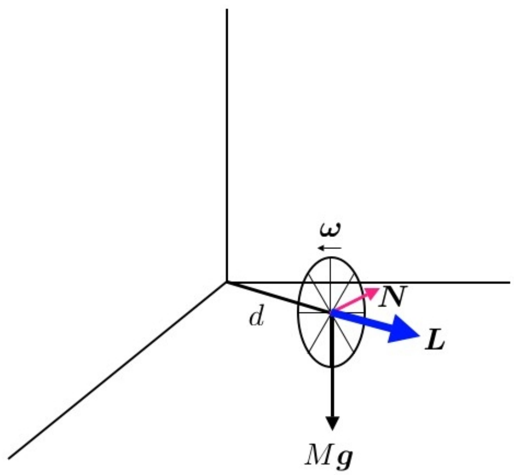

Figure 2.38: Precession of a spinning bicycle wheel.

The angular momentum is

$$
\begin{aligned}
\boldsymbol{L} &= \sum_i m_i \, \boldsymbol{r}_i \times \left(\frac{d\boldsymbol{r}_i}{dt}\right)_{\text{inertial}} \\
&= \sum_i m_i \, \boldsymbol{r}_i \times (\boldsymbol{\omega} \times \boldsymbol{r}_i) = I_{\alpha\beta} \, \omega_\beta \ .
\end{aligned} \tag{2.630}
$$

The diagonal elements of $I_{\alpha\beta}$ are called the *moments of inertia*, while the off-diagonal elements are called the *products of inertia.*

2.17.2 Coordinate Transformations

Consider the basis transformation

$$
\hat{\mathbf{e}}'_\alpha = \mathcal{R}_{\alpha\beta} \, \hat{\mathbf{e}}_\beta \ . \tag{2.631}
$$

We demand $\hat{\mathbf{e}}'_\alpha \cdot \hat{\mathbf{e}}'_\beta = \delta_{\alpha\beta}$, which means $\mathcal{R} \in \mathrm{O}(d)$ is an orthogonal matrix, *i.e.* $\mathcal{R}^{\mathrm{t}} = \mathcal{R}^{-1}$. Thus the inverse transformation is $\boldsymbol{e}_\alpha = \mathcal{R}^{\mathrm{t}}_{\alpha\beta} \boldsymbol{e}'_\beta$. Consider next a general vector $\boldsymbol{A} = A_\beta \, \hat{\mathbf{e}}_\beta$. Expressed in terms of the new basis $\{\hat{\mathbf{e}}'_\alpha\}$, we have

$$
\boldsymbol{A} = A_\beta \, \overbrace{\mathcal{R}^{\mathrm{t}}_{\beta\alpha} \, \hat{\mathbf{e}}'_\alpha}^{\hat{\mathbf{e}}_\beta} = \overbrace{\mathcal{R}_{\alpha\beta} A_\beta}^{A'_\alpha} \, \hat{\mathbf{e}}'_\alpha \tag{2.632}
$$

Thus, the components of $\boldsymbol{A}$ transform as $A'_\alpha = \mathcal{R}_{\alpha\beta} \, A_\beta$. This is true for any vector.

Under a rotation, the density $\rho(\boldsymbol{r})$ must satisfy $\rho'(\boldsymbol{r}') = \rho(\boldsymbol{r})$. This is the transformation rule for scalars. The inertia tensor therefore obeys

$$\begin{aligned}
I'_{\alpha\beta} &= \int \! d^3r' \, \rho'(\boldsymbol{r}') \, \Big[\boldsymbol{r}'^2 \, \delta_{\alpha\beta} - r'_\alpha \, r'_\beta \Big] \\
&= \int \! d^3r \, \rho(\boldsymbol{r}) \, \Big[\boldsymbol{r}^2 \, \delta_{\alpha\beta} - \big(\mathcal{R}_{\alpha\mu} r_\mu\big) \big(\mathcal{R}_{\beta\nu} r_\nu\big) \Big] \\
&= \mathcal{R}_{\alpha\mu} \, I_{\mu\nu} \, \mathcal{R}^{\mathrm{t}}_{\nu\beta} \ .
\end{aligned} \tag{2.633}$$

I.e. $I' = \mathcal{R} I \mathcal{R}^{\mathrm{t}}$, the transformation rule for tensors. The angular frequency $\boldsymbol{\omega}$ is a vector, so $\omega'_\alpha = \mathcal{R}_{\alpha\mu} \, \omega_\mu$. The angular momentum $\boldsymbol{L}$ also transforms as a vector. The kinetic energy is $T = \frac{1}{2} \, \boldsymbol{\omega}^{\mathrm{t}} \cdot I \cdot \boldsymbol{\omega}$, which transforms as a scalar.

2.17.3 The Case of No Fixed Point

If there is no fixed point, we can let $\boldsymbol{r}'$ denote the distance from the center-of-mass (CM), which will serve as the instantaneous origin in the body-fixed frame. We then adopt the definitions from Fig. 2.34, where $\boldsymbol{R}$ is the CM position of the rotating body, as observed in an inertial frame, and is computed from by the expression

$$\boldsymbol{R} = \frac{1}{M} \sum_i m_i \, \boldsymbol{\rho}_i = \frac{1}{M} \int \! d^3r \, \rho(\boldsymbol{r}) \ , \tag{2.634}$$

where the total mass is of course

$$M = \sum_i m_i = \int \! d^3r \, \rho(\boldsymbol{r}) \ . \tag{2.635}$$

The kinetic energy and angular momentum are then

$$T = \tfrac{1}{2} M \dot{\boldsymbol{R}}^2 + \tfrac{1}{2} I_{\alpha\beta} \, \omega_\alpha \, \omega_\beta \tag{2.636}$$

$$L_\alpha = \epsilon_{\alpha\beta\gamma} M R_\beta \dot{R}_\gamma + I_{\alpha\beta} \, \omega_\beta \ , \tag{2.637}$$

where $I_{\alpha\beta}$ is given in eqs. 2.628 and 2.629, where the origin is the CM.

2.17.4 Parallel Axis Theorem

Suppose $I_{\alpha\beta}$ is given in a body-fixed frame. If we displace the origin in the body-fixed frame by $\boldsymbol{b}$, then let $I_{\alpha\beta}(\boldsymbol{b})$ be the inertial tensor with respect to the new origin. We have

$$I_{\alpha\beta}(\boldsymbol{b}) = \sum_i m_i \Big\{ (\boldsymbol{r}_i^2 + 2\boldsymbol{b} \cdot \boldsymbol{r}_i + \boldsymbol{b}^2) \, \delta_{\alpha\beta} - (r_{i,\alpha} + b_\alpha)(r_{i,\beta} + b_\beta) \Big\} \ . \tag{2.638}$$

If $\boldsymbol{r}_i$ is measured with respect to the CM, then

$$\sum_i m_i \, \boldsymbol{r}_i = 0 \tag{2.639}$$

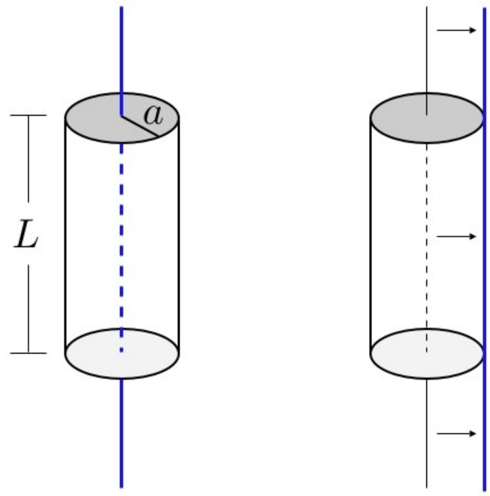

Figure 2.39: Application of the parallel axis theorem to a cylindrically symmetric mass distribution.

and

$$I_{\alpha\beta}(\boldsymbol{b}) = I_{\alpha\beta}(0) + M\big(\boldsymbol{b}^2\,\delta_{\alpha\beta} - b_\alpha b_\beta\big) \ , \tag{2.640}$$

a result known as the *parallel axis theorem.*

As an example of the theorem, consider the situation depicted in Fig. 2.39, where a cylindrically symmetric mass distribution is rotated about is symmetry axis, and about an axis tangent to its side. The component I_{Rzz} of the inertia tensor is easily computed when the origin lies along the symmetry axis:

$$\begin{aligned} I_{zz} &= \int d^3r\,\rho(\boldsymbol{r})\,(\boldsymbol{r}^2 - z^2) = \rho L \cdot 2\pi \int\limits_0^a dr_\perp\, r_\perp^3 \\ &= \tfrac{\pi}{2}\rho L a^4 = \tfrac{1}{2} M a^2 \ , \end{aligned} \tag{2.641}$$

where $M = \pi a^2 L \rho$ is the total mass. If we compute I_{zz} about a vertical axis which is tangent to the cylinder, the parallel axis theorem tells us that

$$I'_{zz} = I_{zz} + Ma^2 = \tfrac{3}{2} Ma^2 \ . \tag{2.642}$$

Doing this calculation by explicit integration of $\int dm\, r_\perp^2$ would be tedious!

`Example:` Compute the CM and the inertia tensor for the planar right triangle of Fig. 2.40, assuming it to be of uniform two-dimensional mass density ρ.

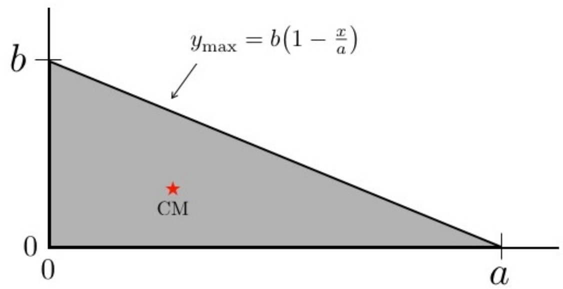

Figure 2.40: A planar mass distribution in the shape of a triangle.

Solution: The total mass is $M = \frac{1}{2}\rho\, ab$. The x-coordinate of the CM is then

$$\begin{aligned} X &= \frac{1}{M}\int\limits_0^a \! dx \int\limits_0^{y_{\rm max}(x)} \!\! dy\,\rho\, x = \frac{\rho}{M}\int\limits_0^a \! dx\, b\big(1 - \tfrac{x}{a}\big)\, x \\ &= \frac{\rho\, a^2 b}{M}\int\limits_0^1 \! du\, u(1-u) = \frac{\rho\, a^2 b}{6M} = \tfrac{1}{3}\, a \ . \end{aligned} \tag{2.643}$$

Clearly we must then have $Y = \frac{1}{3}\, b$, which may be verified by explicit integration.

