## Waves

What is a wave? Whitham says: any recognizable signal transferred from one part of a medium to another, with a recognizable velocity of propagation. E.g. maximum, discontinuity, etc.

Two broad classes of wave systems:

- hyperbolic: arising from hyperbolic PDEs, which may be treated using the method of characteristics. Examples:

$$\varphi_{tt} = c_0^2 \varphi_{xx} \quad , \quad \varphi_t + c_0 \varphi_x = 0 \quad \text{(simplest)}$$

- dispersive: classification based on form of sol$^{\underline{n}}$, rather than on form of equation(s). Linear dispersive systems admit sol$^{\underline{ns}}$ of the form $\varphi(x,t) = A\cos(kx - \omega t)$ with $\omega = \omega(k)$.

  We require $\omega''(k) \neq 0$.

Note that these categories are not mutually exclusive. E.g.

$$\frac{1}{c^2}\varphi_{tt} = \varphi_{xx} + k_0^2 \varphi \qquad \text{(KG eqn)}$$

is hyperbolic but also dispersive in that it admits a sol$^{\underline{n}}$ $A\cos(kx-\omega t)$ with $\omega = \pm c\sqrt{k^2 + k_0^2}$.

### Hyperbolic Systems

The first nonlinear PDE we'll study will be

$$\varphi_t + c(\varphi)\varphi_x = 0$$

Its sol$^{\underline{n}}$ will be a sort of template for hyperbolic systems in general. Main effect of nonlinearity: shocks. Mathematical theory: method of characteristics.

## Method of Characteristics

Consider the quasilinear PDE,

$$a_1(\vec{x},\varphi)\frac{\partial\varphi}{\partial x_1} + a_2(\vec{x},\varphi)\frac{\partial\varphi}{\partial x_2} + \dots + a_n(\vec{x},\varphi)\frac{\partial\varphi}{\partial x_n} = b(\vec{x},\varphi)$$

This is called "quasilinear" because it is linear in the $\frac{\partial\varphi}{\partial x_j}$.

A sol$^{\underline{n}}$ is of the form $\varphi = \varphi(\vec{x})$.

Now consider a curve $\{x_j(s)\}$ parameterized by $s$.

The variation of $\varphi$ along this curve is

$$\frac{d\varphi}{ds} = \frac{\partial\varphi}{\partial x_1}\frac{dx_1}{ds} + \dots + \frac{\partial\varphi}{\partial x_n}\frac{dx_n}{ds}$$

Let the curve $\vec{x}(s)$ be defined by

$$\frac{dx_j}{ds} = a_j(\vec{x},\varphi)$$

Then

$$\frac{d\varphi}{ds} = b(\vec{x}(s),\varphi)$$

and the PDE has been converted into a set of $(n+1)$ ODEs.

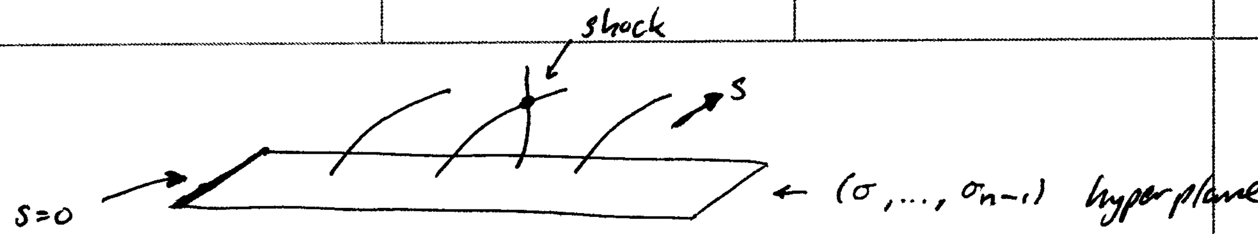
To integrate, we must apply initial conditions of the form

$$g(\vec{x}(0),\varphi) = 0$$

This defines an $(n-1)$-dimensional hypersurface, which may be parameterized by $\{\sigma_1,\dots,\sigma_{n-1}\}$, viz.

$$x_j(s=0) = h_j(\sigma_1,\dots,\sigma_{n-1}) \qquad j=1,\dots,n$$

$$\varphi(s=0) = f(\sigma_1,\dots,\sigma_{n-1})$$




Characteristics are specified by $\{\sigma_1, \ldots, \sigma_{n-1}\}$, and $s$ is the (scaled) distance along the characteristic. When two characteristics cross, there is a <u>shock</u>.

<u>Example</u>

Solve

$$\varphi_t + t^2 \varphi_x = -x\varphi$$

subject to

$$\varphi(x, t=0) = f(x)$$

We define the curve $(x(s), t(s))$ by

$$\frac{dt}{ds} = 1 \quad , \quad \frac{dx}{ds} = t^2$$

so that

$$\frac{d\varphi}{ds} = -x\varphi$$

The initial data are expressed parametrically as

$$t(s=0) = 0$$

$$x(s=0) = \sigma$$

$$\varphi(s=0) = f(\sigma)$$

We have

$$\frac{dt}{ds} = 1 \implies t(s, \sigma) = s$$

Thus,

$$\frac{dx}{ds} = t^2 = s^2 \quad ; \quad x(s=0) = \sigma$$

$$\Rightarrow \quad x(s,\sigma) = \sigma + \tfrac{1}{3}s^3$$

Finally,

$$\frac{d\varphi}{ds} = -x\varphi = -(\sigma + \tfrac{1}{3}s^3)\varphi \quad ; \quad \varphi(s=0) = f(\sigma)$$

$$\Rightarrow \quad \varphi(s,\sigma) = f(\sigma)\ \exp\left(-\tfrac{1}{12}s^4 - s\sigma\right)$$

In this case, we may eliminate $\sigma$, since $\sigma = x - \frac{1}{3}s^3$ and $s = t$:

$$\varphi(x,t) = f(x - \tfrac{1}{3}t^3)\ \exp\left(\tfrac{1}{4}x^4 - xy\right)$$

This method may be extended to more general quasilinear PDEs of the form

$$A_{ij}\ \frac{\partial \varphi_j}{\partial t} + B_{ij}\ \frac{\partial \varphi_j}{\partial x} + q_i = 0 \qquad i,j = 1,\dots,n$$

where $A_{ij} = A_{ij}(x,t,\vec{\varphi})$, $B_{ij} = B_{ij}(x,t,\vec{\varphi})$, $q_i = q_i(x,t,\vec{\varphi})$, provided there exist $n$ linearly independent real-valued vectors $v_i^{(k)}$ and $n$ non-zero real-valued two-dimensional vectors $(\alpha^{(k)}, \beta^{(k)})$ such that

$$\sum_{i,j=1}^{n} v_i^{(k)}\left[A_{ij}\ \alpha^{(k)} - B_{ij}\,\beta^{(k)}\right] = 0$$

This is the general condition for hyperbolicity of quasilinear PDEs

We will start our investigation of nonlinear hyperbolic systems by considering the simplest such example,

$$\varphi_t + c(\varphi)\,\varphi_x = 0$$

This equation provides a laboratory of concepts applicable to more complex systems.

## Dispersive Waves

These aren't as easily classified as hyperbolic waves. Generally, dispersive waves involve wave trains. For linear systems, sol$^{\underline{n}}$s are of the form

$$\varphi(x,t) = A\cos(kx - \omega t + \delta)$$

with

$$\omega = \omega(k) \qquad \text{(dispersion relation)}$$

Examples:

beam eq$^{\underline{n}}$: $\quad \varphi_{tt} + \gamma^2 \varphi_{xxxx} = 0 \quad \Rightarrow \quad \omega = \pm \gamma k^2$

Korteweg-de Vries: $\quad \varphi_t + c_0 \varphi_x + \nu \varphi_{xxx} = 0 \quad \Rightarrow \quad \omega = c_0 k - \nu k^3$

linear Boussinesq: $\quad \varphi_{tt} - \alpha^2 \varphi_{xx} - \beta^2 \varphi_{xxtt} = 0 \quad \Rightarrow \quad \omega = \pm \dfrac{\alpha k}{\sqrt{1 + \beta k^2}}$

For water waves at constant depth, $h$, we will show

$$\omega(k) = \pm\sqrt{gk \tanh kh} = \begin{cases} \sqrt{gk} & \text{if } h \gg \lambda \\ \sqrt{gh}\, k & \text{if } h \ll \lambda \end{cases}$$

Thus, the velocity of shallow water waves is $c = \sqrt{gh}$.
For $h = 5\,m \ll \lambda$, $c \simeq 7\,m/s = 16$ mph. For $h = 5000\,m \ll \lambda$, $c \simeq 220\,m/s \simeq 500$ mph. What sort of disturbance can create a pulse with $\lambda \gg 5\,km$? An undersea earthquake. The resulting deep ocean wave is basically a shallow water wave with a huge wavelength. This is called a tsunami.
As the tsunami comes ashore, $h$ decreases, but energy is conserved, hence the amplitude of the wave must increase.

Energy density $\quad S \propto \dfrac{\partial\varphi}{\partial x}\dfrac{\partial\varphi}{\partial t} = k\omega A^2 = \left(\dfrac{2\pi}{\lambda}\right)^2 \sqrt{gh}\, A^2$

$$\Rightarrow A \propto h^{-1/4} \Rightarrow \frac{A(h_1)}{A(h_0)} = \left(\frac{h_0}{h_1}\right)^{1/4} = \left(\frac{5000\,m}{5\,m}\right)^{1/4} \simeq 5.6$$

A general sol$^{\underline{n}}$ to a linear dispersive system is obtained by superposition:

$$\varphi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sum_i A_i(k) \cos(kx - \omega_i(k)t + \delta_i)$$

where $\omega_i(k)$ is the dispersion relation for the $i^{th}$ branch.
Recall

$$v_p(k) = \frac{\omega(k)}{k} \quad = \quad \text{phase velocity} = \text{velocity of component harmonic}$$

$$v_g(k) = \frac{\partial \omega}{\partial k} \quad = \quad \text{group velocity} = \text{velocity of "wave packet"}$$

Nonuniform Media

Assuming a sol$^{\underline{n}}$ of the form

$$\varphi(x,t) = A \cos\theta$$

with

$$k = \frac{\partial \theta}{\partial x} \quad , \quad \omega = -\frac{\partial \theta}{\partial t}$$

we then have

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0$$

But if $\omega = \omega(k)$, then $\frac{\partial \omega}{\partial x} = \frac{\partial \omega}{\partial k}\frac{\partial k}{\partial x} = c(k)\frac{\partial k}{\partial x}$, so

$$\frac{\partial k}{\partial t} + c(k)\frac{\partial k}{\partial x} = 0$$

This is a hyperbolic eqn! Thus, hyperbolic eqns have relevance to the theory of dispersive systems.

Nonlinear Dispersion

We've seen how nonlinear oscillators, e.g.

$$\ddot{x} + \omega_0^2 x = \epsilon h(x)$$

nonlinear

may be analyzed using methods such as Poincaré-Lindstedt and multiple time scale analysis. One finds that the oscillation frequency $\omega$ is a function of the oscillation amplitude: $\omega = \omega(A)$. Similarly, one can obtain a dispersion $\omega = \omega(k, A)$ which is amplitude-dependent. As an example, consider the KdV eqn,

$$\varphi_t + c_0(1 + l^{-1}\varphi)\varphi_x + \gamma \varphi_{xxx} = 0$$

Here $l = \frac{2}{3} h_0$, where $h_0$ is the depth. $\varphi(x,t)$ is the amplitude for long-wavelength water waves; $\gamma = \frac{1}{6} c_0 h_0^2$; $c_0 = \sqrt{g h_0}$. We start by writing $\eta \equiv \varphi / h_0$, a dimensionless amplitude. Then

$$\eta_t + c_0 \eta_x + \gamma \eta_{xxx} = -\frac{3}{2} c_0 \eta \eta_x$$

Assume a sol$^n$ of the form

$$\eta = \epsilon \eta_1(\theta) + \epsilon^2 \eta_2(\theta) + \epsilon^3 \eta_3(\theta) + \dots$$

where $\epsilon$ is an expansion parameter and

$$\theta = kx - \omega t$$

$$\Rightarrow \partial_t = -\omega \partial_\theta \quad , \quad \partial_x = k \partial_\theta$$

We further assume

$$\omega(k) = \omega_0(k) + \epsilon\,\omega_1(k) + \epsilon^2\omega_2(k) + \dots$$

Now write the equation and expand in powers of $\epsilon$:

$$\left[(\omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots - kc_0)\frac{\partial}{\partial\theta} - \gamma k^3\frac{\partial^3}{\partial\theta^3}\right](\epsilon\eta_1 + \epsilon^2\eta_2 + \epsilon^3\eta_3 + \dots)$$
$$= \frac{3}{2}kc_0\,(\epsilon\eta_1 + \epsilon^2\eta_2 + \epsilon^3\eta_3 + \dots)\frac{\partial}{\partial\theta}(\epsilon\eta_1 + \epsilon^2\eta_2 + \epsilon^3\eta_3 + \dots)$$

We find:

$$\mathcal{O}(\epsilon^1): \qquad (\omega_0 - c_0k)\,\eta_1' - \gamma k^3\eta_1''' = 0$$

$$\mathcal{O}(\epsilon^2): \qquad (\omega_0 - c_0k)\,\eta_2' - \gamma k^3\eta_2''' = \frac{3}{2}c_0k\,\eta_1\eta_1' - \omega_1\eta_1'$$

$$\mathcal{O}(\epsilon^3): \qquad (\omega_0 - c_0k)\,\eta_3' - \gamma k^3\eta_3''' = \frac{3}{2}c_0k\,(\eta_1\eta_2)' - \omega_1\eta_2' - \omega_2\eta_1'$$

This procedure is recognized as the Poincaré-Lindstedt method!
From the first eqn, we have

$$\eta_1(\theta) = \cos\theta$$
$$\omega_0(k) = c_0k - \gamma k^3$$

The second equation gives

$$-\gamma k^3\left(\frac{\partial}{\partial\theta} + \frac{\partial^3}{\partial\theta^3}\right)\eta_2(\theta) = -\frac{3}{4}c_0k\cdot\sin 2\theta + \omega_1\sin\theta$$

The $\sin 2\theta$ term is already nonresonant, so we set $\omega_1 = 0$, in which case

$$\eta_2(\theta) = B\cos 2\theta$$
$$B = \frac{c_0}{8\gamma k^2} = \frac{3}{4k^2h_0^2} \qquad (\gamma = \tfrac{1}{6}c_0h_0^2)$$

Finally, the third eqn gives

$$-\gamma k^3\left(\frac{\partial}{\partial\theta}+\frac{\partial^3}{\partial\theta^3}\right)\eta_3 = \frac{3}{2}C_0 k\,\frac{\partial}{\partial\theta}\left(\frac{C_0}{8\gamma k^2}\cos\theta\cos 2\theta\right) + \omega_2 \sin\theta$$

$$= \frac{3C_0^2}{32\gamma k}\,\frac{\partial}{\partial\theta}\left(\cos 3\theta + \cos\theta\right) + \omega_2\sin\theta$$

$$= -\frac{9C_0^2}{32\gamma k}\sin 3\theta - \frac{3C_0^2}{32\gamma k}\sin\theta + \omega_2\sin\theta$$

The secular term on the RHS is eliminated by setting

$$\omega_2 = \frac{3C_0^2}{32\gamma k} = \frac{9C_0}{16kh_0^2}$$

In which case we obtain $\eta_3(\theta)$:

$$\eta_3(\theta) = C\cos 3\theta$$

$$-\gamma k^3(-3+27)\,C = -\frac{9C_0^2}{32\gamma k}$$

$$\Rightarrow C = \frac{3}{256}\,\frac{C_0^2}{\gamma^2 k^4} = \frac{27}{64k^4h_0^4}$$

Putting it all together,

$$\frac{\varphi}{h_0} = \epsilon\cos\theta + \frac{3\epsilon^2}{4k^2h_0^2}\cos 2\theta + \frac{27\epsilon^3}{64k^4h_0^4}\cos 3\theta + \ldots$$

$$\omega(k) = C_0 k\cdot\left\{1 - \frac{1}{6}k^2h_0^2 + \frac{9\epsilon^2}{16k^2h_0^2} + \ldots\right\}$$

In fact, Korteweg and deVries found a set of exact, analytic sol$^{\underline{ns}}$ to

$$\varphi_t + (c_0 + c_1\varphi)\varphi_x + \gamma\varphi_{xxx} = 0$$

in terms of the Jacobian elliptic function $cn(\theta)$, where $\theta = kx - \omega t$. These sol$^{\underline{ns}}$ correspond to periodic wave trains, and $\omega = \omega(k, A)$, as Stokes showed in his perturbation expansion. Furthermore, in the limit $\lambda \to \infty$, the solutions take the form

$$\varphi(x,t) = A\,\mathrm{sech}^2\left(\left(\frac{c_1 A}{12\gamma}\right)^{1/2}(x - ut)\right) = A\,\mathrm{sech}^2\left(\frac{x-ut}{\xi}\right)$$

$$u = c_0 + \frac{1}{3}c_1 A \quad ; \quad \xi = \left(\frac{12\gamma}{c_1 A}\right)^{1/2}$$

The period is infinite. This <u>soliton</u> solution describes a single hump propagating with velocity $u = u(A)$. The width $\xi$ of the soliton is inversely proportional to $A^{1/2}$. Other PDEs which combine nonlinearity and dispersion also yield soliton sol$^{\underline{ns}}$, such as

$$\phi_{tt} - \phi_{xx} = \sin\phi \qquad \text{(sine-Gordon eqn)}$$

$$i\psi_t = -\psi_{xx} - |\psi|^2\psi \qquad \text{(nonlinear Schrödinger eqn)}$$

## Linear Waves

Linear systems are special in that solutions may be superposed. Suppose $L\varphi = 0$, where $L$ is a linear operator. Then

$$L\varphi_1 = L\varphi_2 = 0 \Rightarrow L(\varphi_1 + \varphi_2) = 0$$

so $\varphi = \varphi_1 + \varphi_2$ is a sol$^{\underline{n}}$ if $\varphi_1$ and $\varphi_2$ are sol$^{\underline{ns}}$. For <u>nonlinear</u> systems, this is no longer necessarily the case.

- Continuity eqn: $\dfrac{\partial \rho}{\partial t} + \dfrac{\partial j}{\partial x} = 0$ ; $j = c_0 \rho \Rightarrow$

$$\frac{\partial \rho}{\partial t} + c_0 \frac{\partial \rho}{\partial x} = 0$$

Let $\xi \equiv x - c_0 t$ , $\eta = x + c_0 t$ . Then

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \quad ; \quad \frac{\partial}{\partial t} = -c_0 \frac{\partial}{\partial \xi} + c_0 \frac{\partial}{\partial \eta}$$

$$\Rightarrow \frac{\partial}{\partial t} + c_0 \frac{\partial}{\partial x} = 2c_0 \frac{\partial}{\partial \eta} \quad ; \quad -\frac{\partial}{\partial t} + c_0 \frac{\partial}{\partial x} = 2c_0 \frac{\partial}{\partial \xi}$$

Thus,

$$\frac{\partial \rho}{\partial \eta} = 0 \Rightarrow \rho = f(\xi) = f(x - c_0 t)$$

which describes a signal propagating with velocity $v = c_0$.

- Helmholtz eqn:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\varphi = 0$$
$$= 4 \frac{\partial^2}{\partial \eta \, \partial \xi} \varphi$$

$$\Rightarrow \varphi = f(x - c_0 t) + g(x + c_0 t)$$

$$\varphi(x,t) = \frac{1}{2}\varphi(x - c_0 t, 0) + \frac{1}{2}\varphi(x + c_0 t, 0) + \frac{1}{2c_0}\int_{x-c_0 t}^{x+c_0 t} d\xi \, \dot{\varphi}(\xi, 0)$$

- Dispersive linear equation :

$$\frac{\partial \varphi}{\partial t} + c_0 \frac{\partial \varphi}{\partial x} + \gamma \frac{\partial^3 \varphi}{\partial x^3} = 0 \qquad \text{(linearized KdV)}$$

Fourier transform :

$$\hat{\varphi}(k,\omega) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt \;\; \varphi(x,t)\, e^{-i(kx-\omega t)}$$

$$\varphi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \;\; \hat{\varphi}(k,\omega)\; e^{+i(kx-\omega t)}$$

$$\partial_t \leftrightarrow -i\omega \qquad , \qquad \partial_x \leftrightarrow +ik$$

$$-i(\omega - c_0 k + \gamma k^3)\, \hat{\varphi}(k,\omega) = 0$$

Thus,

$$\varphi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \; \hat{A}(k)\, e^{ikx}\, e^{-i\omega(k)t}$$

$$\omega(k) = c_0 k - \gamma k^3$$

$$\hat{\varphi}(k,\omega) = 2\pi\, \delta(\omega - \omega(k)) \cdot \hat{A}(k)$$

Initial conditions :

$$\hat{A}(k) = \int_{-\infty}^{\infty} dx \;\; \varphi(x,0)\, e^{-ikx}$$

Note that $\varphi(x,t) = x - c_0 t$ and $\varphi(x,t) = (x-c_0 t)^2$ are also solutions, but they diverge as $x \to \pm\infty$.

- Diffusion equation:

$$\varphi_t = \varphi_{xx}$$

This arises from continuity,

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0$$

plus a constitutive relation

$$j = -D\partial_x \rho \qquad (D = \text{diffusion coefficient})$$

yielding

$$\rho_t = D\rho_{xx}$$

One can now rescale to eliminate $D$.

We Laplace transform in $t$:

$$\tilde{\varphi}(x,s) \equiv \int_0^\infty dt\, e^{-st}\varphi(x,t)$$

Note that

$$(s-\partial_x^2)\tilde{\varphi}(x,s) = \int_0^\infty dt\, e^{-st}(s-\partial_x^2)\varphi(x,t)$$

$$= \int_0^\infty dt \left\{ e^{-st}(\partial_t - \partial_x^2)\varphi(x,t) - \frac{\partial}{\partial t}\left[e^{-st}\varphi(x,t)\right]\right\}$$

$$= \varphi(x, t=0) \equiv f(x)$$

Now Fourier transform:

$$(s+k^2)\hat{\tilde{\varphi}}(k,s) = \hat{f}(k)$$

$$\hat{\tilde{\varphi}}(k,s) = \frac{\hat{f}(k)}{s+k^2}$$

Inverse Laplace transform

$$\hat{\varphi}(k,t) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i}\, \frac{\hat{f}(k)}{s+k^2}\, e^{st} = \hat{f}(k)\, e^{-k^2 t}$$

Inverse Fourier transform:

$$\varphi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \hat{f}(k)\, e^{ikx} e^{-k^2 t} = \int_{-\infty}^{\infty} dx' \frac{1}{2\sqrt{\pi t}}\, e^{-(x-x')^2/4t} f(x')$$

## General Discussion : $2^{nd}$ Order PDEs

Consider a PDE in two independent variables $(x,y)$. This PDE, which we write as

$$A\varphi_{xx} + 2B\varphi_{xy} + C\varphi_{yy} = f(x,y,\varphi_x,\varphi_y) \qquad x,y \in \Omega$$

This PDE is supplemented by boundary conditions. Typically, the boundary conditions fall into one of three broad categories:

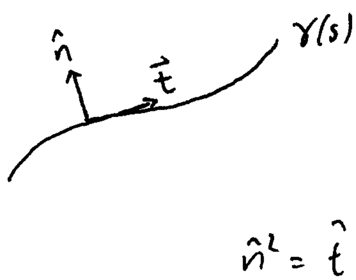
- Dirichlet conditions : $\varphi$ specified on $\partial\Omega$
- Neumann conditions : $\hat{n}\cdot\vec{\nabla}\varphi$ specified on $\partial\Omega$ ($\hat{n}$ = surface normal)
- Cauchy conditions : both $\varphi$ and $\varphi_n$ specified on $\partial\Omega$

For an second order ODE for $\psi(x)$, specifying $\psi(x_0)$ and $\psi'(x_0)$ is sufficient to determine $\psi''(x_0)$ and all higher derivatives at $x_0$, so a Taylor series expansion may be constructed to generate $\psi(x)$ in the vicinity of $x_0$.

What about PDEs? Does specification of $\varphi$ and $\varphi_n$ along a boundary curve suffice to compute higher derivatives and construct a Taylor series? Suppose the boundary curve is given parametrically, as a curve $\gamma(s)$

$$(x,y) = (x(s),y(s)) = \gamma(s)$$

Here, $s$ can be the arc length along the curve. Suppose we're given $\varphi$ and $\varphi_n$ along the curve.




$$\hat{n} = \left(-\frac{dy}{ds}, +\frac{dx}{ds}\right) = \text{normal}$$

$$\hat{t} = \left(+\frac{dx}{ds}, +\frac{dy}{ds}\right) = \text{tangent}$$

$$\hat{n}^2 = \hat{t}^2 = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = \frac{dx^2+dy^2}{ds^2} = 1$$

Thus,

$$\partial_n \varphi = \hat{n} \cdot \vec{\nabla}\varphi = -\varphi_x\, y'(s) + \varphi_y\, x'(s)$$

Now

$$\frac{d\varphi}{ds} = \varphi_x\, x'(s) + \varphi_y\, y'(s) \equiv \partial_s \varphi$$

allows us to write

$$\begin{aligned} \varphi_x &= -\varphi_n\, y'(s) + \varphi_s\, x'(s) \\ \varphi_y &= +\varphi_n\, x'(s) + \varphi_s\, y'(s) \end{aligned}$$

What about the second derivatives? There are three of them: $\varphi_{xx}$, $\varphi_{xy}$, and $\varphi_{yy}$. Differentiating the above pair gives

$$\begin{aligned} \varphi_{sx} &= \varphi_{xx}\, x'(s) + \varphi_{xy}\, y'(s) \\ \varphi_{sy} &= \varphi_{xy}\, x'(s) + \varphi_{yy}\, y'(s) \end{aligned}$$

Thus,

$$\begin{pmatrix} x'(s) & y'(s) & 0 \\ 0 & x'(s) & y'(s) \\ A & 2B & C \end{pmatrix} \begin{pmatrix} \varphi_{xx} \\ \varphi_{xy} \\ \varphi_{yy} \end{pmatrix} = \begin{pmatrix} \varphi_{sx} \\ \varphi_{sy} \\ f \end{pmatrix}$$

This can be solved for the second derivatives if the determinant is nonzero:

$$\begin{aligned} \det &= A\, y'(s)^2 - 2B\, x'(s)\, y'(s) + C\, x'(s)^2 \\ &= x'(s)^2 \cdot \left\{ A \left(\frac{dy}{dx}\right)^2 - 2B \frac{dy}{dx} + C \right\} \end{aligned}$$

The equation $\det = 0$ defines two directions,

$$\frac{dy}{dx} = B \pm \sqrt{B^2 - AC}$$

which are called "characteristic directions". Note that these two directions are defined at each $(x,y)$ point -- not just on the boundary. We see that the second derivatives are determined so long as the boundary curve is not anywhere tangent to a characteristic.

Since $\varphi_{sx}$, $\varphi_{sy}$, $\varphi_x$, $\varphi_y$, and $\varphi$ are all determined by $\varphi(s)$ and $\varphi_n(s)$ (along the boundary), in general we have that the solution is determined by Cauchy boundary conditions <u>if</u> $\gamma(s)$ is nowhere tangent to a characteristic.

Classification:

hyperbolic : $B^2 > AC$ $\Rightarrow$ characteristic curves are <u>real</u>

parabolic : $B^2 = AC$ $\Rightarrow$ characteristics <u>real</u> and <u>degenerate</u>

elliptic : $B^2 < AC$ $\Rightarrow$ characteristic curves are <u>complex</u>

What types of boundary conditions are to be applied in each case?

| Equation | Condition(s) | Boundary |
|---|---|---|
| Hyperbolic | Cauchy | Open |
| Parabolic | Dirichlet or Neumann | Closed |
| Elliptic | Dirichlet or Neumann | Open |

## Waves in First Order Hyperbolic Equations

Simplest PDE:

$$\partial_t \rho + c_0 \partial_x \rho = 0 \quad \Rightarrow \quad \rho = f(x - c_0 t)$$

More interesting:

$$\boxed{\rho_t + c(\rho)\rho_x = 0}$$

which arises from

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0 \quad , \quad j = j(\rho) \Rightarrow c(\rho) = j'(\rho)$$

This is quite nontrivial, and in fact a surprising number of physical problems lead to this or a generalization thereof.

Suppose we consider the curve

$$\frac{dx}{dt} = c(\rho(x,t)) \Rightarrow x = x(t)$$

Then along this curve,

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial x}\frac{dx}{dt} + \frac{\partial \rho}{\partial t} = c(\rho)\frac{\partial \rho}{\partial x} + \frac{\partial \rho}{\partial t} = 0$$

Thus, $\rho$ is a constant along the curve, which means that the curve is a straight line! Note that in the case

$$\varphi_t + c(\varphi)\varphi_x = g(\varphi)$$

we have

$$\frac{dx}{dt} = c(\varphi) \Rightarrow \frac{d\varphi}{dt} = g(\varphi)$$

$$-t_0 + t = \int_{\varphi_0}^{\varphi} \frac{d\varphi'}{g(\varphi')} \Rightarrow x - x_0 = \int_{t_0}^{t} dt'\, c(\varphi(t')) = \int_{\varphi_0}^{\varphi} d\varphi' \frac{c(\varphi')}{g(\varphi')}$$

Characteristics no longer are straight lines!