Since the figure is planar, $z = 0$ everywhere, hence $I_{xz} = I_{zx} = 0$, $I_{yz} = I_{zy} = 0$, and also $I_{zz} = I_{xx} + I_{yy}$. We now compute the remaining independent elements:

$$\begin{aligned} I_{xx} &= \rho\int\limits_0^a \! dx \int\limits_0^{y_{\rm max}} \! dy\, y^2 = \rho\int\limits_0^a \! dx\, \tfrac{1}{3}\, y^3_{\rm max}(x) \\ &= \tfrac{1}{3}\rho\, ab^3 \int\limits_0^1 \! du\, (1-u)^3 = \tfrac{1}{12}\rho\, ab^3 = \tfrac{1}{6} M b^2 \end{aligned} \tag{2.644}$$

$$\begin{aligned} I_{xy} &= -\rho\int\limits_0^a \! dx \int\limits_0^{y_{\rm max}} \! dy\, xy = -\rho\int\limits_0^a \! dx\, \tfrac{1}{2} x\, y^2_{\rm max}(x) \\ &= -\tfrac{1}{2}\rho\, a^2 b^2 \int\limits_0^1 \! du\, u\, (1-u)^2 = -\tfrac{1}{24}\rho\, a^2 b^2 = -\tfrac{1}{12} M ab \ . \end{aligned} \tag{2.645}$$

Thus,

$$I = \tfrac{1}{6} M \begin{pmatrix} b^2 & -\frac{1}{2}ab & 0 \\ -\frac{1}{2}ab & a^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{pmatrix} \ . \tag{2.646}$$

2.17.5 Principal Axes of Inertia

We found that an orthogonal transformation to a new set of axes $\hat{\mathbf{e}}'_\alpha = \mathcal{R}_{\alpha\beta}\hat{\mathbf{e}}_\beta$ entails $I' = \mathcal{R} I \mathcal{R}^{\rm t}$ for the inertia tensor. Since $I = I^{\rm t}$ is manifestly a symmetric matrix, it can be brought to diagonal form by such an orthogonal transformation. To find $\mathcal{R}$, follow this recipe:

1. Find the diagonal elements of I' by setting $P(\lambda) = 0$, where

$$P(\lambda) = \det\big(\lambda \cdot 1 - I\big) \ , \tag{2.647}$$

   is the characteristic polynomial for I, and 1 is the unit matrix.

2. For each eigenvalue λ_b, solve the d equations

$$I_{\mu\nu}\,\psi^b_\nu = \lambda_b\,\psi^b_\mu \ . \tag{2.648}$$

   Here, ψ^b_ν is the $\nu^{\rm th}$ component of the $b^{\rm th}$ eigenvector. Since $(\lambda \cdot 1 - I)$ is degenerate, these equations are linearly dependent, which means that the first $d-1$ components may be determined in terms of the $d^{\rm th}$ component.

3. Because $I = I^{\rm t}$, eigenvectors corresponding to different eigenvalues are orthogonal. In cases of degeneracy, the eigenvectors may be chosen to be orthogonal, *e.g.* via the Gram-Schmidt procedure.

4. Due to the underdetermined aspect to step 2, we may choose an arbitrary normalization for each eigenvector. It is conventional to choose the eigenvectors to be orthonormal: $\psi^a_\mu\,\psi^b_\mu = \delta^{ab}$.

5. The matrix $\mathcal{R}$ is explicitly given by $\mathcal{R}_{a\mu} = \psi^a_\mu$, the matrix whose row vectors are the eigenvectors $\boldsymbol{\psi}^a$. Of course $\mathcal{R}^{\rm t}$ is then the corresponding matrix of column vectors.

6. The eigenvectors form a complete basis. The resolution of unity may be expressed as

$$\delta_{\mu\nu} = \psi^a_\mu\,\psi^a_\nu \qquad (\text{sum on } a) \ . \tag{2.649}$$

As an example, consider the inertia tensor for a general planar mass distribution, which is of the form

$$I = \begin{pmatrix} I_{xx} & I_{xy} & 0 \\ I_{yx} & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix} \ , \tag{2.650}$$

where $I_{yx} = I_{xy}$ and $I_{zz} = I_{xx} + I_{yy}$. Define

$$A = \tfrac{1}{2}\big(I_{xx} + I_{yy}\big) \tag{2.651}$$

$$B = \sqrt{\tfrac{1}{4}\big(I_{xx} - I_{yy}\big)^2 + I_{xy}^2} \tag{2.652}$$

$$\vartheta = \tan^{-1}\left(\frac{2I_{xy}}{I_{xx} - I_{yy}}\right) \ , \tag{2.653}$$

so that

$$I = \begin{pmatrix} A + B\cos\vartheta & -B\sin\vartheta & 0 \\ -B\sin\vartheta & A - B\cos\vartheta & 0 \\ 0 & 0 & 2A \end{pmatrix} \ , \tag{2.654}$$

The characteristic polynomial is found to be

$$P(\lambda) = (\lambda - 2A)\Big[(\lambda - A)^2 - B^2\Big] \ , \tag{2.655}$$

which gives $\lambda_1 = A$, $\lambda_{2,3} = A \pm B$. The corresponding normalized eigenvectors are

$$\boldsymbol{\psi}^1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad , \quad \boldsymbol{\psi}^2 = \begin{pmatrix} \cos\frac{1}{2}\vartheta \\ -\sin\frac{1}{2}\vartheta \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{\psi}^3 = \begin{pmatrix} \sin\frac{1}{2}\vartheta \\ -\cos\frac{1}{2}\vartheta \\ 0 \end{pmatrix} \tag{2.656}$$

and therefore

$$\mathcal{R}^{\rm t} = \begin{pmatrix} 0 & \cos\frac{1}{2}\vartheta & \sin\frac{1}{2}\vartheta \\ 0 & -\sin\frac{1}{2}\vartheta & \cos\frac{1}{2}\vartheta \\ 1 & 0 & 0 \end{pmatrix} \ . \tag{2.657}$$

2.17.6 Euler's Equations

Let us now choose our coordinate axes to be the principal axes of inertia, with the CM at the origin. We may then write

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \quad , \quad I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \quad \Longrightarrow \quad \boldsymbol{L} = \begin{pmatrix} I_1\,\omega_1 \\ I_2\,\omega_2 \\ I_3\,\omega_3 \end{pmatrix} \ . \tag{2.658}$$

The equations of motion are

$$\begin{aligned} \boldsymbol{N}^{\rm ext} &= \left(\frac{d\boldsymbol{L}}{dt}\right)_{\rm inertial} \\ &= \left(\frac{d\boldsymbol{L}}{dt}\right)_{\rm body} + \boldsymbol{\omega} \times \boldsymbol{L} \\ &= I\,\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (I\,\boldsymbol{\omega}) \ . \end{aligned}$$

Thus, we arrive at *Euler's equations:*

$$I_1\,\dot\omega_1 = (I_2 - I_3)\,\omega_2\,\omega_3 + N_1^{\rm ext} \tag{2.659}$$

$$I_2\,\dot\omega_2 = (I_3 - I_1)\,\omega_3\,\omega_1 + N_2^{\rm ext} \tag{2.660}$$

$$I_3\,\dot\omega_3 = (I_1 - I_2)\,\omega_1\,\omega_2 + N_3^{\rm ext} \ . \tag{2.661}$$

These are coupled and nonlinear. Also note the fact that the external torque must be evaluated along body-fixed principal axes. We can however make progress in the case

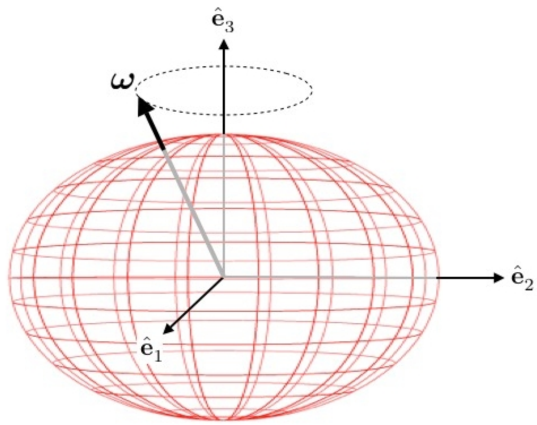

Figure 2.41: Wobbling of a torque-free symmetric top.

where $\boldsymbol{N}^{\text{ext}} = 0$, *i.e.* when there are no external torques. This is true for a body in free space, or in a uniform gravitational field. In the latter case,

$$\boldsymbol{N}^{\text{ext}} = \sum_i \boldsymbol{r}_i \times (m_i\, \boldsymbol{g}) = \Big(\sum_i m_i \boldsymbol{r}_i\Big) \times \boldsymbol{g} \ , \tag{2.662}$$

where $\boldsymbol{g}$ is the uniform gravitational acceleration. In a body-fixed frame whose origin is the CM, we have $\sum_i m_i \boldsymbol{r}_i = 0$, and the external torque vanishes!

Precession of torque-free symmetric tops: Consider a body which has a symmetry axis $\hat{\mathbf{e}}_3$. This guarantees $I_1 = I_2$, but in general we still have $I_1 \neq I_3$. In the absence of external torques, the last of Euler's equations says $\dot{\omega}_3 = 0$, so ω_3 is a constant. The remaining two equations are then

$$\dot{\omega}_1 = \left(\frac{I_1 - I_3}{I_1}\right) \omega_3\, \omega_2 \quad , \quad \dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_1}\right) \omega_3\, \omega_1 \ . \tag{2.663}$$

I.e.$\dot{\omega}_1 = -\Omega\, \omega_2$ and $\dot{\omega}_2 = +\Omega\, \omega_1$, with

$$\Omega = \left(\frac{I_3 - I_1}{I_1}\right) \omega_3 \ , \tag{2.664}$$

which are the equations of a harmonic oscillator. The solution is easily obtained:

$$\omega_1(t) = \omega_\perp \cos\big(\Omega t + \delta\big) \quad , \quad \omega_2(t) = \omega_\perp \sin\big(\Omega t + \delta\big) \quad , \quad \omega_3(t) = \omega_3 \ , \tag{2.665}$$

where $\omega_\perp$ and δ are constants of integration, and where $|\boldsymbol{\omega}| = (\omega_\perp^2 + \omega_3^2)^{1/2}$. This motion is sketched in Fig. 2.41. Note that the perpendicular components of $\boldsymbol{\omega}$ oscillate harmonically, and that the angle $\boldsymbol{\omega}$ makes with respect to $\hat{\mathbf{e}}_3$ is $\lambda = \tan^{-1}(\omega_\perp/\omega_3)$.