But for $\rho_t + c(\rho)\rho_x = 0$, we have

$$\frac{dx}{dt} = c(\rho) \quad \longrightarrow \quad \text{curve } \gamma$$

$$\frac{d\rho}{dt} = 0 \quad \text{on } \gamma$$

$$\Rightarrow \quad x = c(\rho)t + x_0$$

For initial conditions, we take

$$\rho(x, t=0) = f(x)$$

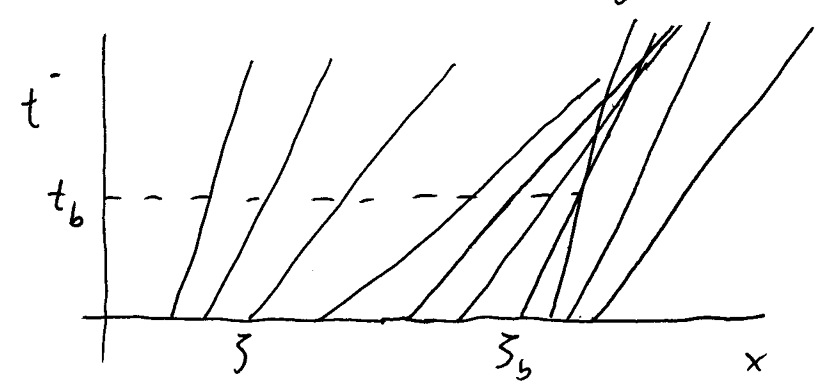
So, if one of the curves $\gamma$ intersects $t=0$ at $x = \xi$, then $\rho = f(\xi)$ along the entire curve $\gamma$, hence the slope of $\gamma$ is $c(f(\xi))$. Let us define $g(\xi) \equiv c(f(\xi))$, a known $f^{n}$ computed from $c(\rho)$ and the initial conditions $\rho(x,t=0) = f(x)$. The eqn of the characteristic is then

$$x = \xi + g(\xi)t$$

This is one characteristic curve, starting at $x(t=0) = \xi$. By varying $\xi$, we construct a whole family of such curves:

$$\rho = f(\xi) \quad , \quad c = c(f(\xi)) = g(\xi)$$

$$\text{on} \quad x(t) = \xi + g(\xi)t$$



We can check explicitly that this is a sol$^{\underline{n}}$ :

$$x = \xi + g(\xi)t \xrightarrow[\text{invert}]{} \xi(x,t)$$

tells where a point $(x,t)$ intersects $t=0$ line by moving along characteristics

$$\downarrow$$

$$\rho = \rho(\xi) = \rho(\xi(x,t)) = \rho(x,t)$$

$$\rho_t = f'(\xi)\xi_t \quad , \quad \rho_x = f'(\xi)\xi_x$$

$$0 = \frac{\partial}{\partial x}\left[\xi + g(\xi)t - x\right] = \xi_x + g'(\xi)\xi_x t - 1$$

$$0 = \frac{\partial}{\partial t}\left[\xi + g(\xi)t - x\right] = \xi_t + g'(\xi)\xi_t t + g(\xi)$$

$$\Rightarrow \quad \rho_x = \frac{f'(\xi)}{1+g'(\xi)t} \quad , \quad \rho_t = -\frac{g(\xi)f'(\xi)}{1+g'(\xi)t}$$

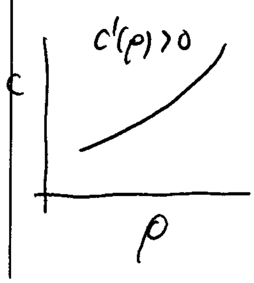
Hence,

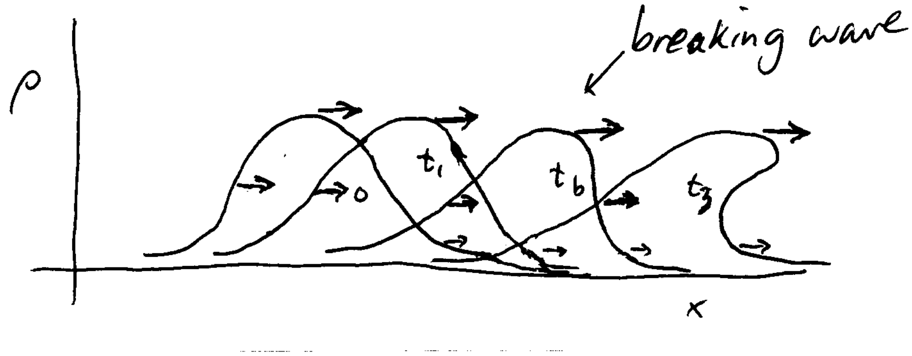
$$\rho_t + c(\rho)\rho_x = 0$$

Since $c(\rho) = g(\xi)$.

For general HPDEs, as we've seen, the solution does not remain constant along characteristics ; that is a special feature of $\rho_t + c(\rho)\rho_x = 0$.

Recall Whitham's definition of a wave : a recognizable feature propagates at finite velocity. The characteristics satisfy this definition : different values of $\rho$ propagate with their own velocity $c(\rho)$.



The sol$^n$ can be constructed by splitting the curve $\rho(x,t=0)$ into level sets of constant $\rho$, and shifting each such set to the right a distance $c(\rho)t$. This generates $\rho(x,t)$. For $c(\rho) = c_0$, the entire curve is shifted uniformly.

We sketch the behavior in the case $c'(\rho) > 0$. The wave "breaks" at a time $t = t_b$ when $\rho(x,t)$ develops an infinite slope. From our analytic sol$^n$, we see this happens for

$$t = -\frac{1}{g'(\xi)}$$

The wave first breaks at $t = t_b$, where

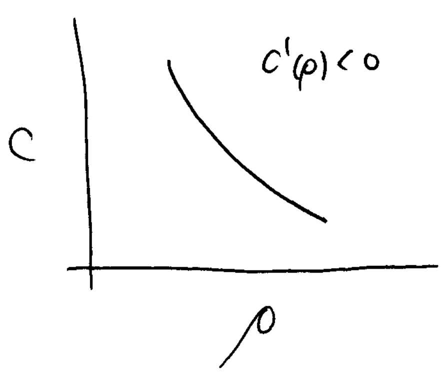
$$t_b = \min_{\substack{\xi \\ g'(\xi)<0}} \left[ -\frac{1}{g'(\xi)} \right] = -\frac{1}{g'(\xi_b)}$$

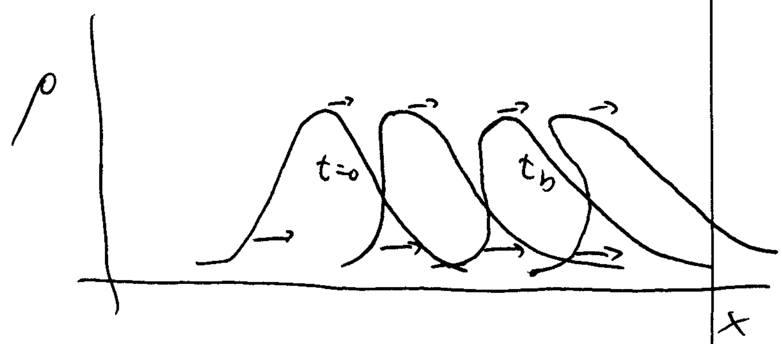
Note

$$g'(\xi) = c'(\rho)\,\rho'(\xi) \qquad \text{(evaluate at } t=0\text{)}$$

So

$$c' < 0 \implies \text{need } \rho' > 0 \text{ to break}$$
$$c' > 0 \implies \text{need } \rho' < 0 \text{ to break}$$



Another way to analyze this is to compare neighboring characteristics:

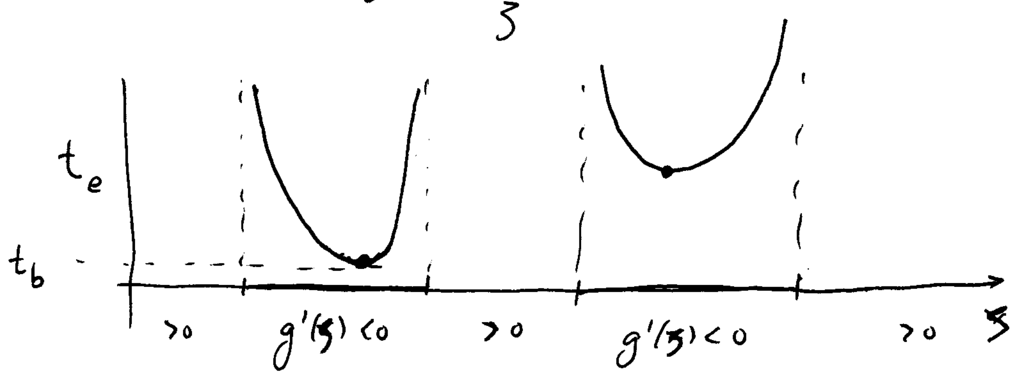
$$x_1(t) = \xi + g(\xi)\,t \qquad (\xi \text{ characteristic})$$

$$x_2(t) = \xi + \delta\xi + g(\xi+\delta\xi)\,t \qquad (\xi+\delta\xi \text{ characteristic})$$
$$= \xi + g(\xi)t + \left(1 + g'(\xi)t\right)\delta\xi$$

For these characteristics to cross, we must have

$$x_1(t) = x_2(t) \Rightarrow t = -\frac{1}{g'(\xi)}$$

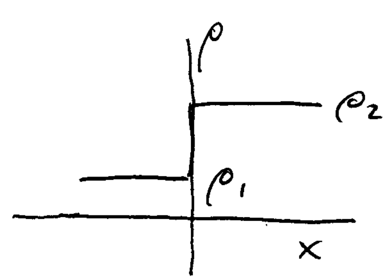
Thus, an envelope is formed, $t_e(\xi) \equiv -\dfrac{1}{g'(\xi)}$, for all characteristics for which $g'(\xi) < 0$. The wave first breaks at $t = t_b = \min\limits_{\xi} t_e(\xi)$.




It is possible that different regions of the wave break at different times.

As an example, consider the case

$$f(x) = \rho(x,0) = \begin{cases} \rho_1, & x<0 \\ \rho_2, & x>0 \end{cases}$$



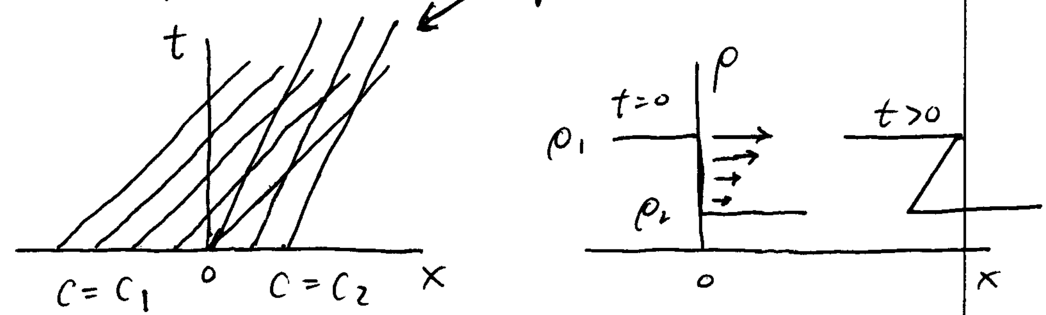

($\rho_2 > \rho_1$ shown)

Then

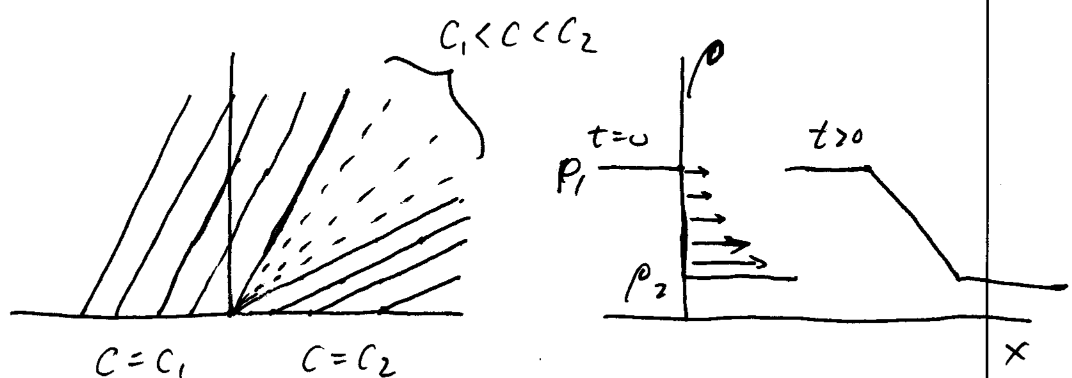
$$g(\xi) = \begin{cases} c_1 = c(\rho_1) , & \xi < 0 \\ c_2 = c(\rho_2) , & \xi > 0 \end{cases}$$

If $c_1 > c_2$, the wave starts to break immediately.
If $c_1 < c_2$, the discontinuity vanishes immediately and is replaced by a linear interpolation : slope $t/x = 1/c$

• $c_1 > c_2$ :




• $c_1 < c_2$ :




In the second case, there is a "fan" of characteristics emanating from $\xi = 0$, described by

$$x(t) = ut \quad ; \quad c_1 < u < c_2$$

Thus, the complete sol$^{\underline{n}}$ for $c(x,t)$ is

$$c(x,t) = \begin{cases} c_2 & \text{for } x/t > c_2 & \text{maximum} \\ x/t & \text{for } c_1 < x/t < c_2 & \text{interpolation} \\ c_1 & \text{for } x/t < c_1 & \text{minimum} \end{cases}$$

Usually, in physical settings, $\rho(x,t)$ is single-valued. In such cases, when the wave breaks, the multivalued sol$^{\underline{n}}$ ceases to be applicable. (NB this is so even for water waves, where one might even find meaning in multivalued $h(x,t)$ height function.) Oftentimes this means some important physics has been left out. E.g. neglecting viscosity and heat conduction, the eqns of gas dynamics have breaking sol$^{\underline{ns}}$ similar to those we've just studied. But when gradients are steep -- just before breaking -- the effects of $\nu$ and $\kappa$ are no longer negligible. (These coeff$^{\underline{s}}$ multiply higher derivative terms, and even if they are small, their influence is magnified greatly in regions of steep gradients.) The shock wave is then a thin region in which $\nu$ and $\kappa$ are crucially important; flow varies rapidly within this region. This thin shock region is idealized into a discontinuity in the inviscid limit of the theory; all that remains are a set of "shock conditions" which relate the discontinuities of various quantities across the shock(s).

<u>Kinematic Waves</u>

Let $\rho(x,t)$ be a density and $j(x,t)$ the current density. Continuity says

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0$$

The <u>flow</u> <u>velocity</u> is $v = j/\rho$. This is not, however the velocity of wave propagation. The integral version of continuity is

$$\frac{d}{dt}(\text{stuff}) = \text{rate in} - \text{rate out}$$

$$\frac{d}{dt}\int_{x_a}^{x_b} dx\, \rho(x,t) = j(x_a,t) - j(x_b,t)$$

Simplest case:

$$j = J(\rho)$$

Then

$$\rho_t + c(\rho)\rho_x = 0$$

$$c(\rho) = J'(\rho) \qquad\qquad v(\rho) = \frac{J(\rho)}{\rho}$$

This is not a bad approximation in the case of <u>vehicular traffic</u>, as we shall discuss. (Cars are conserved!) But not complete.

Other examples:

- <u>Flood waves in rivers:</u> $\rho(x,t) \longrightarrow A(x,t)$ cross-sectional area

  $j(x,t)$ = volume flux of water
  constitutive relation: $j = J(A)$ an approximation to what is in fact a complex fluid flow problem; as in case of traffic, this is semi-empirical.

  Also: glacier flow

- Chromatography / exchange processes in chemical engineering

$$\rho_f = \text{density of flowing material (carried by fluid)}$$
$$\rho_s = \text{density of non-flowing sediment}$$
$$j = v\rho_f \quad ; \quad v = \text{fluid velocity (constant)}$$

Continuity:

$$\frac{\partial}{\partial t}\overbrace{(\rho_f + \rho_s)}^{\rho} + \frac{\partial}{\partial x}(v\rho_f) = 0$$

Deposition rate eqn:

$$\frac{\partial \rho_s}{\partial t} = \overbrace{k_1(\alpha - \rho_s)\rho_f}^{\text{on}} - \overbrace{k_2(\beta - \rho_f)\rho_s}^{\text{off}}$$

This is a simple "master eqn" with rate constants $k_1$ and $k_2$. If the reaction rates are large, then we can assume we rapidly flow to deposition equilibrium, where $\partial_t \rho_s = 0$, so that

$$\rho_s = \frac{\alpha k_1 \rho_f}{\beta k_2 + (k_1 - k_2)\rho_f} = \rho - \rho_f$$

This implies a relation $\rho_f = \rho_f(\rho)$, hence

$$j = v\rho_f = J(\rho)$$

When changes become rapid, just before breaking, the term $\frac{\partial \rho_s}{\partial t}$ can no longer be neglected.