For the earth, $(I_3 - I_1)/I_1 \approx \frac{1}{305}$, so $\omega_3 \approx \omega$, and $\Omega \approx \omega/305$, yielding a precession period of 305 days, or roughly 10 months. Astronomical observations reveal such a precession, known as the *Chandler wobble*. For the earth, the precession angle is $\lambda_{\text{Chandler}} \simeq 6 \times 10^{-7}$

rad, which means that the North Pole moves by about 4 meters during the wobble. The Chandler wobble has a period of about 14 months, so the naïve prediction of 305 days is off by a substantial amount. This discrepancy is attributed to the mechanical properties of the earth: elasticity and fluidity. The earth is not solid![8]

`Asymmetric tops:` Next, consider the torque-free motion of an asymmetric top, where $I_1 \neq I_2 \neq I_3 \neq I_1$. Unlike the symmetric case, there is no conserved component of $\boldsymbol{\omega}$. True, we can invoke conservation of energy and angular momentum,

$$E = \tfrac{1}{2} I_1\, \omega_1^2 + \tfrac{1}{2} I_2\, \omega_2^2 + \tfrac{1}{2} I_3\, \omega_3^2 \tag{2.666}$$

$$\boldsymbol{L}^2 = I_1^2\, \omega_1^2 + I_2^2\, \omega_2^2 + I_3^2\, \omega_3^2 \ , \tag{2.667}$$

and, in principle, solve for ω_1 and ω_2 in terms of ω_3, and then invoke Euler's equations (which must honor these conservation laws). However, the nonlinearity greatly complicates matters and in general this approach is a dead end.

We can, however, find a *particular* solution quite easily – one in which the rotation is about a single axis. Thus, $\omega_1 = \omega_2 = 0$ and $\omega_3 = \omega_0$ is indeed a solution for all time, according to Euler's equations. Let us now perturb about this solution, to explore its stability. We write

$$\boldsymbol{\omega} = \omega_0\, \hat{\mathbf{e}}_3 + \delta\boldsymbol{\omega} \ , \tag{2.668}$$

and we invoke Euler's equations, linearizing by dropping terms quadratic in $\delta\boldsymbol{\omega}$. This yield

$$I_1\, \delta\dot\omega_1 = (I_2 - I_3)\, \omega_0\, \delta\omega_2 + \mathcal{O}(\delta\omega_2\, \delta\omega_3) \tag{2.669}$$

$$I_2\, \delta\dot\omega_2 = (I_3 - I_1)\, \omega_0\, \delta\omega_1 + \mathcal{O}(\delta\omega_3\, \delta\omega_1) \tag{2.670}$$

$$I_3\, \delta\dot\omega_3 = 0 + \mathcal{O}(\delta\omega_1\, \delta\omega_2) \ . \tag{2.671}$$

Taking the time derivative of the first equation and invoking the second, and *vice versa,* yields

$$\delta\ddot\omega_1 = -\Omega^2\, \delta\omega_1 \quad , \quad \delta\ddot\omega_2 = -\Omega^2\, \delta\omega_2 \ , \tag{2.672}$$

with

$$\Omega^2 = \frac{(I_3 - I_2)(I_3 - I_1)}{I_1\, I_2} \cdot \omega_0^2 \ . \tag{2.673}$$

The solution is then $\delta\omega_1(t) = C\, \cos(\Omega t + \delta)$.

If $\Omega^2 > 0$, then Ω is real, and the deviation results in a harmonic precession. This occurs if I_3 is either the largest or the smallest of the moments of inertia. If, however, I_3 is the middle moment, then $\Omega^2 < 0$, and Ω is purely imaginary. The perturbation will in general increase exponentially with time, which means that the initial solution to Euler's equations is *unstable* with respect to small perturbations. This result can be vividly realized using a tennis racket, and sometimes goes by the name of the "tennis racket theorem."

[8] The earth is a layered like a *Mozartkugel,* with a solid outer shell, an inner fluid shell, and a solid (iron) core.

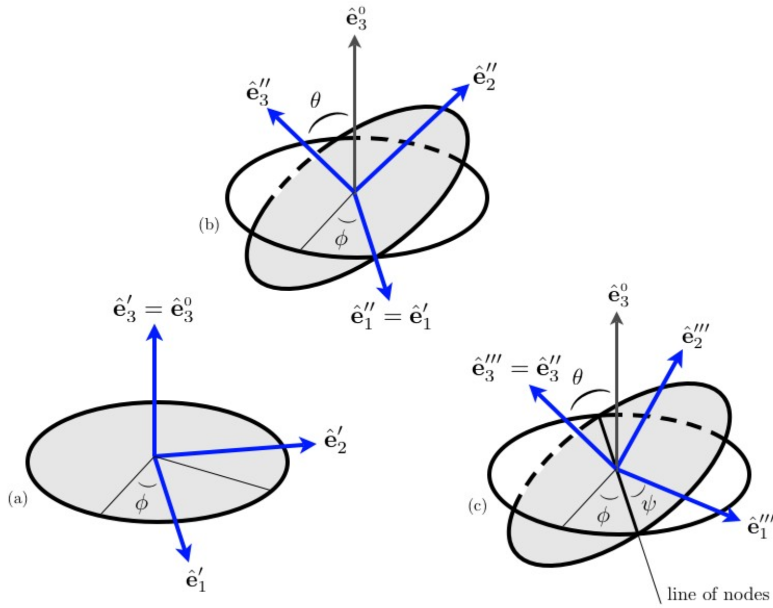

Figure 2.42: A general rotation, defined in terms of the Euler angles $\{\phi, \theta, \psi\}$. Three successive steps of the transformation are shown.

2.17.7 Euler's Angles

In d dimensions, an orthogonal matrix $\mathcal{R} \in \mathrm{O}(d)$ has $\frac{1}{2}d(d-1)$ independent parameters. To see this, consider the constraint $\mathcal{R}^{\mathrm{t}}\mathcal{R} = 1$. The matrix $\mathcal{R}^{\mathrm{t}}\mathcal{R}$ is manifestly symmetric, so it has $\frac{1}{2}d(d+1)$ independent entries (*e.g.* on the diagonal and above the diagonal). This amounts to $\frac{1}{2}d(d+1)$ constraints on the d^2 components of $\mathcal{R}$, resulting in $\frac{1}{2}d(d-1)$ freedoms. Thus, in $d = 3$ rotations are specified by three parameters. The *Euler angles* $\{\phi, \theta, \psi\}$ provide one such convenient parameterization.

A general rotation $\mathcal{R}(\phi, \theta, \psi)$ is built up in three steps. We start with an orthonormal triad $\hat{\mathbf{e}}_\mu^0$ of body-fixed axes. The first step is a rotation by an angle ϕ about $\hat{\mathbf{e}}_3^0$:

$$\hat{\mathbf{e}}_\mu' = \mathcal{R}_{\mu\nu}\big(\hat{\mathbf{e}}_3^0, \phi\big)\,\hat{\mathbf{e}}_\nu^0 \quad , \quad \mathcal{R}\big(\hat{\mathbf{e}}_3^0, \phi\big) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.674}$$

This step is shown in panel (a) of Fig. 2.42. The second step is a rotation by θ about the new axis $\hat{\mathbf{e}}_1'$:

$$\hat{\mathbf{e}}_\mu'' = \mathcal{R}_{\mu\nu}\big(\hat{\mathbf{e}}_1', \theta\big)\,\hat{\mathbf{e}}_\nu' \quad , \quad \mathcal{R}\big(\hat{\mathbf{e}}_1', \theta\big) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{2.675}$$

This step is shown in panel (b) of Fig. 2.42. The third and final step is a rotation by ψ

about the new axis $\hat{\mathbf{e}}_3''$:

$$\hat{\mathbf{e}}_\mu''' = \mathcal{R}_{\mu\nu}\big(\hat{\mathbf{e}}_3'', \psi\big)\, \hat{\mathbf{e}}_\nu'' \quad , \quad \mathcal{R}\big(\hat{\mathbf{e}}_3'', \psi\big) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.676}$$

This step is shown in panel (c) of Fig. 2.42. Putting this all together,

$$\mathcal{R}(\phi,\theta,\psi) = \mathcal{R}\big(\hat{\mathbf{e}}_3'', \phi\big)\, \mathcal{R}\big(\hat{\mathbf{e}}_1', \theta\big)\, \mathcal{R}\big(\hat{\mathbf{e}}_3^0, \psi\big) \tag{2.677}$$

$$= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\psi\cos\phi - \sin\psi\cos\theta\sin\phi & \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\psi\cos\theta\sin\phi & -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} .$$

Next, we'd like to relate the components $\omega_\mu = \boldsymbol{\omega}\cdot\hat{\mathbf{e}}_\mu$ (with $\hat{\mathbf{e}}_\mu \equiv \hat{\mathbf{e}}_\mu'''$) of the rotation in the body-fixed frame to the derivatives $\dot\phi$, $\dot\theta$, and $\dot\psi$. To do this, we write

$$\boldsymbol{\omega} = \dot\phi\, \hat{\mathbf{e}}_\phi + \dot\theta\, \hat{\mathbf{e}}_\theta + \dot\psi\, \hat{\mathbf{e}}_\psi \ , \tag{2.678}$$

where

$$\hat{\mathbf{e}}_3^0 = \hat{\mathbf{e}}_\phi = \sin\theta\sin\psi\, \hat{\mathbf{e}}_1 + \sin\theta\cos\psi\, \hat{\mathbf{e}}_2 + \cos\theta\, \hat{\mathbf{e}}_3 \tag{2.679}$$

$$\hat{\mathbf{e}}_\theta = \cos\psi\, \hat{\mathbf{e}}_1 - \sin\psi\, \hat{\mathbf{e}}_2 \qquad \text{(“line of nodes”)} \tag{2.680}$$

$$\hat{\mathbf{e}}_\psi = \hat{\mathbf{e}}_3 \ . \tag{2.681}$$

This gives

$$\omega_1 = \boldsymbol{\omega}\cdot\hat{\mathbf{e}}_1 = \dot\phi\, \sin\theta\sin\psi + \dot\theta\cos\psi \tag{2.682}$$

$$\omega_2 = \boldsymbol{\omega}\cdot\hat{\mathbf{e}}_2 = \dot\phi\, \sin\theta\cos\psi - \dot\theta\, \sin\psi \tag{2.683}$$

$$\omega_3 = \boldsymbol{\omega}\cdot\hat{\mathbf{e}}_3 = \dot\phi\cos\theta + \dot\psi \ . \tag{2.684}$$

Note that

$$\dot\phi \leftrightarrow \text{precession} \quad , \quad \dot\theta \leftrightarrow \text{nutation} \quad , \quad \dot\psi \leftrightarrow \text{axial rotation} \ . \tag{2.685}$$

The general form of the kinetic energy is then

$$\begin{aligned} T = {}& \tfrac{1}{2} I_1 \big(\dot\phi\, \sin\theta\sin\psi + \dot\theta\cos\psi\big)^2 \\ & + \tfrac{1}{2} I_2 \,\big(\dot\phi\, \sin\theta\cos\psi - \dot\theta\, \sin\psi\big)^2 + \tfrac{1}{2} I_3 \,\big(\dot\phi\cos\theta + \dot\psi\big)^2 \ . \end{aligned} \tag{2.686}$$

Note that

$$\boldsymbol{L} = p_\phi\, \hat{\mathbf{e}}_\phi + p_\theta\, \hat{\mathbf{e}}_\theta + p_\psi\, \hat{\mathbf{e}}_\psi \ , \tag{2.687}$$

which may be verified by explicit computation.