Recall example of <u>relaxation oscillations</u> (slow + fast dynamics)

Relaxation oscillations:

$$\ddot{x} + \underbrace{\mu\, \Phi(x)\, \dot{x} + \omega_0^2 x}_{\equiv\, \mu y} = 0 \qquad \mu >> 1$$

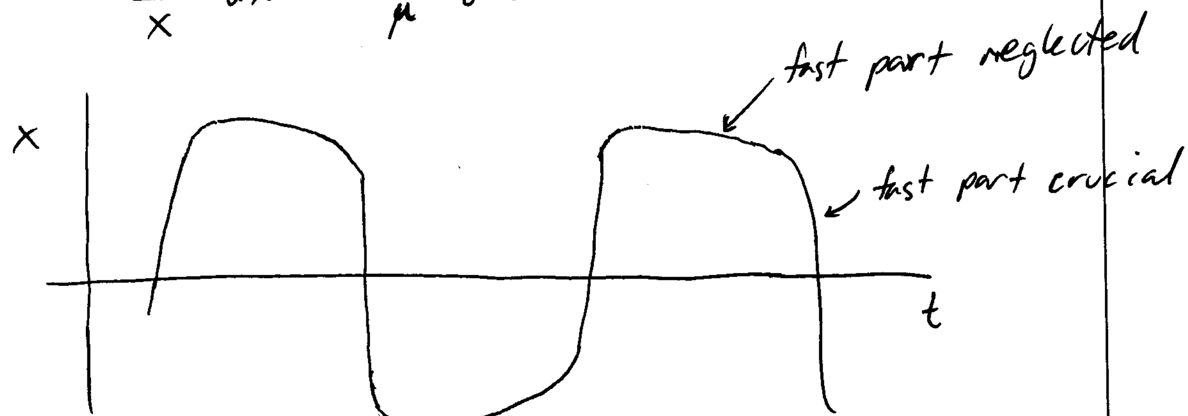
$$\dot{y} = -\frac{1}{\mu}\omega_0^2 x \qquad \text{(slow)}$$

$$\dot{x} = \mu\left(y - F(x)\right) \qquad \text{(fast)}$$

Assume fast eqn runs its course, so $y = F(x)$, then

$$\dot{y} = F'\dot{x} = -\frac{1}{\mu}\omega_0^2 x$$

$$-\frac{\Phi(x)}{x}\, dx = \frac{1}{\mu}\omega_0^2\, dt \qquad \rightarrow \quad t(x) \rightarrow x(t)$$

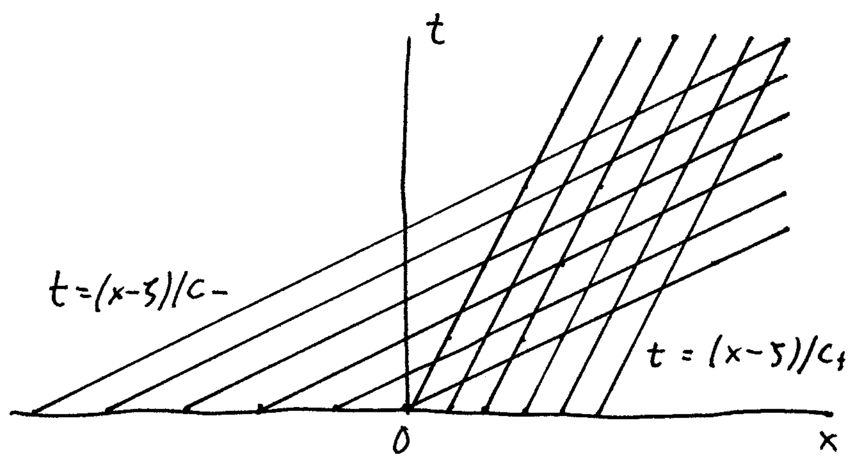


## Example

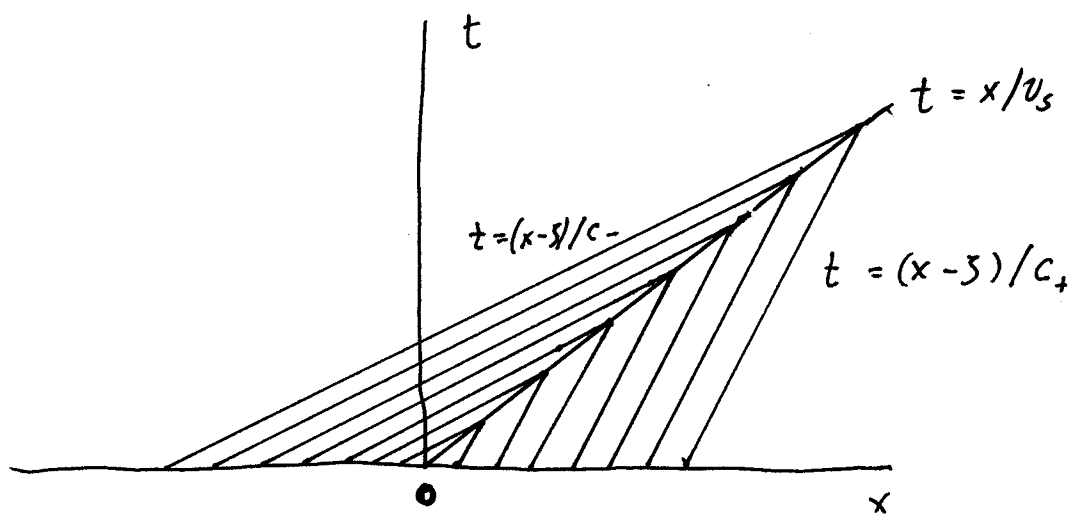
$$\rho(x,0) = \begin{cases} \rho_- & x < 0 \\ \rho_+ & x > 0 \end{cases}$$

$c_- > c_+ \Rightarrow$ shock propagates

Plot the characteristics:




Now replot with the shock wave:




In these sketches: $c_- = 2$, $c_+ = \frac{1}{2} \Rightarrow v_s = \frac{5}{4}$ (for quadratic $J(\rho)$)

A useful way to derive the shock condition is to shift to a frame of reference moving with instantaneous velocity $v_s$. In this frame there is a "wind" with $j_w = -v_s \rho$:

| lab frame | | shock frame | |
|---|---|---|---|
| | $\xrightarrow{v_s}$ | | |
| $\rho = \rho_-$ | $\rho = \rho_+$ | $\rho = \rho_-$ | $\rho = \rho_+$ |
| $j = j_-$ | $j = j_+$ | $j = j_- - v_s \rho_-$ | $j = j_+ - v_s \rho_+$ |

Conservation of current now means

$$j_- - v_s \rho_- = j_+ - v_s \rho_+$$

which is equivalent to the shock condition

<u>Shock Structure</u>

Let's add a term to $J(\rho)$ to make it more physical:

$$j = J(\rho, \rho_x) = \underbrace{J(\rho)}_{\text{steepening and breaking}} - \overbrace{\nu \rho_x}^{\text{diffusion}}$$

The second term provides a counterflow to oppose gradients in $\rho$. Thus, we have

$$\rho_t + c(\rho)\rho_x = \nu \rho_{xx} \qquad ; \qquad c(\rho) = J'(\rho)$$

Note: if $c(\rho) \propto \rho$, i.e. $J(\rho) \propto \rho^2$, we recover <u>Burgers' eqn</u>

$$\varphi_t + \varphi \varphi_x = \nu \varphi_{xx}$$

We'll study this in detail later on.

Even if $\nu$ is small, clearly it's importance is magnified in the vicinity of a shock, where $|\rho_x| \to \infty$. In the vicinity of a shock, the term $-\nu\rho_x$ dominates the relation $j = J(\rho) - \nu\rho_x$. If $\nu$ is very small, this is the only region where it is important; elsewhere $\nu$ may be neglected, or its effects treated perturbatively.

Look for a sol$^{n}$

$$\rho = \rho(x - v_s t) \quad ; \quad \xi \equiv x - v_s t$$

$$\Rightarrow \quad (c(\rho) - v_s)\,\rho_\xi = \nu\,\rho_{\xi\xi}$$

Integrate once:

$$J(\rho) - v_s\rho + \underbrace{A}_{\text{constant}} = \nu\rho_\xi$$

Integrate twice:

$$\xi = \nu \int^{\rho} \frac{d\rho'}{J(\rho') - v_s\rho' + A} \qquad \text{(implicitly gives } \rho(\xi)\text{)}$$

If $\rho(\xi = \infty) = \rho_+$ and $\rho(\xi = -\infty) = \rho_-$, then we must have

$$J(\rho_+) - v_s\rho_+ + A = 0$$

$$J(\rho_-) - v_s(\rho_-) + A = 0$$

$$\Rightarrow \quad v_s = \frac{J(\rho_+) - J(\rho_-)}{\rho_+ - \rho_-} \qquad A = \frac{\rho_- J_+ - \rho_+ J_-}{\rho_+ - \rho_-}$$


exactly as before!

Special case : quadratic $J(\rho)$

$$J(\rho) = \alpha\rho^2 + \beta\rho + \gamma$$
$$c(\rho) = 2\alpha\rho + \beta$$

We may write

$$J(\rho) - v_s\rho + A = + \alpha(\rho-\rho_+)(\rho-\rho_-)$$
$$v_s = \beta + \alpha(\rho_+ + \rho_-) \quad , \quad A = \alpha\rho_+\rho_- - \gamma$$

Then

$$\xi = -\nu\int^{\rho}\frac{d\rho'}{\alpha(\rho'-\rho_-)(\rho_+-\rho')} = \frac{\nu}{\alpha(\rho_- - \rho_+)}\ln\frac{\rho_- - \rho}{\rho - \rho_+}$$

$$\Rightarrow \quad \rho(x,t) = \frac{\rho_- + \rho_+\exp\left[\alpha(\rho_- - \rho_+)(x - v_s t)/\nu\right]}{1 + \exp\left[\alpha(\rho_- - \rho_+)(x - v_s t)/\nu\right]}$$

We consider $\alpha > 0$, so $c'(\rho) > 0$. Thus,

$$\rho(x,t) = \begin{cases} \rho_+ & x - v_s t \gg \delta \\ \rho_- & x - v_s t \ll -\delta \end{cases}$$

with

$$\delta = \frac{\nu}{\alpha(\rho_- - \rho_+)}$$

the thickness of the shock region. In the limit $\nu \to 0$, the shock is discontinuous. All that remains is the "shock condition"

$$v_s = \alpha(\rho_+ + \rho_-) + \beta = \frac{1}{2}(c_+ + c_-)$$

## Weak Shocks

We found

$$v_s = \frac{J(\rho_-) - J(\rho_+)}{\rho_- - \rho_+} \quad ; \quad \bar{\rho} \equiv \tfrac{1}{2}(\rho_- + \rho_+)$$

Suppose the shock is <u>weak</u>, i.e. $\Delta\rho = \rho_- - \rho_+$ is small.
Then we may Taylor expand:

$$J(\rho_-) = J(\bar{\rho} + \tfrac{1}{2}\Delta\rho) = J(\bar{\rho}) + \tfrac{1}{2}\Delta\rho \cdot J'(\bar{\rho}) + \tfrac{1}{8}(\Delta\rho)^2 J''(\bar{\rho}) + \cdots$$

$$J(\rho_+) = J(\bar{\rho} - \tfrac{1}{2}\Delta\rho) = J(\bar{\rho}) - \tfrac{1}{2}\Delta\rho\, J'(\bar{\rho}) + \tfrac{1}{8}(\Delta\rho)^2 J''(\bar{\rho}) + \cdots$$

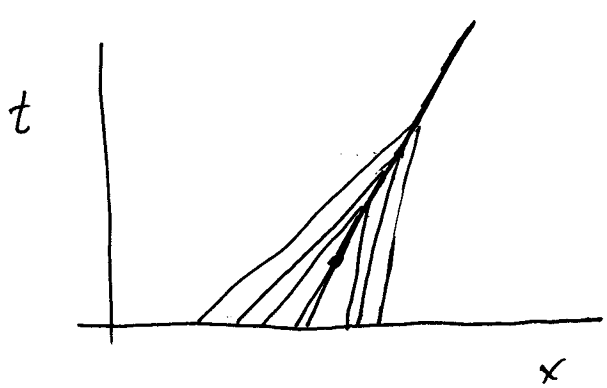
$$\Rightarrow \quad J(\rho_-) - J(\rho_+) = \Delta\rho \cdot J'(\bar{\rho}) + \cdots$$

$$c(\rho_-) = J'(\bar{\rho} + \tfrac{1}{2}\Delta\rho) = J'(\bar{\rho}) + \tfrac{1}{2}\Delta\rho\, J''(\bar{\rho}) + \cdots$$

$$c(\rho_+) = J'(\bar{\rho} - \tfrac{1}{2}\Delta\rho) = J'(\bar{\rho}) - \tfrac{1}{2}\Delta\rho\, J''(\bar{\rho}) + \cdots$$

Thus,

$$\begin{aligned} v_s &= J'(\bar{\rho}) + \mathcal{O}((\Delta\rho)^2) \\ &= \tfrac{1}{2}(c_+ + c_-) + \mathcal{O}((\Delta\rho)^2) \end{aligned}$$



## Breaking Condition

A continuous wave breaks if in any region $\frac{dc}{dx} < 0$ (i.e. compression). When a shock forms,

$$c_- > v_s > c_+$$

where $c_+$ = velocity ahead of shock, $c_-$ = velocity behind shock, and $v_s$ = velocity of shock. Shocks with $c_+ > c_-$ can be fit, but they can never form from any continuous initial data.

## Shock Fitting

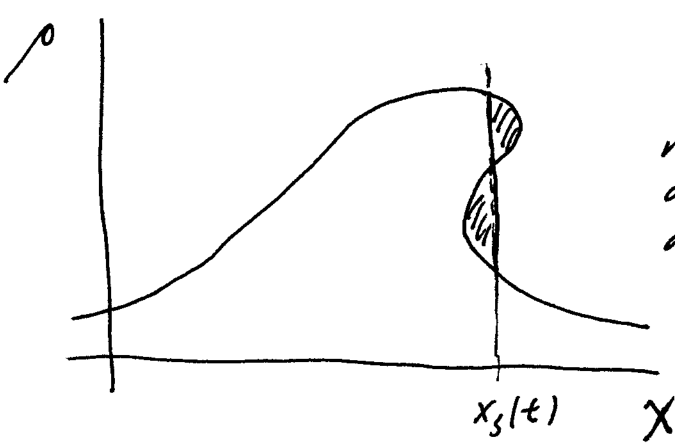
We now consider how to fit discontinuous shocks, satisfying

$$v_s = \frac{J(\rho_-) - J(\rho_+)}{\rho_- - \rho_+} \quad ; \quad c(\rho) = J'(\rho)$$

into the continuous sol$^{\underline{n}}$,

$$\rho = f(\xi) \qquad g(\xi) = c(f(\xi))$$
$$x = \xi + g(\xi)t$$

such that multivalued parts of the continuous sol$^{\underline{n}}$ are replaced with a shock discontinuity. Important: $\int dx\, \rho$ is conserved.




Conservation of $\int dx\, \rho$ means that the continuous and discontinuous curves define the same area.