Torque-free symmetric top: A body falling in a gravitational field experiences no net torque about its CM:

$$\boldsymbol{N}^{\text{ext}} = \sum_i \boldsymbol{r}_i \times (-m_i\, \boldsymbol{g}) = \boldsymbol{g} \times \sum_i m_i\, \boldsymbol{r}_i = 0 \ . \tag{2.688}$$

For a symmetric top with $I_1 = I_2$, we have

$$T = \tfrac{1}{2} I_1 \big(\dot\theta^2 + \dot\phi^2\,\sin^2\!\theta\big) + \tfrac{1}{2} I_3 \big(\dot\phi \cos\theta + \dot\psi\big)^2 \ . \tag{2.689}$$

The potential is cyclic in the Euler angles, hence the equations of motion are

$$\frac{d}{dt}\,\frac{\partial T}{\partial(\dot\phi,\dot\theta,\dot\psi)} = \frac{\partial T}{\partial(\phi,\theta,\psi)} \ . \tag{2.690}$$

Since ϕ and ψ are cyclic in T, their conjugate momenta are conserved:

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = I_1\, \dot\phi\, \sin^2\!\theta + I_3\, (\dot\phi \cos\theta + \dot\psi)\, \cos\theta \tag{2.691}$$

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3\, (\dot\phi \cos\theta + \dot\psi) \ . \tag{2.692}$$

Note that $p_\psi = I_3\, \omega_3$, hence ω_3 is constant, as we have already seen.

To solve for the motion, it is convenient to choose $\hat{\boldsymbol{L}} = \hat{\mathbf{e}}_3^0 = \hat{\mathbf{e}}_\phi$. Thus, $p_\phi = L$. Since $\hat{\mathbf{e}}_\phi \cdot \hat{\mathbf{e}}_\psi = \cos\theta$, we have that $p_\psi = L\cos\theta$. On the other hand, $\hat{\mathbf{e}}_\phi \cdot \hat{\mathbf{e}}_\theta = 0$, which means $p_\theta = 0$. From the equations of motion,

$$\dot p_\theta = I_1\, \ddot\theta = \big(I_1\, \dot\phi \cos\theta - p_\psi\big)\, \dot\phi \sin\theta \ , \tag{2.693}$$

hence we must have

$$\dot\theta = 0 \quad , \quad \dot\phi = \frac{p_\psi}{I_1 \cos\theta} \ . \tag{2.694}$$

Finally, from the equation for p_ψ, we conclude

$$\dot\psi = \frac{p_\psi}{I_1} - \frac{p_\psi}{I_3} = \left(\frac{I_1 - I_3}{I_3}\right) \omega_3 \ , \tag{2.695}$$

which recapitulates (2.664), with $\dot\psi = -\Omega$.

2.17.8 Symmetric Top with One Point Fixed

Consider the case of a symmetric top with one point fixed, as depicted in Fig. 2.43. The Lagrangian is

$$L = \tfrac{1}{2} I_1 \big(\dot\theta^2 + \dot\phi^2\,\sin^2\!\theta\big) + \tfrac{1}{2} I_3 \big(\dot\phi \cos\theta + \dot\psi\big)^2 - Mg\ell\cos\theta \ . \tag{2.696}$$

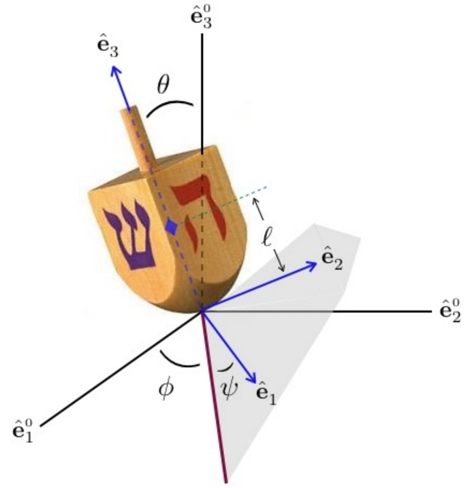

Figure 2.43: A *dreidl* is a symmetric top. The four-fold symmetry axis guarantees $I_1 = I_2$. The blue diamond represents the center-of-mass.

Here, ℓ is the distance from the fixed point to the CM, and the inertia tensor is defined along principal axes whose origin lies at the fixed point (not the CM!). Gravity now supplies a torque, but as in the torque-free case, the Lagrangian is still cyclic in ϕ and ψ, so

$$p_\phi = (I_1 \sin^2\!\theta + I_3 \cos^2\!\theta)\,\dot\phi + I_3 \cos\theta\,\dot\psi \tag{2.697}$$

$$p_\psi = I_3 \cos\theta\,\dot\phi + I_3\,\dot\psi \tag{2.698}$$

are each conserved. We can invert these relations to obtain $\dot\phi$ and $\dot\psi$ in terms of $\{p_\phi, p_\psi, \theta\}$:

$$\dot\phi = \frac{p_\phi - p_\psi\,\cos\theta}{I_1\,\sin^2\!\theta} \quad , \quad \dot\psi = \frac{p_\psi}{I_3} - \frac{(p_\phi - p_\psi\,\cos\theta)\,\cos\theta}{I_1\,\sin^2\!\theta} \ . \tag{2.699}$$

In addition, since $\partial L/\partial t = 0$, the total energy is conserved:

$$E = T + U = \tfrac{1}{2} I_1\,\dot\theta^2 + \overbrace{\frac{(p_\phi - p_\psi\,\cos\theta)^2}{2I_1\,\sin^2\!\theta} + \frac{p_\psi^2}{2I_3} + Mg\ell\cos\theta}^{U_{\rm eff}(\theta)} \ , \tag{2.700}$$

where the term under the brace is the effective potential $U_{\rm eff}(\theta)$.

The problem thus reduces to the one-dimensional dynamics of $\theta(t)$, *i.e.*

$$I_1\,\ddot\theta = -\frac{\partial U_{\rm eff}}{\partial\theta} \ , \tag{2.701}$$

with

$$U_{\rm eff}(\theta) = \frac{(p_\phi - p_\psi\,\cos\theta)^2}{2I_1\,\sin^2\!\theta} + \frac{p_\psi^2}{2I_3} + Mg\ell\cos\theta \ . \tag{2.702}$$

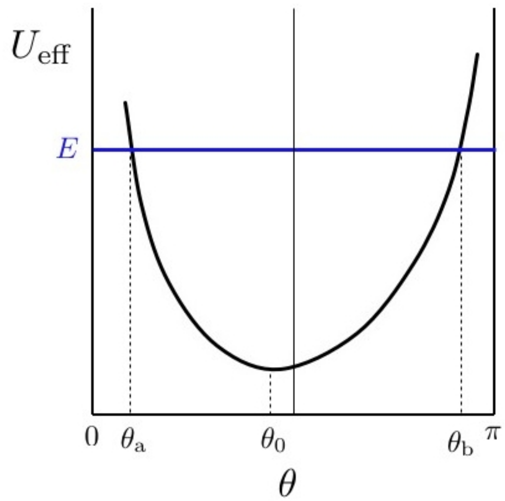

Figure 2.44: The effective potential of eq. 2.705.

Using energy conservation, we may write

$$dt = \pm\sqrt{\frac{I_1}{2}}\,\frac{d\theta}{\sqrt{E - U_{\rm eff}(\theta)}} \ . \tag{2.703}$$

and thus the problem is reduced to quadratures:

$$t(\theta) = t(\theta_0) \pm \sqrt{\frac{I_1}{2}} \int\limits_{\theta_0}^{\theta} \! d\vartheta \, \frac{1}{\sqrt{E - U_{\rm eff}(\vartheta)}} \ . \tag{2.704}$$

We can gain physical insight into the motion by examining the shape of the effective potential,

$$U_{\rm eff}(\theta) = \frac{(p_\phi - p_\psi \, \cos\theta)^2}{2I_1 \, \sin^2\!\theta} + Mg\ell\cos\theta + \frac{p_\psi^2}{2I_3} \ , \tag{2.705}$$

over the interval $\theta \in [0, \pi]$. Clearly $U_{\rm eff}(0) = U_{\rm eff}(\pi) = \infty$, so the motion must be bounded. What is not yet clear, but what is nonetheless revealed by some additional analysis, is that $U_{\rm eff}(\theta)$ has a single minimum on this interval, at $\theta = \theta_0$. The turning points for the θ motion are at $\theta = \theta_{\rm a}$ and $\theta = \theta_{\rm b}$, where $U_{\rm eff}(\theta_{\rm a}) = U_{\rm eff}(\theta_{\rm b}) = E$. Clearly if we expand about θ_0 and write $\theta = \theta_0 + \eta$, the η motion will be harmonic, with

$$\eta(t) = \eta_0 \, \cos(\varOmega t + \delta) \quad , \quad \varOmega = \sqrt{\frac{U''_{\rm eff}(\theta_0)}{I_1}} \ . \tag{2.706}$$

EXERCISE: Prove that the effective potential exhibits the generic features depicted in Fig. 2.44.

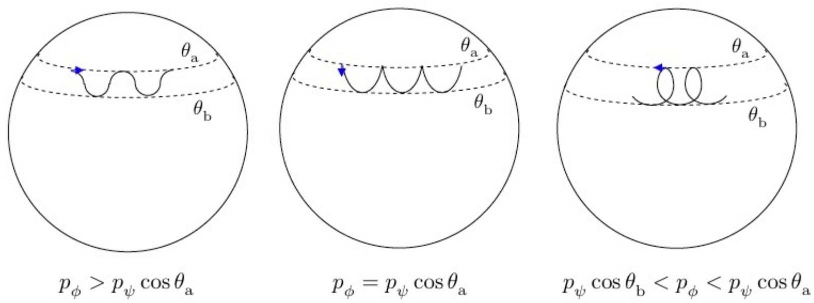

Figure 2.45: Precession and nutation of the symmetry axis of a symmetric top.

To apprehend the full motion of the top in an inertial frame, let us follow the symmetry axis $\hat{\mathbf{e}}_3$:

$$\hat{\mathbf{e}}_3 = \sin\theta \sin\phi \,\hat{\mathbf{e}}_1^0 - \sin\theta\cos\phi\,\hat{\mathbf{e}}_2^0 + \cos\theta\,\hat{\mathbf{e}}_3^0 \ . \tag{2.707}$$

Once we know $\theta(t)$ and $\phi(t)$ we're done. The motion $\theta(t)$ is described above: θ oscillates between turning points at θ_a and θ_b. As for $\phi(t)$, we have already derived the result

$$\dot{\phi} = \frac{p_\phi - p_\psi \, \cos\theta}{I_1 \, \sin^2\!\theta} \ . \tag{2.708}$$

Thus, if $p_\phi > p_\psi \cos\theta_a$, then $\dot{\phi}$ will remain positive throughout the motion. If, on the other hand, we have

$$p_\psi \cos\theta_b < p_\phi < p_\psi \cos\theta_a \ , \tag{2.709}$$

then $\dot{\phi}$ changes sign at an angle $\theta^* = \cos^{-1}\big(p_\phi/p_\psi\big)$. The motion is depicted in Fig. 2.45. An extensive discussion of this problem is given in H. Goldstein, *Classical Mechanics*.

2.17.9 Rolling and Skidding Motion of Real Tops

The material in this section is based on the corresponding sections from V. Barger and M. Olsson, *Classical Mechanics: A Modern Perspective*. This is an excellent book which contains many interesting applications and examples.

Rolling tops – In most tops, the point of contact rolls or skids along the surface. Consider the peg end top of Fig. 2.46, executing a circular rolling motion, as sketched in Fig. 2.47. There are three components to the force acting on the top: gravity, the normal force from the surface, and friction. The frictional force is perpendicular to the CM velocity, and results in centripetal acceleration of the top:

$$f = M\varOmega^2\rho \le \mu M g \ , \tag{2.710}$$

where $\varOmega$ is the frequency of the CM motion and μ is the coefficient of friction. If the above inequality is violated, the top starts to slip.

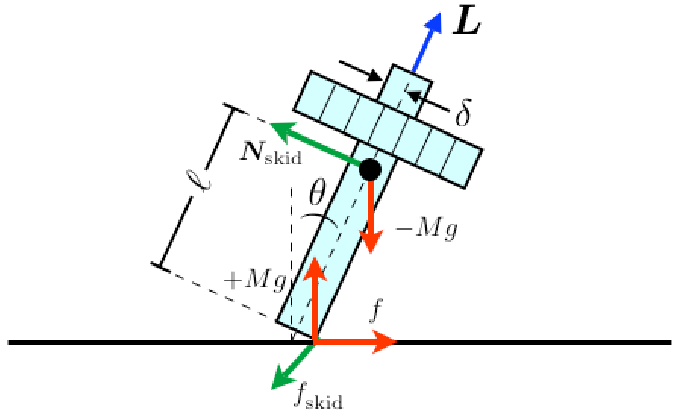

Figure 2.46: A top with a peg end. The frictional forces $\boldsymbol{f}$ and $\boldsymbol{f}_{\rm skid}$ are shown. When the top rolls without skidding, $\boldsymbol{f}_{\rm skid} = 0$.

The frictional and normal forces combine to produce a torque $N = Mg\ell \sin\theta - f\ell\cos\theta$ about the CM[9]. This torque is tangent to the circular path of the CM, and causes $\boldsymbol{L}$ to precess. We assume that the top is spinning rapidly, so that $\boldsymbol{L}$ very nearly points along the symmetry axis of the top itself. (As we'll see, this is true for slow precession but not for fast precession, where the precession frequency is proportional to ω_3.) The precession is then governed by the equation

$$\begin{aligned} N &= Mg\ell \sin\theta - f\ell\cos\theta \\ &= \big|\dot{\boldsymbol{L}}\big| = \big|\boldsymbol{\varOmega}\times\boldsymbol{L}\big| \approx \varOmega\, I_3\,\omega_3 \sin\theta\ , \end{aligned} \tag{2.711}$$

where $\hat{\mathbf{e}}_3$ is the instantaneous symmetry axis of the top. Substituting $f = M\varOmega^2\rho$,

$$\frac{Mg\ell}{I_3\,\omega_3}\left(1 - \frac{\varOmega^2\rho}{g}\,\operatorname{ctn}\theta\right) = \varOmega\ , \tag{2.712}$$

which is a quadratic equation for $\varOmega$. We supplement this with the 'no slip' condition,

$$\omega_3\,\delta = \varOmega\,\big(\rho + \ell\sin\theta\big)\ , \tag{2.713}$$

resulting in two equations for the two unknowns $\varOmega$ and ρ.

Substituting for $\rho(\varOmega)$ and solving for $\varOmega$, we obtain

$$\varOmega = \frac{I_3\,\omega_3}{2M\ell^2\cos\theta}\left\{1 + \frac{Mg\ell\delta}{I_3}\,\operatorname{ctn}\theta \pm \sqrt{\left(1 + \frac{Mg\ell\delta}{I_3}\,\operatorname{ctn}\theta\right)^2 - \frac{4M\ell^2}{I_3}\cdot\frac{Mg\ell}{I_3\,\omega_3^2}}\ .\right. \tag{2.714}$$

This in order to have a real solution we must have

$$\omega_3 \geq \frac{2M\ell^2\sin\theta}{I_3\sin\theta + Mg\ell\delta\cos\theta}\,\sqrt{\frac{g}{\ell}}\ . \tag{2.715}$$

[9]Gravity of course produces no net torque about the CM.

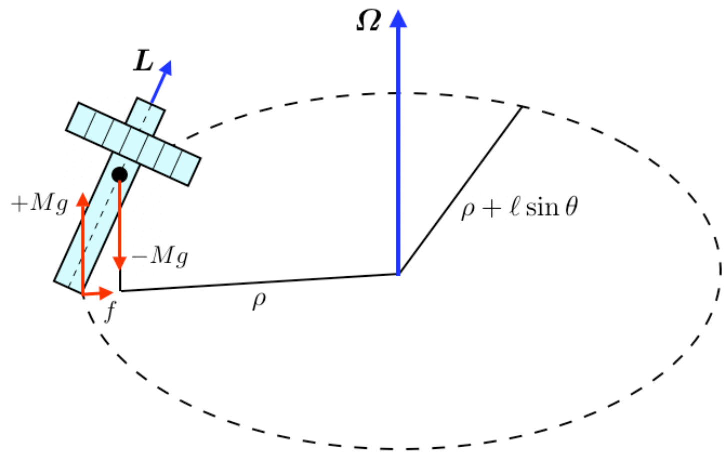

Figure 2.47: Circular rolling motion of the peg top.

If the inequality is satisfied, there are two possible solutions for Ω, corresponding to fast and slow precession. Usually one observes slow precession. Note that it is possible that $\rho < 0$, in which case the CM and the peg end lie on opposite sides of a circle from each other.

Skidding Tops – A skidding top experiences a frictional force which opposes the skidding velocity, until $v_{\rm skid} = 0$ and a pure rolling motion sets in. This force provides a torque which makes the top *rise:*

$$\dot\theta = -\frac{N_{\rm skid}}{L} = -\frac{\mu M g \ell}{I_3\,\omega_3} \ . \tag{2.716}$$

Suppose $\delta \approx 0$. in which case $\rho + \ell \sin\theta = 0$ and the point of contact remains fixed. Now recall the effective potential for a symmetric top with one point fixed:

$$U_{\rm eff}(\theta) = \frac{(p_\phi - p_\psi\,\cos\theta)^2}{2I_1\,\sin^2\!\theta} + \frac{p_\psi^2}{2I_3} + Mg\ell\cos\theta \ . \tag{2.717}$$

We demand $U'_{\rm eff}(\theta_0) = 0$, which yields

$$\cos\theta_0 \cdot \beta^2 - p_\psi \sin^2\!\theta_0 \cdot \beta + Mg\ell I_1 \sin^4\!\theta_0 = 0 \ , \tag{2.718}$$

where

$$\beta \equiv p_\phi - p_\psi \cos\theta_0 = I_1 \sin^2\!\theta_0\,\dot\phi \ . \tag{2.719}$$

Solving the quadratic equation for β, we find

$$\dot\phi = \frac{I_3\,\omega_3}{2I_1\cos\theta_0}\left(1 \pm \sqrt{1 - \frac{4Mg\ell I_1\cos\theta_0}{I_3^2\,\omega_3^2}}\right) \ . \tag{2.720}$$

This is simply a recapitulation of eqn. 2.714, with $\delta = 0$ and with $M\ell^2$ replaced by I_1. Note $I_1 = M\ell^2$ by the parallel axis theorem if $I_1^{\rm CM} = 0$. But to the extent that $I_1^{\rm CM} \neq 0$, our treatment of the peg top was incorrect. It turns out to be OK, however, if the precession is slow, *i.e.* if $\Omega/\omega_3 \ll 1$.

On a level surface, $\cos\theta_0 > 0$, and therefore we must have

$$\omega_3 \geq \frac{2}{I_3}\sqrt{Mg\ell I_1\cos\theta_0} \ . \tag{2.721}$$

Thus, if the top spins too slowly, it cannot maintain precession. Eqn. 2.720 says that there are two possible precession frequencies. When ω_3 is large, we have

$$\dot\phi_{\rm slow} = \frac{Mg\ell}{I_3\,\omega_3} + \mathcal{O}(\omega_3^{-1}) \qquad , \qquad \dot\phi_{\rm fast} = \frac{I_3\,\omega_3}{I_1\cos\theta_0} + \mathcal{O}(\omega_3^{-3}) \ . \tag{2.722}$$

Again, one usually observes slow precession.