## Quadratic $J(\rho)$

We consider

$$J(\rho) = \alpha\rho^2 + \beta\rho + \gamma$$
$$c(\rho) = 2\alpha\rho + \beta$$
$$c'(\rho) = 2\alpha > 0$$

The shock velocity is then

$$v_s = \frac{1}{2}(c_+ + c_-)$$

where $c_\pm = c(\rho_\pm) = c(\rho(x_s^\pm))$, where $x_s$ is the position of the shock. Since $c \propto \rho$, we can eliminate $\rho$ in terms of $c$. Thus,

$$\rho_t + c\rho_x = 0 \Rightarrow c_t + cc_x = 0$$

or

$$c_t + (\tfrac{1}{2}c^2)_x = 0$$

$$\frac{\partial c}{\partial t} + \frac{\partial (c)}{\partial x} = 0 \qquad E(c) \equiv \frac{1}{2}c^2$$

Thus,

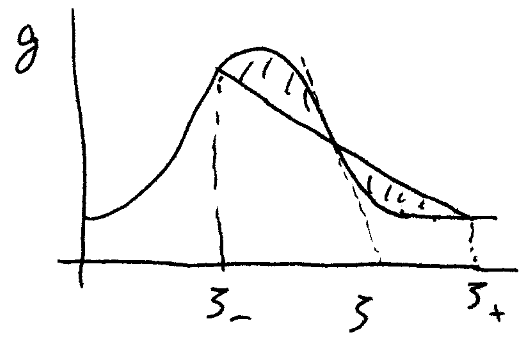
$$v_s = \frac{E(c_-) - E(c_+)}{c_- - c_+} = \frac{1}{2}(c_- + c_+)$$

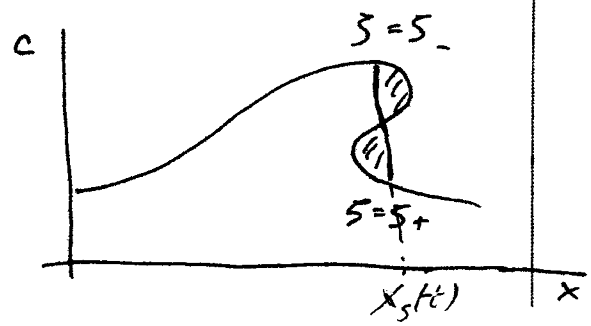
The continuous sol$\underline{n}$ is

$$c = g(\xi)$$
$$x = \xi + g(\xi)t$$

Shock fitting thus requires

$$v_s = \frac{1}{2}\left[c(\xi_+) + c(\xi_-)\right]$$




Equal area construction: $(\zeta, c) \to (\zeta + ct, c)$ preserves area, straight lines

Recipe:

(i) Sketch $g(\zeta)$

(ii) Draw a straight line on this curve from $\zeta_-$ to $\zeta_+$ which obeys the equal area law,

$$\frac{1}{2}(\zeta_+ - \zeta_-)\big(g(\zeta_+) + g(\zeta_-)\big) = \int_{\zeta_-}^{\zeta_+} d\zeta \, g(\zeta)$$

(iii) This line evolves into the shock front after time $t$, where

$$x_s(t) = \zeta_- + g(\zeta_-)t = \zeta_+ + g(\zeta_+)t$$

$$\Rightarrow \; t = -\frac{\zeta_+ - \zeta_-}{g(\zeta_+) - g(\zeta_-)}$$

(iv) The position of the shock at this time is $x_s(t)$.

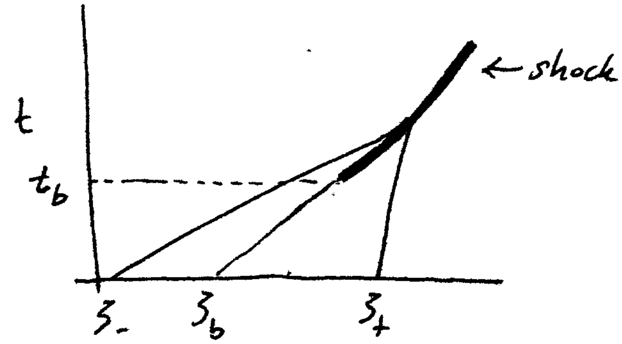
(v) Alternately, we can fix $t$ and solve for $\zeta_\pm$

(vi) The break time is given by finding the steepest chord satisfying the equal area law:

$$t_B = -\frac{1}{g'(\zeta_B)}$$

(vii) If $g(\infty) = g(-\infty)$, then the shock strength vanishes in the limit of infinite time.

NB: (ii) fails when $J'''(\rho) \neq 0$




<u>A Hump's Fate</u>

Suppose $g(\zeta) = c_0$ for $|\zeta| > L$. We have

$$\frac{1}{2}\{g(\zeta_+) + g(\zeta_-) - 2c_0\}(\zeta_+ - \zeta_-) = \int_{\zeta_-}^{\zeta_+} d\zeta\,\{g(\zeta) - c_0\}$$

Eventually, the quantity $\zeta_+$ passes the point $x = L$, so $g(\zeta_+) = c_0$. Then

$$\frac{1}{2}\{g(\zeta_-) - c_0\}(\zeta_+ - \zeta_-) = \int_{\zeta_-}^{L} d\zeta\,\{g(\zeta) - c_0\} \qquad (\zeta_+ > L)$$

$$t = \frac{\zeta_+ - \zeta_-}{g(\zeta_-) - c_0}$$

hence

$$\frac{1}{2}\{g(\zeta_-) - c_0\}^2 t = \int_{\zeta_-}^{L} d\zeta\,\{g(\zeta) - c_0\} > 0$$

The shock position is

$$x_s(t) = \zeta_- + g(\zeta_-)t$$

As $t \to \infty$, we must have $\zeta_- \to -L$, so

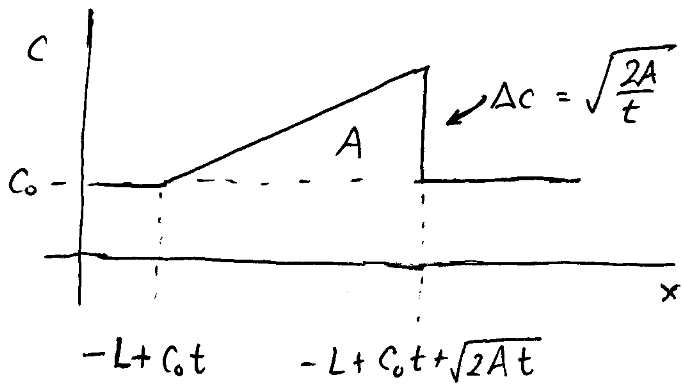
$$\frac{1}{2}\{g(\zeta_-) - c_0\}^2 t \simeq \int_{-L}^{L} d\zeta\,\{g(\zeta) - c_0\} \equiv A \quad (= \text{area})$$

Thus,

$$g(\xi_-) - c_0 \approx \sqrt{\frac{2A}{t}}$$

$$\Rightarrow \begin{cases} x_s(t) \approx -L + c_0 t + \sqrt{2At} \\ v_s(t) \approx c_0 + \sqrt{\dfrac{A}{2t}} \end{cases}$$

The shock strength is $g(\xi_-) - c_0 \approx \sqrt{\frac{2A}{t}}$.




NB: $A = \frac{1}{2}\sqrt{\frac{2A}{t}} \cdot \sqrt{2At}$ ✓

Behind the shock, we have

$$\left.\begin{aligned} c &= g(\xi) \\ x &= \xi + g(\xi)t \end{aligned}\right\} \xrightarrow[\xi \to -L]{} c = \frac{x+L}{t}$$

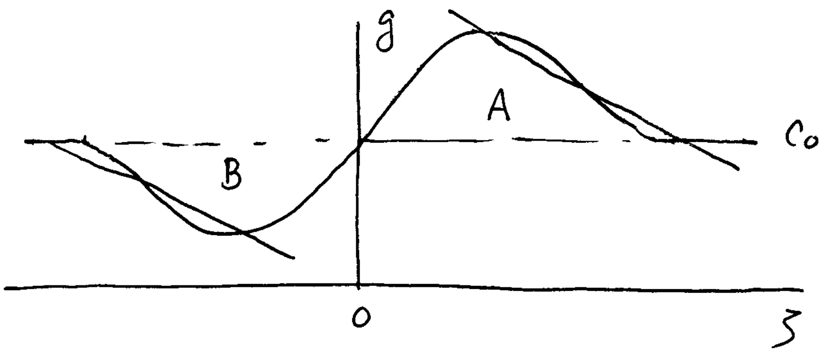
$$c_0 t - L < x < c_0 t - L + \sqrt{2At}$$

Note:

- details of original distribution are lost; all that remains is $A$
- $v_s(t) \approx c_0 + \sqrt{\frac{A}{2t}} \to c_0$, $\quad \Delta c = \sqrt{\frac{2A}{t}} \to 0$

## N - Wave

Consider $g(\xi)$ as shown below:

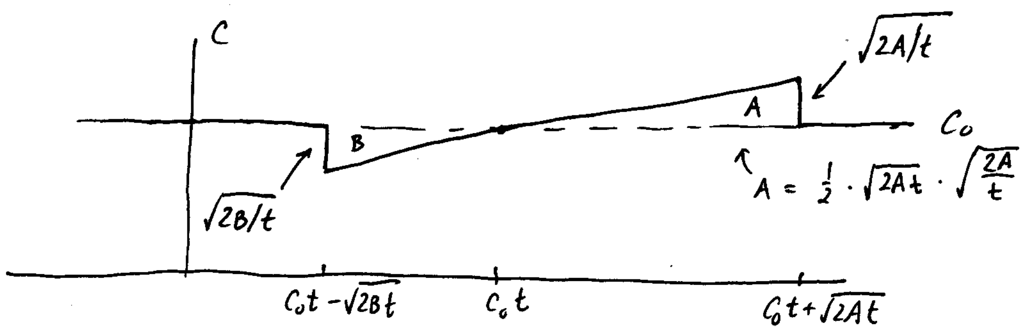



Now there are two shocks, since there are two compression regions where $g'(\xi) < 0$. As $t \to \infty$, $(\xi_-, \xi_+)_A \to (0, \infty)$ for the A shock and $(\xi_-, \xi_+)_B \to (-\infty, 0)$ for the B shock. Asymptotically,

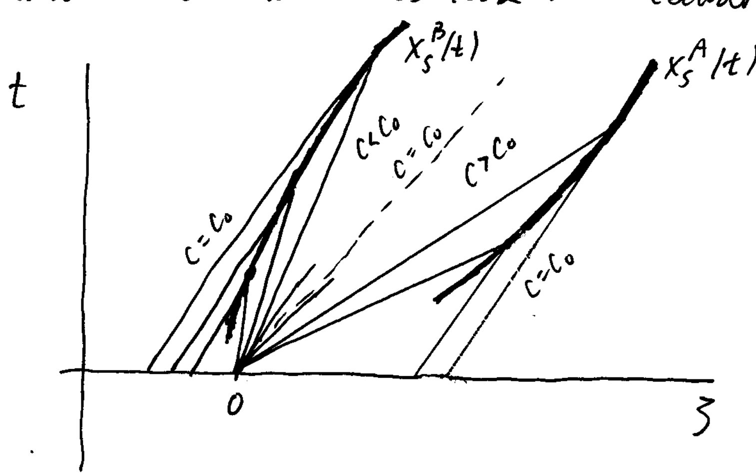
$$x_s^A \approx c_0 t + \sqrt{2At} \qquad \Delta c_A \approx \sqrt{\frac{2A}{t}}$$

$$x_s^B \approx c_0 t - \sqrt{2Bt} \qquad \Delta c_B \approx -\sqrt{\frac{2B}{t}}$$

Asymptotically, we have




Here's what the characteristics look like (beware fig. 2.13 in Whitham)



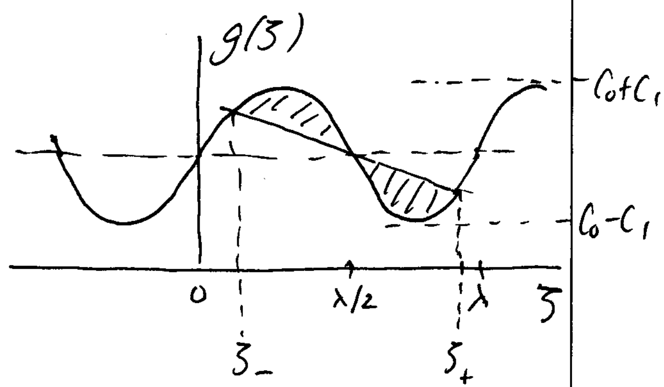

asymptotic $(t \to \infty)$
N-wave characteristics

## Periodic Wave

Next consider a periodic wave, with

$$g(\zeta) = c_0 + c_1 \sin\left(\frac{2\pi\zeta}{\lambda}\right)$$

We consider a single period $0 \le \zeta \le \lambda$. Then we have




$$\frac{1}{2}(\zeta_+-\zeta_-)\left\{c_1\sin\left(\frac{2\pi\zeta_+}{\lambda}\right)+c_1\sin\left(\frac{2\pi\zeta_-}{\lambda}\right)+2c_0\right\}$$
$$= \int_{\zeta_-}^{\zeta_+} d\zeta\left[c_0+c_1\sin\left(\frac{2\pi\zeta}{\lambda}\right)\right]$$

$$\Rightarrow \quad (\zeta_+-\zeta_-)\sin\left[\frac{\pi}{\lambda}(\zeta_++\zeta_-)\right]\cos\left[\frac{\pi}{\lambda}(\zeta_+-\zeta_-)\right] = \frac{\lambda}{\pi}\sin\left[\frac{\pi}{\lambda}(\zeta_+-\zeta_-)\right]\sin\left[\frac{\pi}{\lambda}(\zeta_++\zeta_-)\right]$$

The solution we seek is $\zeta_++\zeta_-=\lambda$, as is apparent from the figure. Then

$$x_s(t) = \zeta_- + g(\zeta_-)\,t = \zeta_+ + g(\zeta_+)\,t$$

$$t = \frac{\zeta_+-\zeta_-}{c_1\left[\sin\left(\frac{2\pi\zeta_-}{\lambda}\right)-\sin\left(\frac{2\pi\zeta_+}{\lambda}\right)\right]} = \frac{\zeta_+-\zeta_-}{2c_1\sin\left[\frac{\pi}{\lambda}(\zeta_+-\zeta_-)\right]}$$

$$x_s(t) = \frac{1}{2}(\zeta_++\zeta_-) + \frac{1}{2}\left\{\sin\left(\frac{2\pi\zeta_+}{\lambda}\right)+\sin\left(\frac{2\pi\zeta_-}{\lambda}\right)\right\}t$$
$$= \frac{\lambda}{2} + c_0 t$$

The shock discontinuity is

$$\Delta c = c(\zeta_+) - c(\zeta_-) = c_1\sin\left(\frac{2\pi\zeta_+}{\lambda}\right) - c_1\sin\left(\frac{2\pi\zeta_-}{\lambda}\right)$$
$$= 2c_1\sin\left(\frac{\pi}{\lambda}(\zeta_+-\zeta_-)\right)$$

Define

$$\theta \equiv \frac{\pi}{\lambda}(\zeta_+ - \zeta_-)$$

so that

$$x_s(t) = \frac{1}{2}\lambda + c_0 t$$

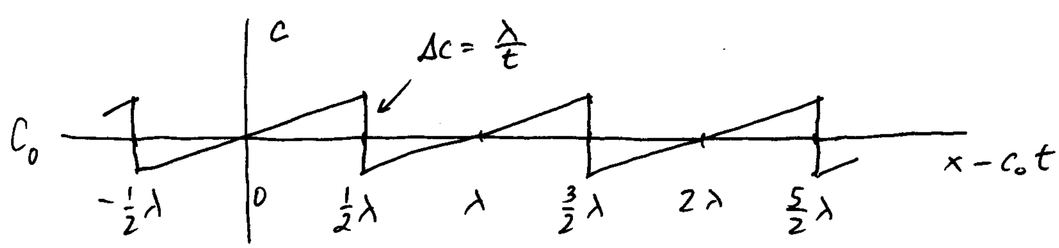
$$v_s(t) = c_0$$

$$\frac{\Delta c}{c_0} = \frac{2c_1}{c_0}\sin\theta \qquad \text{with} \qquad t = \frac{\lambda}{2\pi c_1}\frac{\theta}{\sin\theta}$$

Thus,

| $\theta$ | $t$ | $\Delta c$ | |
|---|---|---|---|
| $0$ | $t_b = \frac{\lambda}{2\pi c_1}$ | $0$ | shock starts at $t = t_b = \frac{\lambda}{2\pi c_1}$ |
| $\frac{\pi}{2}$ | $\frac{\lambda}{4c_1}$ | $\frac{2c_1}{c_0}$ | maximum shock discontinuity |
| $\pi$ | $\infty$ | $0$ | asymptotically, shock disappears |

As $t \to \infty$, we have $\sin\theta \sim \frac{\lambda}{2c_1 t} \Rightarrow \Delta c(t \to \infty) = \frac{\lambda}{t}$.