A top with $\omega_3 > \frac{2}{I_3}\sqrt{Mg\ell I_1}$ may 'sleep' in the vertical position with $\theta_0 = 0$. Due to the constant action of frictional forces, ω_3 will eventually drop below this value, at which time the vertical position is no longer stable. The top continues to slow down and eventually falls.

2.17.10 Tippie-Top

A particularly nice example from the Barger and Olsson book is that of the tippie-top, a truncated sphere with a peg end, sketched in Fig. 2.48 The CM is close to the center of curvature, which means that there is almost no gravitational torque acting on the top.

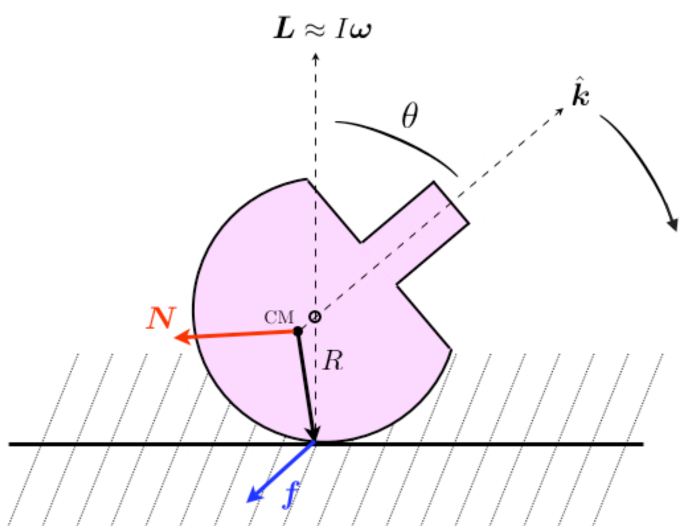

Figure 2.48: The tippie-top behaves in a counterintuitive way. Once started spinning with the peg end up, the peg axis rotates downward. Eventually the peg scrapes the surface and the top rises to the vertical in an inverted orientation.

The frictional force $\boldsymbol{f}$ opposes slipping, but as the top spins $\boldsymbol{f}$ rotates with it, and hence the time-averaged frictional force $\langle \boldsymbol{f} \rangle \approx 0$ has almost no effect on the motion of the CM. A similar argument shows that the frictional torque, which is nearly horizontal, also time averages to zero:

$$\left\langle \frac{d\boldsymbol{L}}{dt} \right\rangle_{\text{inertial}} \approx \quad 0 \ . \tag{2.723}$$

In the *body*-fixed frame, however, $\boldsymbol{N}$ is roughly constant, with magnitude $N \approx \mu M g R$, where R is the radius of curvature and μ the coefficient of sliding friction. Now we invoke

$$\boldsymbol{N} = \frac{d\boldsymbol{L}}{dt}\bigg|_{\text{body}} + \quad \boldsymbol{\omega} \times \boldsymbol{L} \ . \tag{2.724}$$

The second term on the RHS is very small, because the tippie-top is almost spherical, hence inertia tensor is very nearly diagonal, and this means

$$\boldsymbol{\omega} \times \boldsymbol{L} \approx \boldsymbol{\omega} \times I\boldsymbol{\omega} = 0 \ . \tag{2.725}$$

Thus, $\dot{\boldsymbol{L}}_{\text{body}} \approx \boldsymbol{N}$, and taking the dot product of this equation with the unit vector $\hat{\boldsymbol{k}}$, we obtain

$$-N \sin\theta = \hat{\boldsymbol{k}} \cdot \boldsymbol{N} = \frac{d}{dt}\Big(\hat{\boldsymbol{k}} \cdot \boldsymbol{L}_{\text{body}}\Big) = -L \sin\theta\, \dot{\theta} \ . \tag{2.726}$$

Thus,

$$\dot{\theta} = \frac{N}{L} \approx \frac{\mu M g R}{I\omega} \ . \tag{2.727}$$

Once the stem scrapes the table, the tippie-top rises to the vertical just like any other rising top.

2.18 Coupled Oscillations

2.18.1 General Nonlinear Problem

For a set of n generalized coordinates $\{q_1, \ldots, q_n\}$ the kinetic energy is a quadratic function of the velocities,

$$T = \tfrac{1}{2}\, T_{ij}(q_1, \ldots, q_n)\, \dot{q}_i\, \dot{q}_j \ , \tag{2.728}$$

where the sum on i and j from 1 to n is implied. For example, expressed in terms of polar coordinates (r, θ, ϕ), the matrix T_{ij} is

$$T_{ij} = m \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\!\theta \end{pmatrix} \quad \Longrightarrow \quad T = \tfrac{1}{2}\, m\big(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\!\theta\, \dot{\phi}^2\big) \ . \tag{2.729}$$

The potential $U(q_1, \ldots, q_n)$ is some function of the generalized coordinates. The Euler-Lagrange equations then give the equations of motion

$$T_{ij}(q)\, \ddot{q}_j + \left(\frac{\partial T_{ij}}{\partial q_k} - \frac{\partial T_{kj}}{\partial q_i} \right) \dot{q}_j\, \dot{q}_k = -\frac{\partial U}{\partial q_i} \ , \tag{2.730}$$

which is a set of coupled nonlinear second order ODEs.

2.18.2 Expansion about Static Equilibrium

Small oscillation theory begins with the identification of a static equilibrium $\{\bar{q}_1, \ldots, \bar{q}_n\}$, which satisfies the n nonlinear equations

$$\left. \frac{\partial U}{\partial q_i} \right|_{q=\bar{q}} = 0 \ . \tag{2.731}$$

Once an equilibrium is found (note that there may be more than one static equilibrium), we expand about this equilibrium, writing

$$q_i \equiv \bar{q}_i + \eta_i \ . \tag{2.732}$$

The coordinates $\{\eta_1, \ldots, \eta_n\}$ represent the *displacements relative to equilibrium.*

We next expand the Lagrangian to quadratic order in the generalized displacements, yielding

$$L = \tfrac{1}{2} \mathrm{T}_{ij}\, \dot{\eta}_i\, \dot{\eta}_j - \tfrac{1}{2} \mathrm{V}_{ij}\, \eta_i \eta_j \ , \tag{2.733}$$

where

$$\mathrm{T}_{ij} = \left. \frac{\partial^2 T}{\partial \dot{q}_i\, \partial \dot{q}_j} \right|_{q=\bar{q}} \quad , \qquad \mathrm{V}_{ij} = \left. \frac{\partial^2 U}{\partial q_i\, \partial q_j} \right|_{q=\bar{q}} \ . \tag{2.734}$$

Writing $\boldsymbol{\eta}^{\mathrm{t}}$ for the row-vector $(\eta_1, \ldots, \eta_n)$, we may suppress indices and write

$$L = \tfrac{1}{2}\, \dot{\boldsymbol{\eta}}^{\mathrm{t}}\, \mathrm{T}\, \dot{\boldsymbol{\eta}} - \tfrac{1}{2}\, \boldsymbol{\eta}^{\mathrm{t}}\, \mathrm{V}\, \boldsymbol{\eta} \ , \tag{2.735}$$

where T and V are the constant matrices of eqn. 2.734.

2.18.3 Method of Small Oscillations

The idea behind the method of small oscillations is to effect a coordinate transformation from the generalized displacements $\boldsymbol{\eta}$ to a new set of coordinates $\boldsymbol{\xi}$, which render the Lagrangian particularly simple. All that is required is a linear transformation,

$$\eta_i = \mathrm{A}_{ia}\, \xi_a \ , \tag{2.736}$$

where both i and a run from 1 to n. The $n \times n$ matrix A_{ia} is known as the *modal matrix.* With the substitution $\boldsymbol{\eta} = \mathrm{A}\,\boldsymbol{\xi}$ (hence $\boldsymbol{\eta}^{\mathrm{t}} = \boldsymbol{\xi}^{\mathrm{t}}\,\mathrm{A}^{\mathrm{t}}$, where $\mathrm{A}^{\mathrm{t}}_{ai} = \mathrm{A}_{ia}$ is the matrix transpose), we have

$$L = \tfrac{1}{2}\dot{\boldsymbol{\xi}}^{\mathrm{t}}\,\mathrm{A}^{\mathrm{t}}\,\mathrm{T}\,\mathrm{A}\,\dot{\boldsymbol{\xi}} - \boldsymbol{\xi}^{\mathrm{t}}\,\mathrm{A}^{\mathrm{t}}\,\mathrm{V}\,\mathrm{A}\,\boldsymbol{\xi}\ . \tag{2.737}$$

We now choose the matrix A such that

$$\mathrm{A}^{\mathrm{t}}\,\mathrm{T}\,\mathrm{A} = \mathbf{1} \tag{2.738}$$

$$\mathrm{A}^{\mathrm{t}}\,\mathrm{V}\,\mathrm{A} = \mathrm{diag}\,(\omega_1^2, \ldots, \omega_n^2)\ . \tag{2.739}$$

With this choice of A, the Lagrangian decouples:

$$L = \tfrac{1}{2}\sum_{a=1}^{n}\Big(\dot{\xi}_a^2 - \omega_a^2\,\xi_a^2\Big)\ , \tag{2.740}$$

with the solution

$$\xi_a(t) = C_a\,\cos(\omega_a t) + D_a\sin(\omega_a t)\ , \tag{2.741}$$

where $\{C_1, \ldots, C_n\}$ and $\{D_1, \ldots, D_n\}$ are $2n$ constants of integration. Note that

$$\boldsymbol{\xi} = \mathrm{A}^{-1}\boldsymbol{\eta} = \mathrm{A}^{\mathrm{t}}\mathrm{T}\,\boldsymbol{\eta}\ . \tag{2.742}$$

In terms of the original generalized displacements, the solution is

$$\eta_i(t) = \sum_{a=1}^{n}\mathrm{A}_{ia}\Big\{C_a\,\cos(\omega_a t) + D_a\sin(\omega_a t)\Big\}\ , \tag{2.743}$$

and the constants of integration are linearly related to the initial generalized displacements and generalized velocities:

$$C_a = \mathrm{A}_{ia}\,\mathrm{T}_{ij}\,\eta_j(0) \tag{2.744}$$

$$D_a = \omega_a^{-1}\mathrm{A}_{ia}\,\mathrm{T}_{ij}\,\dot{\eta}_j(0)\ , \tag{2.745}$$

where we have used $\mathrm{A}^{-1} = \mathrm{A}^{\mathrm{t}}\,\mathrm{T}$, from eqn. 2.738.

Note that the normal coordinates have unusual dimensions: $[\boldsymbol{\xi}] = L\sqrt{M}$ where L is length and M is mass.