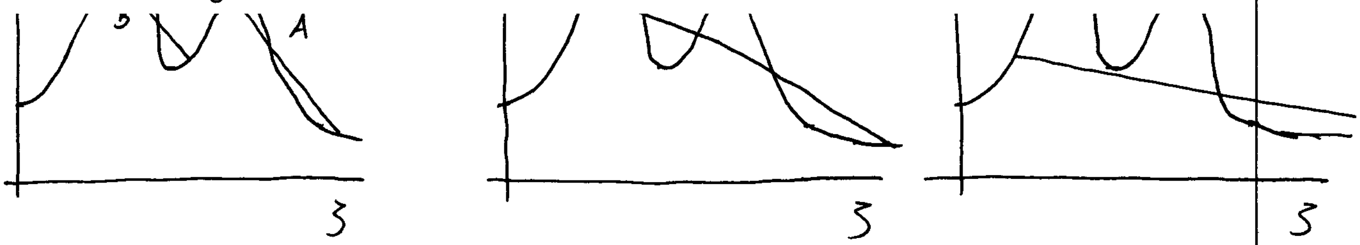



Indeed for <u>any</u> periodic $g(\zeta)$ with $g(\zeta) = g(\zeta + \lambda)$, asymptotically we must have $\zeta_+ - \zeta_- \to \lambda$ as $t \to \infty$, hence

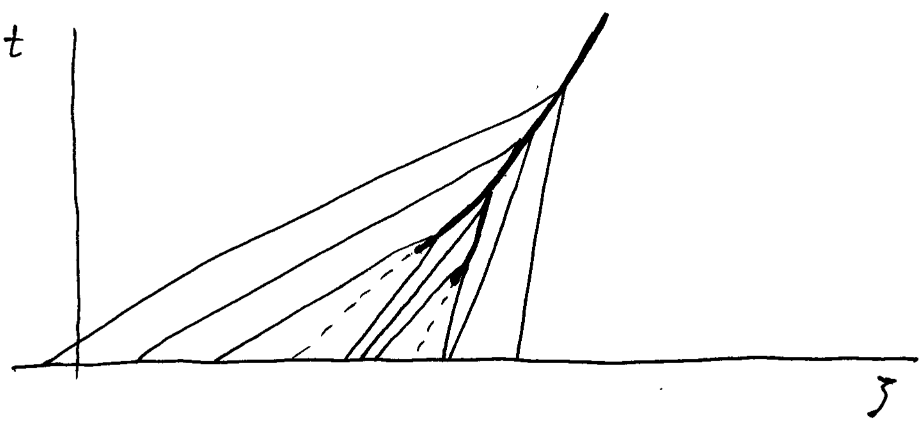
$$\Delta c = \frac{\zeta_+ - \zeta_-}{t} \underset{t\to\infty}{=} \frac{\lambda}{t}$$

## Confluence of Shocks

When several shocks are produced, it is possible for a trailing shock to overtake one which is ahead of it, and to continue on as a single shock.




This leads to the following diagram (beware Whitham fig. 2.17)




Note that

$$v_s^A = \frac{1}{2} g(\xi_+^A) + \frac{1}{2} g(\xi_-^A)$$

$$v_s^B = \frac{1}{2} g(\xi_+^B) + \frac{1}{2} g(\xi_-^B)$$

NB: if $J(\rho)$ quadratic then $c_{12} = \frac{1}{2}(c_1 + c_2)$, $c_{23} = \frac{1}{2}(c_2 + c_3)$, $c_{13} = \frac{1}{2}(c_1 + c_3)$

Merging requires

$$\xi_-^A = \xi_+^B \equiv \xi$$

$$\xi_+^A \equiv \xi_+ \quad ; \quad \xi_-^B \equiv \xi_-$$

and

$$t = \frac{\xi_+ - \xi}{g(\xi)-g(\xi_+)} = \frac{\xi - \xi_-}{g(\xi_-)-g(\xi)}$$

so the slopes are also equal, i.e. the segments connect.

Shock Fitting : General $J(\rho)$

When $J(\rho)$ is quadratic, we have

$$\rho_t + c(\rho)\rho_x = 0 \Rightarrow c_t + cc_x = 0$$

The time evolution

$$(\xi, c) \longrightarrow (\xi + ct, c)$$

preserves areas and also maps lines to lines. In general, though, while

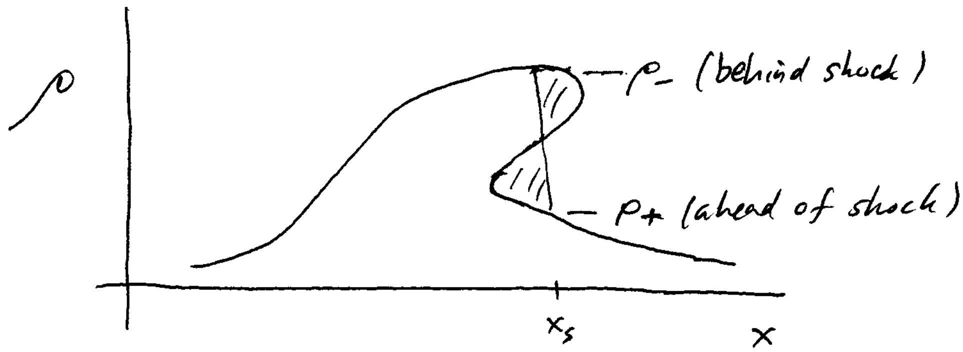
$$(\xi, \rho) \rightarrow (\xi + c(\rho)t, \rho)$$

preserves areas, it doesn't send lines to lines. Thus, the "pre-image" of the shock front is not a simple straight line, so our "linear" equal area construction is no longer useful. Still, we can make progress.

Let $x(\rho,t)$ be the inverse of $\rho(x,t)$, with $\xi(\rho) = x(\rho, t=0)$. Then

$$x(\rho,t) = \xi(\rho) + c(\rho)t$$

Note that $\xi(\rho)$ is in general multivalued. We still have that the shock sol$^{\underline{n}}$ covers the same area as the multivalued $\rho(x,t)$.




We have

$$\int_{\rho_+}^{\rho_-} d\rho \; x(\rho,t) = \int_{\rho_+}^{\rho_-} d\rho \left[ \zeta(\rho) + c(\rho)\, t \right] = (\rho_- - \rho_+)\, x_s$$

Since $c(\rho) = J'(\rho)$, we have

$$(J_+ - J_-) - (\rho_+ - \rho_-)\, x_s = \int_{\rho_+}^{\rho_-} d\rho \; \zeta(\rho)$$

$$= \rho_- \zeta_- - \rho_+ \zeta_+ - \int_{\zeta_+}^{\zeta_-} d\zeta \; \rho(\zeta)$$

Now the shock position is given by

$$x_s = \zeta_- + c_- t = \zeta_+ + c_+ t$$

hence

$$\left\{ (J_+ - J_-) - (\rho_+ c_+ - \rho_- c_-) \right\} = + \frac{c_+ - c_-}{\zeta_+ - \zeta_-} \int_{\zeta_-}^{\zeta_+} d\zeta \; \rho(\zeta)$$

This is useful because $J_\pm$, $\rho_\pm$, $c_\pm$ are all functions of $\zeta_\pm$, hence this gives one equation relating $\zeta_+$ and $\zeta_-$. When $J(\rho)$ is quadratic, it reduces to

$$\frac{1}{2}(c_+ + c_-)(\zeta_+ - \zeta_-) = \int_{\zeta_-}^{\zeta_+} d\zeta \; c(\rho(\zeta))$$

as before. For a hump, we still have $x_s \approx c_0 t + \sqrt{2At}$ and $c - c_0 \approx \sqrt{2A/t}$, with $A = c'(\rho_0) \int_{-L}^{L} d\zeta \left[ \rho(\zeta) - \rho_0 \right]$.

## Linearized Theory

The equation

$$c_t + c c_x = 0$$

subject to

$$c(x,0) = g(x)$$

is solved by

$$x = \xi + g(\xi) t$$
$$c = g(\xi)$$

Suppose that $g(\xi) = c_0 + \epsilon h(\xi)$. Then

$$c = c_0 + \epsilon h(x - ct)$$

To lowest order in $c - c_0$, we have $c = c_0 + \epsilon h(x - c_0 t)$, which has no shocks. However, from

$$c_x = g' - c_x t g' \Rightarrow c_x = \frac{g'}{1 + g' t}$$

we see that $c_x$ diverges for

$$t = t_b = \left( -\frac{1}{g'(\xi)} \right)_{min}$$

Thus, a perturbation expansion cannot converge uniformly. Nevertheless, we may attempt to expand:

$$c = c_0 + \epsilon h(x - c_0 t - \epsilon h(x - ct) t)$$
$$= c_0 + \epsilon h(x - c_0 t) - \epsilon^2 h'(x - c_0 t) h(x - \mathbf{ct}) t + \ldots$$

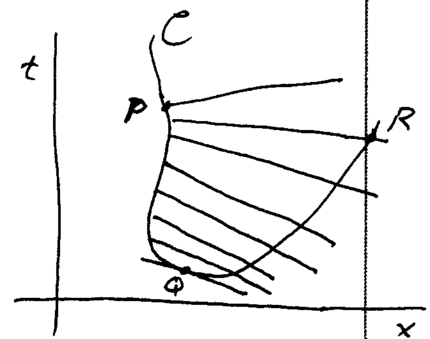
N.B.!

## Signaling Problems

Thus far we've specified initial conditions $\rho(x,0) = f(x)$ along the x-axis. In general, we can imagine specifying initial conditions along some curve $C$ in the $(x,t)$ plane. As before, the solution to $\rho_t + c(\rho)\rho_x = 0$ is obtained by integrating

$$\frac{d\rho}{dt} = 0 \quad , \quad \frac{dx}{dt} = c(\rho)$$

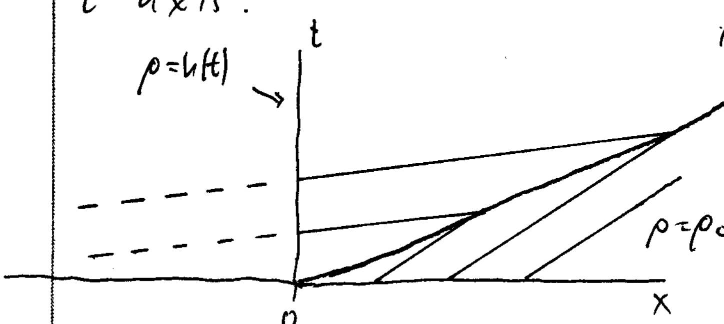
using the initial conditions along $C$. If $C$ intersects characteristics twice, as shown in the figure to the right, then initial data can only be supplied on part of $C$, e.g. from $P$ to $Q$ or from $Q$ to $R$. Because characteristics depend on initial conditions, the acceptable region of $C$ on which initial data can be specified can't always be determined a priori.

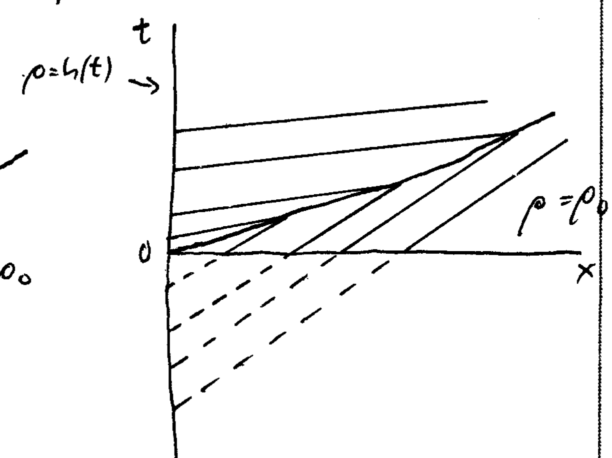



A standard boundary value problem is the so-called <u>signaling problem</u>, for which

$$\rho(x>0, t=0) = \rho_0$$
$$\rho(x=0, t>0) = h(t)$$

Now characteristics emanate from the positive x-axis and positive t-axis.



Again, crossing of characteristics means formation of a shock, else the sol$^{\underline{n}}$ is multivalued. We can treat this problem as before if we extend the characteristics backward into the 'unphysical' regions $t<0$ or $x<0$. E.g. consider instead

$$\rho_t + c(\rho)\,\rho_x = 0$$
$$\rho(x=0,t) = r(t) = \begin{cases} h(t) & \text{if } t>0 \\ \rho_0 & \text{if } t<0 \end{cases}$$

The sol$^{\underline{n}}$ is

$$\begin{aligned} x &= c(r(\tau))\,(t-\tau) \\ &\equiv b(\tau)(t-\tau) \qquad\qquad b(\tau) \equiv c(r(\tau)) \end{aligned}$$

where now $\tau$ labels the characteristic (as $s$ did earlier). Crossing of neighboring characteristics means

$$\begin{aligned} x &= b(\tau)(t-\tau) \\ &= b(\tau+d\tau)(t-\tau-d\tau) \\ &= b(\tau)(t-\tau) + \{b'(\tau)(t-\tau) - b(\tau)\}\,d\tau + \cdots \end{aligned}$$

i.e.

$$t = \tau + \frac{b(\tau)}{b'(\tau)}$$

Only crossings at positive $x$ are relevant. Thus,

$$t_b = \min_{\tau}\left\{\tau + \frac{b(\tau)}{b'(\tau)}\right\}$$

$$x_s = \frac{b^2(\tau)}{b'(\tau)}$$

See Whitham §2.11 for more details.

## Source Terms

Consider the more general case,

$$\rho_t + c\rho_x = \sigma$$

where $c = c(\rho, x, t)$ and $\sigma = \sigma(\rho, x, t)$. Characteristics obey

$$\frac{d\rho}{dt} = \sigma(\rho, x, t), \quad \frac{dx}{dt} = c(\rho, x, t)$$

and the equations are now coupled. In general, characteristics no longer are straight lines. Still, crossing of characteristics means that waves break and shocks develop.

## Damped Example

Consider

$$c_t + c c_x + \alpha c = 0$$

so that

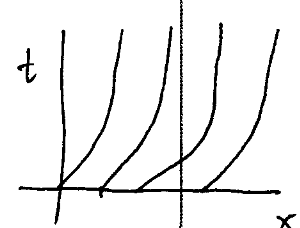
$$\frac{dc}{dt} = -\alpha c \quad , \quad \frac{dx}{dt} = c$$

$$\Rightarrow \begin{cases} c = e^{-\alpha t} f(\xi) \\ \dot{x} = e^{-\alpha t} f(\xi) \quad \text{with} \quad x(t=0) = \xi \end{cases}$$

The sol$^{\underline{n}}$ is

$$x(t) = \xi + \frac{1 - e^{-\alpha t}}{\alpha} f(\xi) \qquad \text{(not a straight line!)}$$

Neighboring characteristics cross if $\frac{dx}{d\xi} = 0$, i.e.




$$1 + \frac{1 - e^{-\alpha t}}{\alpha} f'(\xi) = 0$$

which requires $f'(\xi) < -\alpha$, i.e. a critical slope.

## Moving Source

Consider next

$$C_t + C C_x = \sigma(x-ut)$$

We seek a moving wave sol$^{\underline{n}}$ $c = c(\xi) = c(x-ut)$. Then

$$(c-u)C_\xi = \sigma(\xi)$$

which may be formally integrated to yield

$$\frac{1}{2}(u-c)^2 - \frac{1}{2}(u-c_0)^2 = -\int_\xi^\infty d\xi'\,\sigma(\xi')$$

Consider the supersonic case, with $u > c$. Then

$$c = u - \left\{ (u-c_0)^2 - 2\int_\xi^\infty d\xi'\,\sigma(\xi') \right\}^{1/2}$$

This is a valid sol$^{\underline{n}}$ provided

$$u - c_0 > \sqrt{2\int_\xi^\infty d\xi'\,\sigma(\xi')}$$

for all $\xi$, i.e. if

$$u - c_0 > \sqrt{2\int_{-\infty}^\infty d\xi'\,\sigma(\xi')}$$

Thus, for small source strengths, no shock is required.

If shocks are produced, it is due to transients from initial conditions overtaking the wave, as is also true in the subsonic case.

## General Nonlinear First Order PDEs

Let $\phi = \phi(x_1, \ldots, x_n)$ and define $p_i \equiv \frac{\partial \phi}{\partial x_i}$. A general nonlinear first order PDE may be written

$$H(\vec{p}, \phi, \vec{x}) = 0$$

Now consider a curve $\vec{x}(\lambda)$. We have

$$\frac{d\phi}{d\lambda} = \frac{\partial \phi}{\partial x_i} \frac{dx_i}{d\lambda} = p_i \frac{dx_i}{d\lambda}$$

Notice that

$$\frac{dp_i}{d\lambda} = \frac{\partial^2 \phi}{\partial x_i \partial x_j} \frac{dx_j}{d\lambda}$$

and

$$\frac{\partial}{\partial x_j} H(\vec{p}, \phi, \vec{x}) = \frac{\partial H}{\partial p_i} \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \frac{\partial H}{\partial \phi} \overset{p_j}{\frac{\partial \phi}{\partial x_j}} + \frac{\partial H}{\partial x_j} = 0$$

so that *if* we choose

$$① \quad \frac{dx_j}{d\lambda} = \frac{\partial H}{\partial p_j} \qquad \text{($n$ eqns)}$$

then

$$② \quad \frac{dp_j}{d\lambda} = -p_j \frac{\partial H}{\partial \phi} - \frac{\partial H}{\partial x_j} \qquad \text{($n$ eqns)}$$

We also have

$$③ \quad \frac{d\phi}{d\lambda} = p_j \frac{\partial H}{\partial p_j} \qquad \text{(1 eqn)}$$

Taken together, these provide $(2n+1)$ ODEs which determine the characteristic curve $x_j(\lambda)$, $\phi(\lambda)$, and $p_j(\lambda)$. In principle, the entire sol$^{\underline{n}}$ over the whole region of $(x,t)$ may be obtained in this manner.

The quasi-linear case,

$$H = c_i(\phi, \vec{x})\, p_i - \sigma(\phi, \vec{x})$$

is special, because ① and ③ give

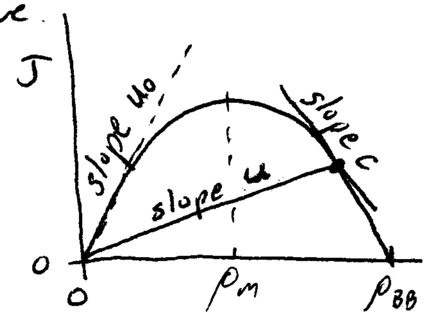
$$(1') \quad \frac{dx_j}{d\lambda} = c(\phi, \vec{x})$$

$$(3') \quad \frac{d\phi}{d\lambda} = p_j c_j = \sigma(\phi, \vec{x})$$

and so $\vec{p}$ drops out and these two eqns may be solved independently.

Traffic Flow

The mathematical theory of vehicular traffic flow was developed by Lighthill and Whitham in the 1950's. The basic idea of LW is to posit a constitutive relation $j = J(\rho)$ of the form shown here. At low values of $\rho$, we have




NB: $u > c$

$$J \sim u_0 \rho$$

where $u_0$ is the <u>free flow</u> velocity, which is typically around 70 mph for American freeways. At larger values of $\rho$, the function $J(\rho)$ turns over, and $J(\rho = \rho_{BB}) = 0$ when the traffic is bumper-to-bumper. In fact, this is an idealization, and the right half of this curve represents flow which is unstable and hysteretic.

For $\rho \lesssim \rho_{BB}$, we can approximate the velocity $u(\rho)$ as follows:

$$u(\rho) \cdot \tau_r = h - l_0$$

$$\tau_r = \text{reaction time}$$

$$h = 1/\rho = \text{headway}$$

$$l_0 = \text{average vehicle length} = 1/\rho_{BB}$$

Thus,

$$u(\rho) \approx \frac{1}{\tau_r}\left(\frac{1}{\rho} - \frac{1}{\rho_{BB}}\right) \approx \frac{\rho_{BB} - \rho}{\tau_r \rho_{BB}^2} \quad \text{if } \rho_{BB} - \rho \ll \rho_{BB}$$

Note that

$$J(\rho) = \rho\, u(\rho)$$

$$\Rightarrow \quad c(\rho) = J'(\rho) = u(\rho) + \rho\, u'(\rho) \quad < \quad u(\rho)$$

N.B.!

hence

$$c(\rho \lesssim \rho_{BB}) \approx \frac{\rho_{BB} - 2\rho}{\tau_r \, \rho_{BB}^2} \approx -\frac{1}{\tau_r \, \rho_{BB}} = -\frac{l_0}{\tau_r}$$

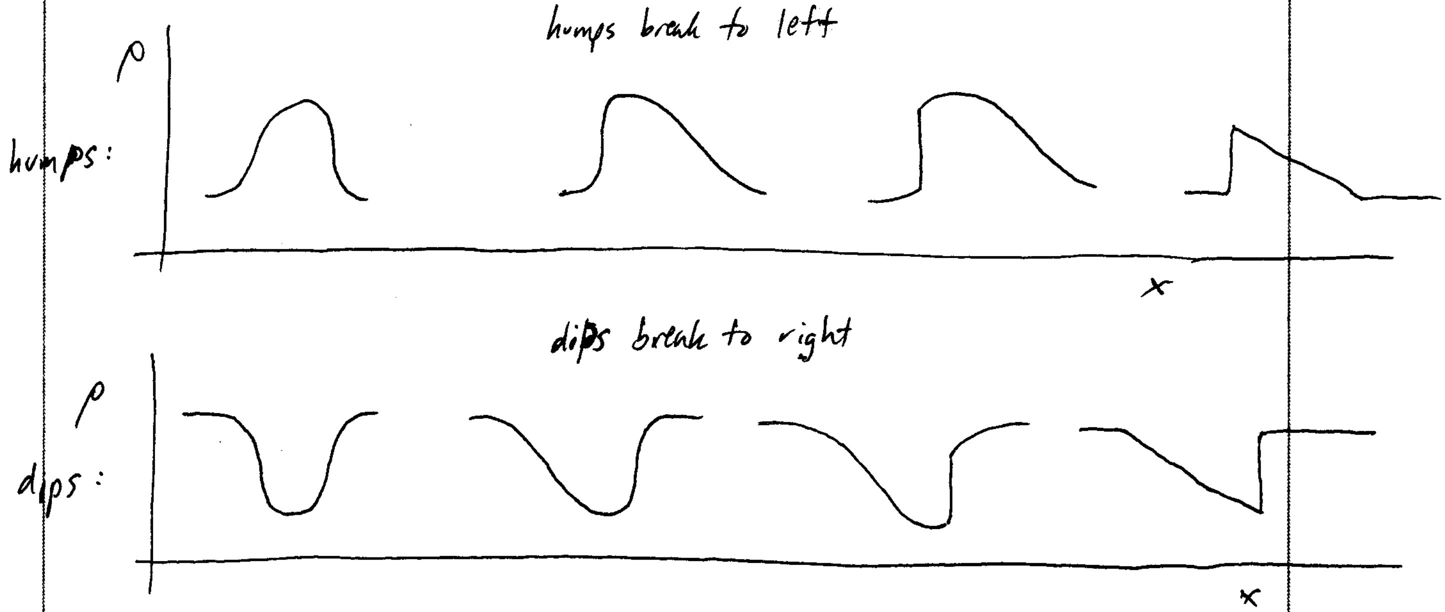
It is crucial to distinguish

$u(\rho)$ = velocity of the flow

$c(\rho)$ = velocity of wave propagation

Note that while $u(\rho) \geq 0$, $c(\rho)$ may be negative. For $\tau_r = 1\,\mathrm{sec}$ and $l_0 = 20\,\mathrm{ft}$, $c_{BB} \approx -14\,\mathrm{mph}$.

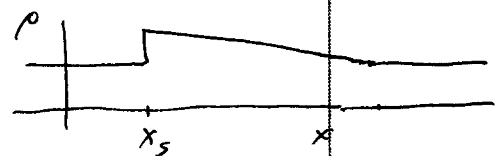
Since $J''(\rho) < 0$, we have $c'(\rho) < 0$, so larger $\rho$ corresponds to smaller $c$. Thus, bulges in traffic break as shown below:




Note also that $c < 0$ for $\rho > \rho_m$, where $J'(\rho_m) = 0$. Another important feature is that $c(\rho) < u(\rho)$, which means that <u>individual drivers pass through the wave disturbance</u>. For a hump-shaped $\rho(x)$, for example, which evolves into a triangle-shape with a trailing shock, drivers slow down rapidly as they enter the jam, and accelerate slowly as they leave it, a result which agrees with common experience. Asymptotically,

- $c \approx \dfrac{x}{t}$ for $c_0 t - \sqrt{2Bt} < x < c_0 t$

- $\rho = \rho_0 + \dfrac{x - c_0 t}{c'(\rho_0)\, t}$




- $B = |c'(\rho_0)| \displaystyle\int_{-\infty}^{\infty} d\xi \, [\rho(\xi) - \rho_0]$

- $x_s(t) = c_0 t - \sqrt{2Bt} \Rightarrow v_s \approx -\sqrt{\dfrac{Bt}{2}}$

- $c_+ - c_0 \approx -\sqrt{\dfrac{2B}{t}}$ , $\rho_+ - \rho_0 \approx + \dfrac{1}{|c'(\rho_0)|}\sqrt{\dfrac{2B}{t}}$

## Examples

In the following examples, we assume

$l_0 = 1/\rho_{BB}$ = effective vehicle length

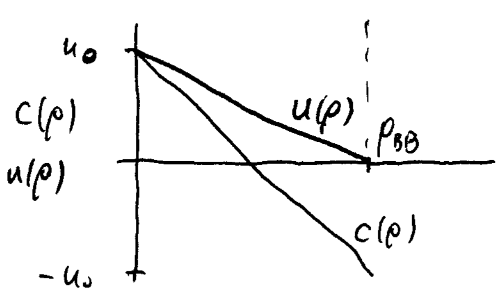
$$J(\rho) = u_0 \rho \left(1 - \frac{\rho}{\rho_{BB}}\right) = u_0 \rho (1 - l_0 \rho)$$

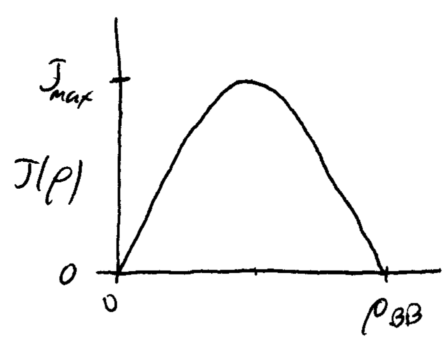
$$u(\rho) = J(\rho)/\rho = u_0 \left(1 - \frac{\rho}{\rho_{BB}}\right)$$

$$c(\rho) = \frac{\partial J}{\partial \rho} = u_0 \left(1 - \frac{2\rho}{\rho_{BB}}\right)$$

$$c(\rho) = 0 \Rightarrow \rho = \frac{1}{2}\rho_{BB} = \frac{1}{2 l_0} \quad ; \quad J_{max} = J(\tfrac{1}{2}\rho_{BB}) = \tfrac{1}{2} u_0 \rho_{BB}$$

Note that $c(0) = u_0$ and $c(\rho_{BB}) = -u_0$.



Green Light

We solve

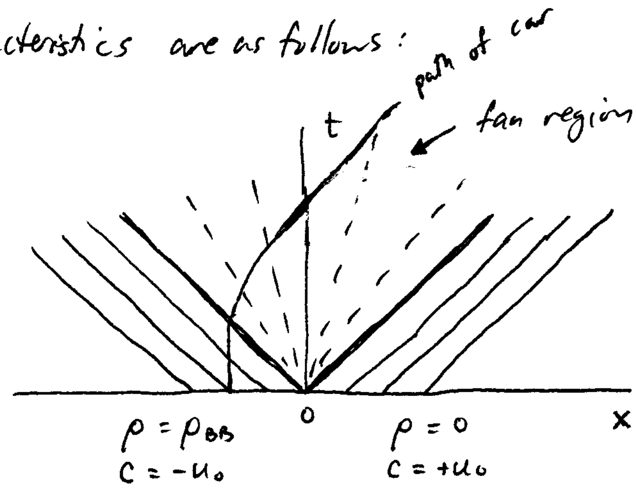
$$\rho_t + c(\rho)\rho_x = 0$$

$$c(\rho) = u_0\left(1 - 2\frac{\rho}{\rho_{BB}}\right)$$

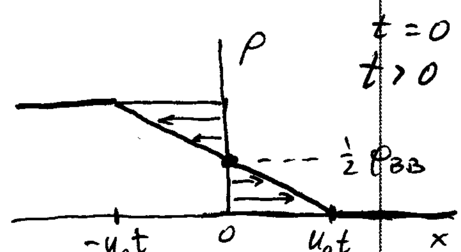
subject to

$$\rho(x,0) = \rho_{BB}\,\Theta(-x)$$

The characteristics are as follows:




$u > c \Rightarrow \frac{1}{u} < \frac{1}{c}$ (for $c > 0$)




In the fan, we have

$$\frac{x}{t} = c(\rho) = u_0\left(1 - \frac{2\rho}{\rho_{BB}}\right)$$

$$\Rightarrow \quad \rho(x,t) = \frac{1}{2}\rho_{BB}\left(1 - \frac{x}{u_0 t}\right) \quad \text{for } -u_0 \le \frac{x}{t} \le u_0$$

What is the motion of a vehicle in this flow?

$$\frac{dx}{dt} = u = \begin{cases} 0 \ ; & x < -u_0 t \\ u_0\left(1 - \frac{\rho}{\rho_{BB}}\right) = \frac{1}{2}u_0 + \frac{x}{2t} \ ; & -u_0 t < x < u_0 t \\ u_0 \ ; & x > u_0 t \end{cases}$$

Initial conditions : $x(t=0) = x_0 < 0$. We integrate until $t = -x_0/u_0$, so

$$x\left(t = -\frac{x_0}{u_0}\right) = x_0 < 0$$

For $-u_0 t < x < u_0 t$, we have

$$t\,\frac{dx}{dt} - \tfrac{1}{2}x = \tfrac{1}{2}u_0 t$$

which is linear but inhomogeneous. The homogeneous equation

$$t\,\frac{dx}{dt} - \tfrac{1}{2}x = 0 \quad \Rightarrow \quad \frac{dx}{x} = \frac{1}{2}\frac{dt}{t}$$

is separable, with sol$^{\underline{n}}$ $x_h(t) = C\,t^{1/2}$. The inhomogeneous eq$^{\underline{n}}$ is solved by $x(t) = u_0 t$, thus the complete sol$^{\underline{n}}$ is

$$x(t) = u_0 t + C\,t^{1/2}$$

We fix $C$ by demanding

$$x(t = -x_0/u_0) = -x_0 + C\sqrt{-\frac{x_0}{u_0}} = x_0$$

$$\Rightarrow \quad C = -2\sqrt{-u_0 x_0} = -2\sqrt{u_0|x_0|}$$

so that

$$x(t) = u_0 t - 2\sqrt{u_0|x_0|\,t}$$

Since $x(t) < u_0 t \;\; \forall\, t$, we never enter the third regime.