2.18.4 Can you really just choose an A so that both these wonderful things happen in 2.738 and 2.739?

Yes.

2.18.5 Er...care to elaborate?

Both T and V are symmetric matrices. Aside from that, there is no special relation between them. In particular, they need not commute, hence they do not necessarily share any eigenvectors. Nevertheless, they may be simultaneously diagonalized as per 2.738 and 2.739. Here's why:

- Since T is symmetric, it can be diagonalized by an orthogonal transformation. That is, there exists a matrix $\mathcal{O}_1 \in \mathrm{O}(n)$ such that

$$\mathcal{O}_1^{\mathrm{t}}\, \mathrm{T}\, \mathcal{O}_1 = \mathrm{T}_{\mathrm{d}} \ , \tag{2.746}$$

  where T_{d} is diagonal.

- We may safely assume that T is positive definite. Otherwise the kinetic energy can become arbitrarily negative, which is unphysical. Therefore, one may form the matrix $\mathrm{T}_{\mathrm{d}}^{-1/2}$ which is the diagonal matrix whose entries are the inverse square roots of the corresponding entries of T_{d}. Consider the linear transformation $\mathcal{O}_1\, \mathrm{T}_{\mathrm{d}}^{-1/2}$. Its effect on T is

$$\mathrm{T}_{\mathrm{d}}^{-1/2}\, \mathcal{O}_1^{\mathrm{t}}\, \mathrm{T}\, \mathcal{O}_1\, \mathrm{T}_{\mathrm{d}}^{-1/2} = 1 \ . \tag{2.747}$$

- Since $\mathcal{O}_1$ and T_{d} are wholly derived from T, the only thing we know about

$$\widetilde{\mathrm{V}} \equiv \mathrm{T}_{\mathrm{d}}^{-1/2}\, \mathcal{O}_1^{\mathrm{t}}\, \mathrm{V}\, \mathcal{O}_1\, \mathrm{T}_{\mathrm{d}}^{-1/2} \tag{2.748}$$

  is that it is explicitly a symmetric matrix. Therefore, it may be diagonalized by some orthogonal matrix $\mathcal{O}_2 \in \mathrm{O}(n)$. As T has already been transformed to the identity, the additional orthogonal transformation has no effect there. Thus, we have shown that there exist orthogonal matrices $\mathcal{O}_1$ and $\mathcal{O}_2$ such that

$$\mathcal{O}_2^{\mathrm{t}}\, \mathrm{T}_{\mathrm{d}}^{-1/2}\, \mathcal{O}_1^{\mathrm{t}}\, \mathrm{T}\, \mathcal{O}_1\, \mathrm{T}_{\mathrm{d}}^{-1/2}\, \mathcal{O}_2 = 1 \tag{2.749}$$

$$\mathcal{O}_2^{\mathrm{t}}\, \mathrm{T}_{\mathrm{d}}^{-1/2}\, \mathcal{O}_1^{\mathrm{t}}\, \mathrm{V}\, \mathcal{O}_1\, \mathrm{T}_{\mathrm{d}}^{-1/2}\, \mathcal{O}_2 = \mathrm{diag}\,(\omega_1^2, \ldots, \omega_n^2) \ . \tag{2.750}$$

  All that remains is to identify the modal matrix $\mathrm{A} = \mathcal{O}_1\, \mathrm{T}_{\mathrm{d}}^{-1/2}\, \mathcal{O}_2$.

2.18.6 Finding the Modal Matrix

While the above proof allows one to construct A by finding the two orthogonal matrices $\mathcal{O}_1$ and $\mathcal{O}_2$, such a procedure is extremely cumbersome. It would be much more convenient if A could be determined in one fell swoop. Fortunately, this is possible.

The eigenfrequencies ω_a are obtained from the equations of motion,

$$\mathrm{A}^{\mathrm{t}}\, \mathrm{T}\, \mathrm{A}\, \ddot{\boldsymbol{\xi}} + \mathrm{A}^{\mathrm{t}}\, \mathrm{V}\, \mathrm{A}\, \boldsymbol{\xi} = 0 \ . \tag{2.751}$$

If $\boldsymbol{\xi}$ oscillates at frequency ω, then $\ddot{\boldsymbol{\xi}} = -\omega^2 \boldsymbol{\xi}$, and a nontrivial solution exists only if the matrix $\omega^2 \mathrm{T} - \mathrm{V}$ is defective, *i.e.* if

$$\det\big(\omega^2\, \mathrm{T} - \mathrm{V}\big) = 0 \ . \tag{2.752}$$

Since T and V are of rank n, the above determinant yields an $n^{\rm th}$ order polynomial whose n roots are the desired squared eigenfrequencies $\{\omega_1^2, \ldots, \omega_n^2\}$.

Once the n eigenfrequencies are obtained, the modal matrix is constructed as follows. Solve the equations

$$\big(\omega_a^2\, \mathrm{T}_{ij} - \mathrm{V}_{ij}\big)\psi_j^a = 0 \tag{2.753}$$

which are a set of $(n-1)$ linearly independent equations among the n components of the eigenvector $\boldsymbol{\psi}^a$. The eigenvectors may be chosen to satisfy a generalized orthogonality relationship, $\psi_i^a\, \mathrm{T}_{ij}\, \psi_j^b = \delta_{ij}$. To see this, let us duplicate eqn. 2.753, replacing a with b, and multiply both equations as follows:

$$\psi_i^b \times \big(\omega_a^2\, \mathrm{T}_{ij} - \mathrm{V}_{ij}\big)\psi_j^a = 0 \tag{2.754}$$

$$\psi_i^a \times \big(\omega_b^2\, \mathrm{T}_{ij} - \mathrm{V}_{ij}\big)\psi_j^b = 0 \ . \tag{2.755}$$

Using the symmetry of T and V, upon subtracting these equations we obtain

$$(\omega_a^2 - \omega_b^2) \sum_{i,j=1}^{n} \psi_i^a\, \mathrm{T}_{ij}\, \psi_j^b = 0 \ , \tag{2.756}$$

where the sums on i and j have been made explicit. This establishes that eigenvectors $\boldsymbol{\psi}^a$ and $\boldsymbol{\psi}^b$ corresponding to distinct eigenvalues $\omega_a^2 \neq \omega_b^2$ are orthogonal: $(\boldsymbol{\psi}^a)^{\rm t}\, \mathrm{T}\, \boldsymbol{\psi}^b = 0$. For degenerate eigenvalues, the eigenvectors are not *a priori* orthogonal, but they may be orthogonalized via application of the Gram-Schmidt procedure. The remaining degrees of freedom - one for each eigenvector - are fixed by imposing the condition of normalization:

$$\psi_i^a \to \psi_i^a \Big/ \sqrt{\psi_k^a\, \mathrm{T}_{kl}\, \psi_l^a} \qquad \Longrightarrow \qquad \psi_i^a\, \mathrm{T}_{ij}\, \psi_j^b = \delta_{ab} \ . \tag{2.757}$$

The modal matrix is just the matrix of eigenvectors: $\mathrm{A}_{ia} = \psi_i^a$.

2.18.7 Example: Double Pendulum

As an example, consider the double pendulum, with $m_1 = m_2 = m$ and $\ell_1 = \ell_2 = \ell$. The kinetic and potential energies are

$$T = m\ell^2 \dot\theta_1^2 + m\ell^2 \cos(\theta_1 - \theta_1)\, \dot\theta_1 \dot\theta_2 + \tfrac{1}{2} m\ell^2 \dot\theta_2^2 \tag{2.758}$$

$$V = -2mg\ell \cos\theta_1 - mg\ell\cos\theta_2 \ , \tag{2.759}$$

leading to

$$\mathrm{T} = \begin{pmatrix} 2m\ell^2 & m\ell^2 \\ m\ell^2 & m\ell^2 \end{pmatrix} \quad , \quad \mathrm{V} = \begin{pmatrix} 2mg\ell & 0 \\ 0 & mg\ell \end{pmatrix} \ . \tag{2.760}$$

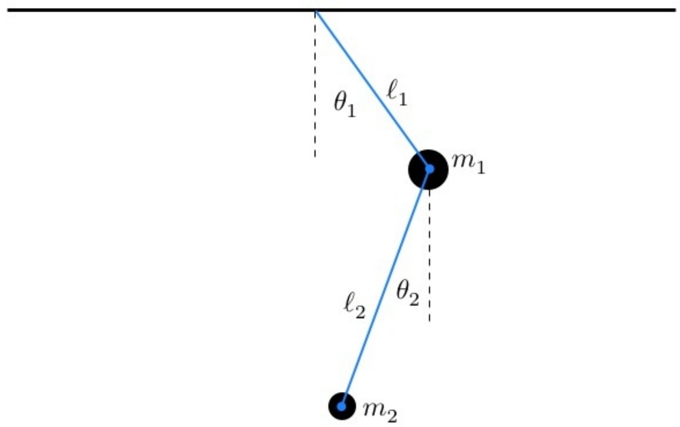

Figure 2.49: The double pendulum.

Then

$$\omega^2 \mathrm{T} - \mathrm{V} = m\ell^2 \begin{pmatrix} 2\omega^2 - 2\omega_0^2 & \omega^2 \\ \omega^2 & \omega^2 - \omega_0^2 \end{pmatrix} \ , \tag{2.761}$$

with $\omega_0 = \sqrt{g/\ell}$. Setting the determinant to zero gives

$$2(\omega^2 - \omega_0^2)^2 - \omega^4 = 0 \quad \Rightarrow \quad \omega^2 = (2 \pm \sqrt{2})\, \omega_0^2 \ . \tag{2.762}$$

We find the unnormalized eigenvectors by setting $(\omega_a^2 \mathrm{T} - V)\, \psi^a = 0$. This gives

$$\psi^+ = C_+ \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \quad , \quad \psi^- = C_- \begin{pmatrix} 1 \\ +\sqrt{2} \end{pmatrix} \ , \tag{2.763}$$

where $C_\pm$ are constants. One can check $\mathrm{T}_{ij}\, \psi_i^a\, \psi_j^b$ vanishes for $a \neq b$. We then normalize by demanding $\mathrm{T}_{ij}\, \psi_i^a\, \psi_j^a = 1$, which determines the coefficients $C_\pm = \frac{1}{2}\sqrt{(2 \pm \sqrt{2})/m\ell^2}$. Thus, the modal matrix is

$$A = \begin{pmatrix} \psi_1^+ & \psi_1^- \\ \psi_2^+ & \psi_2^- \end{pmatrix} = \frac{1}{2\sqrt{m\ell^2}} \begin{pmatrix} \sqrt{2+\sqrt{2}} & \sqrt{2-\sqrt{2}} \\ -\sqrt{4+2\sqrt{2}} & +\sqrt{4-2\sqrt{2}} \end{pmatrix} \ . \tag{2.764}$$

2.18.8 Chain of Mass Points

Next consider an infinite chain of identical masses, connected by identical springs of spring constant k and equilibrium length a. The Lagrangian is

$$\begin{aligned} L &= \tfrac{1}{2} m \sum_n \dot{x}_n^2 - \tfrac{1}{2} k \sum_n (x_{n+1} - x_n - a)^2 \\ &= \tfrac{1}{2} m \sum_n \dot{u}_n^2 - \tfrac{1}{2} k \sum_n (u_{n+1} - u_n)^2 \ , \end{aligned} \tag{2.765}$$

where $u_n \equiv x_n - na - b$ is the displacement from equilibrium of the n^{th} mass. The constant b is arbitrary. The Euler-Lagrange equations are

$$\begin{aligned}\frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{u}_n}\bigg) = m\ddot{u}_n &= \frac{\partial L}{\partial u_n} \\ &= k(u_{n+1} - u_n) - k(u_n - u_{n-1}) \\ &= k(u_{n+1} + u_{n-1} - 2u_n)\ . \end{aligned} \tag{2.766}$$

Now let us assume that the system is placed on a large ring of circumference Na, where $N \gg 1$. Then $u_{n+N} = u_n$ and we may shift to Fourier coefficients,

$$u_n = \frac{1}{\sqrt{N}} \sum_q e^{iqan}\, \hat{u}_q \tag{2.767}$$

$$\hat{u}_q = \frac{1}{\sqrt{N}} \sum_n e^{-iqan}\, u_n\ , \tag{2.768}$$

where $q_j = 2\pi j/Na$, and both sums are over the set $j, n \in \{1, \ldots, N\}$. Expressed in terms of the $\{\hat{u}_q\}$, the equations of motion become

$$\begin{aligned}\ddot{\hat{u}}_q &= \frac{1}{\sqrt{N}} \sum_n e^{-iqna}\, \ddot{u}_n \\ &= \frac{k}{m}\frac{1}{\sqrt{N}} \sum_n e^{-iqan}\, (u_{n+1} + u_{n-1} - 2u_n) \\ &= \frac{k}{m}\frac{1}{\sqrt{N}} \sum_n e^{-iqan}\, (e^{-iqa} + e^{+iqa} - 2)\, u_n \\ &= -\frac{2k}{m}\, \sin^2\big(\tfrac{1}{2}qa\big)\, \hat{u}_q \end{aligned} \tag{2.769}$$

Thus, the $\{\hat{u}_q\}$ *are* the normal modes of the system (up to a normalization constant), and the eigenfrequencies are

$$\omega_q = \frac{2k}{m}\, \big|\sin\big(\tfrac{1}{2}qa\big)\big|\ . \tag{2.770}$$

This means that the modal matrix is

$$\mathrm{A}_{nq} = \frac{1}{\sqrt{Nm}}\, e^{iqan}\ , \tag{2.771}$$

where we've included the $\frac{1}{\sqrt{m}}$ factor for a proper normalization. (The normal modes themselves are then $\xi_q = \mathrm{A}^\dagger_{qn}\mathrm{T}_{nn'}u_{n'} = \sqrt{m}\,\hat{u}_q$. For complex A, the normalizations are $\mathrm{A}^\dagger\mathrm{T}\mathrm{A} = \mathbf{1}$ and $\mathrm{A}^\dagger\mathrm{V}\mathrm{A} = \mathrm{diag}(\omega_1^2, \ldots, \omega_N^2)$.

Note that

$$\mathrm{T}_{nn'} = m\,\delta_{n,n'} \tag{2.772}$$

$$\mathrm{V}_{nn'} = 2k\,\delta_{n,n'} - k\,\delta_{n,n'+1} - k\,\delta_{n,n'-1} \tag{2.773}$$

and that

$$
\begin{aligned}
(\mathrm{A}^\dagger \mathrm{T} \mathrm{A})_{qq'} &= \sum_{n=1}^N \sum_{n'=1}^N \mathrm{A}^*_{nq} \mathrm{T}_{nn'} \mathrm{A}_{n'q'} \\
&= \frac{1}{Nm} \sum_{n=1}^N \sum_{n'=1}^N e^{-iqan}\, m\, \delta_{nn'}\, e^{iq'an'} \\
&= \frac{1}{N} \sum_{n=1}^N e^{i(q'-q)an} = \delta_{qq'} \ , \qquad (2.774)
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathrm{A}^\dagger \mathrm{V} \mathrm{A})_{qq'} &= \sum_{n=1}^N \sum_{n'=1}^N \mathrm{A}^*_{nq} \mathrm{T}_{nn'} \mathrm{A}_{n'q'} \\
&= \frac{1}{Nm} \sum_{n=1}^N \sum_{n'=1}^N e^{-iqan} \Big(2k\, \delta_{n,n'} - k\, \delta_{n,n'+1} - k\, \delta_{n,n'-1}\Big)\, e^{iq'an'} \\
&= \frac{k}{m} \frac{1}{N} \sum_{n=1}^N e^{i(q'-q)an} \big(2 - e^{-iq'a} - e^{iq'a}\big) \\
&= \frac{4k}{m} \sin^2\!\big(\tfrac{1}{2}qa\big)\, \delta_{qq'} = \omega_q^2\, \delta_{qq'} \qquad (2.775)
\end{aligned}
$$

Since $\hat{x}_{q+G} = \hat{x}_q$, where $G = \frac{2\pi}{a}$, we may choose any set of q values such that no two are separated by an integer multiple of G. The set of points $\{jG\}$ with $j \in \mathbf{Z}$ is called the *reciprocal lattice*. For a linear chain, the reciprocal lattice is itself a linear chain[10]. One natural set to choose is $q \in \big[-\frac{\pi}{a}, \frac{\pi}{a}\big]$. This is known as the *first Brillouin zone* of the reciprocal lattice.

Finally, we can write the Lagrangian itself in terms of the $\{u_q\}$. One easily finds

$$
L = \tfrac{1}{2}\, m \sum_q \dot{\hat{u}}^*_q\, \dot{\hat{u}}_q - \, k \sum_q (1 - \cos qa)\, \hat{u}^*_q\, \hat{u}_q \ , \qquad (2.776)
$$

where the sum is over q in the first Brillouin zone. Note that

$$
\hat{u}_{-q} = \hat{u}_{-q+G} = \hat{u}^*_q \ . \qquad (2.777)
$$

This means that we can restrict the sum to half the Brillouin zone:

$$
L = \tfrac{1}{2} m \sum_{q \in [0, \frac{\pi}{a}]} \bigg\{ \dot{\hat{u}}^*_q\, \dot{\hat{u}}_q - \frac{4k}{m} \sin^2\!\big(\tfrac{1}{2}qa\big)\, \hat{u}^*_q\, \hat{u}_q \bigg\} \ . \qquad (2.778)
$$

[10]For higher dimensional Bravais lattices, the reciprocal lattice is often different than the real space ("direct") lattice. For example, the reciprocal lattice of a face-centered cubic structure is a body-centered cubic lattice.

Now $\hat{u}_q$ and $\hat{u}_q^*$ may be regarded as linearly independent, as one regards complex variables z and z^*. The Euler-Lagrange equation for $\hat{u}_q^*$ gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\hat{u}}_q^*}\right) = \frac{\partial L}{\partial \hat{u}_q^*} \quad \Rightarrow \quad \ddot{\hat{u}}_q = -\omega_q^2\, \hat{u}_q \ . \tag{2.779}$$

Extremizing with respect to $\hat{u}_q$ gives the complex conjugate equation.

2.18.9 Continuum Limit

Let us take $N \to \infty$, $a \to 0$, with $L_0 = Na$ fixed. We'll write

$$u_n(t) \longrightarrow u(x = na, t) \tag{2.780}$$

in which case

$$T = \tfrac{1}{2} m \sum_n \dot{u}_n^2 \quad \longrightarrow \quad \tfrac{1}{2} m \int \frac{dx}{a} \left(\frac{\partial u}{\partial t}\right)^2 \tag{2.781}$$

$$V = \tfrac{1}{2} k \sum_n (u_{n+1} - u_n)^2 \quad \longrightarrow \quad \tfrac{1}{2} k \int \frac{dx}{a} \left(\frac{u(x+a) - u(x)}{a}\right)^2 a^2 \tag{2.782}$$

Recognizing the spatial derivative above, we finally obtain

$$\begin{aligned} L &= \int dx\, \mathcal{L}(u, \partial_t u, \partial_x u) \\ \mathcal{L} &= \tfrac{1}{2}\, \mu \left(\frac{\partial u}{\partial t}\right)^2 - \tfrac{1}{2}\, \tau \left(\frac{\partial u}{\partial x}\right)^2 , \end{aligned} \tag{2.783}$$

where $\mu = m/a$ is the linear mass density and $\tau = ka$ is the tension[11]. The quantity $\mathcal{L}$ is the *Lagrangian density*; it depends on the field $u(x,t)$ as well as its partial derivatives $\partial_t u$ and $\partial_x u$[12]. The action is

$$S\big[u(x,t)\big] = \int\limits_{t_a}^{t_b} dt \int\limits_{x_a}^{x_b} dx\, \mathcal{L}(u, \partial_t u, \partial_x u) \ , \tag{2.784}$$

where $\{x_a, x_b\}$ are the limits on the x coordinate. Setting $\delta S = 0$ gives the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\,(\partial_t u)}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial\,(\partial_x u)}\right) = 0 \ . \tag{2.785}$$

For our system, this yields the Helmholtz equation,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \ , \tag{2.786}$$

[11] For a proper limit, we demand μ and τ be neither infinite nor infinitesimal.

[12] $\mathcal{L}$ may also depend explicitly on x and t.

where $c = \sqrt{\tau/\mu}$ is the velocity of wave propagation. This is a linear equation, solutions of which are of the form

$$u(x,t) = C\, e^{iqx}\, e^{-i\omega t} \ , \tag{2.787}$$

where

$$\omega = cq \ . \tag{2.788}$$

Note that in the continuum limit $a \to 0$, the dispersion relation derived for the chain becomes

$$\omega_q^2 = \frac{4k}{m} \sin^2\big(\tfrac{1}{2}qa\big) \longrightarrow \frac{ka^2}{m}\, q^2 = c^2\, q^2 \ , \tag{2.789}$$

and so the results agree.