When does the car pass the light?

$$x(t) = 0 \quad \Rightarrow \quad t = \frac{4|x_0|}{u_0}$$

Let $t = \underset{\nearrow}{-\frac{x_0}{u_0}} + \delta t$. Then

$$x(t_0 + \delta t) = -x_0 + u_0\,\delta t - 2|x_0|\sqrt{1 + \frac{u_0\,\delta t}{|x_0|}}$$

$$= x_0 + \frac{u_0^2}{4|x_0|}(\delta t)^2 + \ldots$$

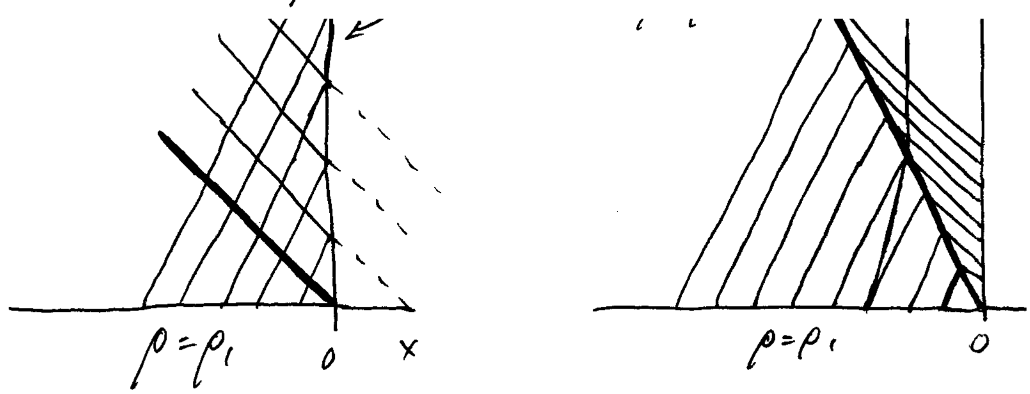
$$\Rightarrow \quad \ddot{x}(t_0^+) = \frac{u_0^2}{2|x_0|} = \text{initial acceleration upon exiting queue}$$

## Red Light

Consider now the initial conditions $\rho(x,0) = \rho_1$ and the "stop condition"

$$u(0,t) = 0$$

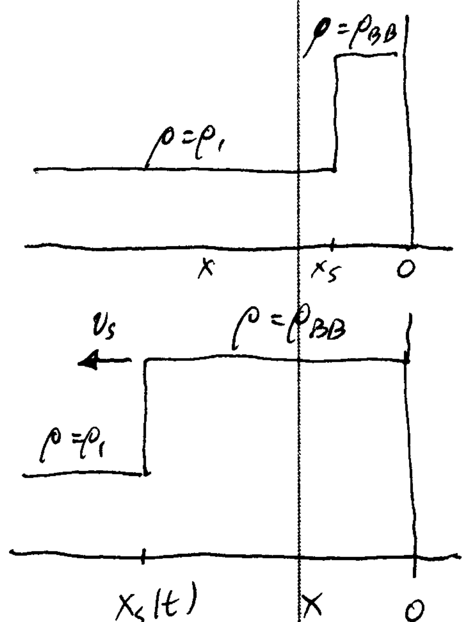
$$\text{i.e. } \rho(0,t) = \rho_{BB}$$




We have

$$x_s(t) = v_s t$$

$$v_s = \frac{J_+ - J_-}{\rho_+ - \rho_-} = -\frac{\rho_1 u(\rho_1)}{\rho_{BB} - \rho_1}$$




Simple derivation:

$$x_n(0) = -n/\rho_1 \qquad (n \geq 1)$$

$$x_n(t) = -n/\rho_1 + u_1 t$$

Car $n$ stops when

$$x_n(t) = -n/\rho_{BB} = x_s$$

$$\Rightarrow \quad t = n(\rho_1^{-1} - \rho_{BB}^{-1})/u_1$$

$$= -x_s\left(\frac{\rho_{BB}}{\rho_1} - 1\right)/u_0$$

$$\Rightarrow \quad x_s(t) = -\frac{\rho_1 u_1 t}{\rho_{BB} - \rho_1} \quad ✓$$

NB: $u(\rho) = u_0\left(1 - \frac{\rho}{\rho_{BB}}\right)$

$\Rightarrow \quad v_s = -u_0 \rho_1/\rho_{BB}$

## More on OVM

Consider the OVM,

$$\ddot{x}_n = \frac{1}{\tau}\left\{ V(x_{n+1} - x_n) - \dot{x}_n \right\}$$

Let's perturb about a steady-state sol$^{\underline{n}}$,

$$x_n^0(t) = an + V(a)\,t$$

Linear stability analysis:

$$x_n \equiv x_n^0 + \delta x_n$$

$$\delta\ddot{x}_n = \frac{1}{\tau}\left\{ V(a + \delta x_{n+1} - \delta x_n) - V(a) - \delta\dot{x}_n \right\}$$

$$= \frac{1}{\tau}\left\{ V'(a)(\delta x_{n+1} - \delta x_n) - \delta\dot{x}_n + \mathcal{O}\left((\delta x_{n+1} - \delta x_n)^2\right) \right\}$$

Try a sol$^{\underline{n}}$ of the form

$$\delta x_n = A\, e^{ikna} e^{-\lambda t}$$

$$\Rightarrow \qquad \lambda^2 = \frac{1}{\tau}\left\{ V'(a)\,(e^{ika} - 1) + \lambda \right\}$$

The sol$^{\underline{n}}$ is

$$\lambda = \frac{1 \pm \sqrt{1 - 4\tau V'(a)(1 - e^{ika})}}{2\tau}$$

Stability requires $\mathrm{Re}\,\lambda > 0$. Let $z \equiv e^{ika}$ ; $|z| = 1$, and let $\alpha \equiv V'(a)$. [The discriminant is

$$D = 1 - 4\alpha\tau\,(1 - z) \equiv r e^{i\theta} = 1 - 4\alpha\tau + 4\alpha\tau\cos ka + 4i\alpha\tau \sin ka$$

$$|D|^2 = \left| 1 - 4\alpha\tau(1 - \cos ka) + 4i\alpha\tau \sin ka \right|^2$$

$$\Rightarrow \quad |D|^2 = \left(1 - 4\alpha\tau + 4\alpha\tau\cos ka\right)^2 + 16\alpha^2\tau^2\sin^2 ka$$
$$= (1-4\alpha\tau)^2 + 8\alpha\tau(1-4\alpha\tau)\cos ka + 16\alpha^2\tau^2 \quad ]$$

Let's write

$$\sqrt{D} = \mu + i\nu \qquad (\mu > 0 \text{ wlog})$$

so that

$$1 - 4\alpha\tau(1-\cos ka) = \mu^2 - \nu^2$$
$$4\alpha\tau\sin ka = 2\mu\nu$$

Now

$$\lambda = \frac{1 \pm \mu \pm i\nu}{2\tau}$$

so stability requires $\mu < 1$. The stability boundary is $\mu = 1$, which means

$$\nu = 2\alpha\tau\sin ka$$

$$\mu^2 - \nu^2 = 1 - 4\alpha^2\tau^2\sin^2 ka$$
$$= 1 - 4\alpha\tau + 4\alpha\tau\cos ka$$

i.e.

$$1 - \cos ka = \alpha\tau\sin^2 ka$$

so that

$$\boxed{\alpha\tau = \frac{1}{2\cos^2(\frac{1}{2}ka)}}$$

The minimum of the RHS is $\frac{1}{2}$, at $k=0$, so stability requires

$$\alpha\tau < \frac{1}{2} \Leftrightarrow \text{stability of steady state sol}^{\underline{n}}$$

Thus, if the reaction time $\tau$ is too long, the steady state sol$^{\underline{n}}$ is unstable.

<u>Traditional CFM</u>

Let $v_n(t) = \dot{x}_n(t)$. Study the model

$$\dot{v}_n(t+\tau) = \frac{1}{\sigma}\{v_{n+1}(t) - v_n(t)\}$$

Fourier transform: $v_n(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}\hat{v}_n(\omega)$. Then

$$-i\omega e^{-i\omega\tau}\hat{v}_n(\omega) = \frac{1}{\sigma}\hat{v}_{n+1}(\omega) - \frac{1}{\sigma}\hat{v}_n(\omega)$$

$$\hat{v}_n(\omega)\cdot\left(1 - i\omega\sigma e^{-i\omega\tau}\right) = \hat{v}_{n+1}(\omega)$$

$$\rightarrow \underset{(\text{trailer})}{\hat{v}_n(\omega)} = \frac{\hat{v}_{n+1}(\omega)}{1-i\omega\sigma e^{-i\omega\tau}} \equiv r(\omega)\,e^{i\theta(\omega)}\underset{(\text{leader})}{\hat{v}_{n+1}(\omega)}$$

We have

$$1-i\omega\sigma e^{-i\omega\tau} = 1-\omega\sigma\sin\omega\tau - i\omega\sigma\cos\omega\tau$$
$$= \sqrt{1-2\omega\sigma\sin\omega\tau+\omega^2\sigma^2}\,\exp\left(-i\tan^{-1}\frac{\omega\sigma\cos\omega\tau}{1-\omega\sigma\sin\omega\tau}\right)$$

$$r(\omega) = \left(1-2\omega\sigma\sin\omega\tau+\omega^2\sigma^2\right)^{-1/2}$$

$$\theta(\omega) = \tan^{-1}\left(\frac{\omega\sigma\cos\omega\tau}{1-\omega\sigma\sin\omega\tau}\right)$$

Stability requires $r<1 \Rightarrow 2\sin\omega\tau < \omega\sigma$.

## Continuum Limit of Car-following Models

$$\underbrace{\cdots\ 4\ 3\ 2\ 1\ 0}_{n} \longrightarrow$$

The generic CFM is

$$\dot{x}_n(t+\tau) = V(x_{n-1}(t) - x_n(t))$$
$$= V(h_n(t))$$

where $h_n(t)$ is the headway, $h_n = x_{n-1} - x_n$. Clearly

$$\dot{h}_n(t) = \dot{x}_{n-1}(t) - \dot{x}_n(t)$$

Now introduce continuous f^ns $v(x,t)$ and $h(x,t)$, with

$$v(x_n, t) = \dot{x}_n(t)$$
$$h\left(\frac{x_{n-1}+x_n}{2}, t\right) = h_n(t) = x_{n-1}(t) - x_n(t)$$

Then

$$v(x_n(t+\tau), t+\tau) = V\left(h\left(x_n + \tfrac{1}{2}h_n, t\right)\right)$$

so

$$v(x + \tau v + \ldots, t+\tau) = V\left(h\left(x_n + \tfrac{1}{2}h_n, t\right)\right)$$

$$\Rightarrow \quad v + \tau v v_x + \tau v_t = V(h) + \tfrac{1}{2} h h_x V'(h)$$

up to $O(\tau^2, h^2)$. Similarly,

$$\frac{d}{dt} h\left(\frac{x_{n-1}+x_n}{2}, t\right) = v(x_{n-1}, t) - v(x_n, t)$$

$$h_t + v h_x = h v_x$$

so we have, with $h = 1/\rho$, and $u(\rho) = V(1/\rho)$,

$$\boxed{\begin{aligned} v + \tau(v_t + v v_x) &= u(\rho) + \frac{u'(\rho)}{2\rho}\rho_x \\ \rho_t + (\rho v)_x &= 0 \end{aligned}}$$

Note that

$$v_t + v v_x = -\frac{1}{\tau}\left\{ v - u(\rho) - \frac{u'(\rho)}{2\rho}\rho_x \right\}$$

The stability criterion is $2\rho_0^2 |u'(\rho_0)|\tau < 1$. To derive this, expand about $\rho_0$, $v_0 = u(\rho_0)$ : $\rho = \rho_0 + \delta\rho$ , $v = v_0 + \delta v$

① $$\delta\rho_t + \rho_0 \delta v_x + v_0 \delta\rho_x = 0$$

② $$\tau(\delta v_t + v_0 \delta v_x) = u'(\rho_0)\delta\rho - \delta v + \frac{u'(\rho_0)}{2\rho_0}\delta\rho_x$$

Eliminate $\delta v$ :

$$\delta v_x = -\frac{1}{\rho_0}(\delta\rho_t + v_0 \delta\rho_x)$$

Differentiate ② wrt $x$ :

$$\begin{aligned} \tau(\delta v_{tx} + v_0 \delta v_{xx}) &= -\frac{\tau}{\rho_0}(\partial_t + v_0\partial_x)^2 \delta\rho \\ &= u'(\rho_0)\delta\rho_x + \frac{u'(\rho_0)}{2\rho_0}\delta\rho_{xx} - \delta v_x \\ &= \frac{1}{\rho_0}\delta\rho_t + \left(u'(\rho_0) + \frac{v_0}{\rho_0}\right)\delta\rho_x + \frac{u'(\rho_0)}{2\rho_0}\delta\rho_{xx} \end{aligned}$$

Thus,

$$0 = (\partial_t + v_0\partial_x)^2\delta\rho + \frac{1}{\tau}\delta\rho_t + \frac{1}{\tau}(v_0 + \rho_0 u'(\rho_0))\delta\rho_x + \frac{u'(\rho_0)}{2\tau}\delta\rho_{xx}$$

i.e. with $c_0 \equiv v_0 + \rho_0 u'(\rho_0)$ and $\tilde{\nu} \equiv -\frac{1}{2}u'(\rho_0)$

$$0 = \tau(\partial_t + v_0\partial_x)^2\delta\rho + \delta\rho_t + c_0\delta\rho_x - \tilde{\nu}\,\delta\rho_{xx}$$

We take

$$\delta\rho(x,t) = A\,e^{ikx}e^{-i\omega t}$$

and derive

$$\tau(\omega - v_0 k)^2 + i(\omega - c_0 k) - \tilde{\nu} k^2 = 0$$

so

$$\omega = \omega_{\pm}(k) = v_0 k - \frac{i}{2\tau} \pm \frac{1}{\tau}\sqrt{\tilde{\nu}\tau k^2 - \frac{1}{4} + i(c_0 - v_0)\tau k}$$

Stability requires $\mathrm{Im}(\omega_{\pm}) < 0$ for both roots. The marginal case is when $\mathrm{Im}(\omega_{+}) = 0$, i.e.

$$\left(\Omega - v_0 k + \frac{i}{2\tau}\right)^2 = \frac{1}{\tau^2}\left(\tilde{\nu}\tau k^2 - \frac{1}{4} - i(v_0 - c_0)\tau k\right)$$

with $\Omega \in \mathbb{R}$. Thus

$$\left.\begin{aligned} (\Omega - v_0 k)^2 &= \tilde{\nu}\tau^{-1} k^2 \\ \Omega - v_0 k &= (c_0 - v_0) k \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \Omega &= c_0 k \\ (v_0 - c_0)^2 &= \tilde{\nu}/\tau \end{aligned}$$

Thus,

$$\text{stability} \Leftrightarrow \quad \frac{\tilde{\nu}}{\tau} > (v_0 - c_0)^2$$

i.e.

$$-\frac{1}{2\tau} u'(\rho_0) > \rho_0^2 \left(u'(\rho_0)\right)^2$$

$$\Rightarrow \quad 2\rho_0^2 \tau \left| u'(\rho_0) \right| < 1$$

## Diffusion Corrections

Suppose we add a diffusive term to $J(\rho)$:

$$j = J(\rho) - \nu \rho_x$$
$$= \rho V(\rho) - \nu \rho_x$$

We model the behavior of the velocity field $u(x,t)$ as

$$\frac{Du}{Dt} = u_t + u u_x = -\frac{1}{\tau}\left(u - V(\rho) + \frac{\nu}{\rho}\rho_x\right)$$

Here, $\tau$ is a time scale, on the order of driver reaction time.

The density still obeys

$$\rho_t + j_x = \rho_t + (\rho u)_x = 0$$

We now have two coupled PDEs.

To get some idea of the behavior, we linearize, writing

$$\rho = \rho_0 + \delta\rho \qquad\qquad u_0 = V(\rho_0)$$
$$u = u_0 + \delta u$$

and obtain

$$\tau(\delta u_t + u_0 \delta u_x) = -\left\{\delta u - V'(\rho_0)\delta\rho + \frac{\nu}{\rho_0}\delta\rho_x\right\}$$
$$\delta\rho_t + u_0 \delta\rho_x + \rho_0 \delta u_x = 0$$

Therefore,

$$\delta u_x = -\frac{1}{\rho_0}\delta\rho_t - \frac{u_0}{\rho_0}\delta\rho_x$$

so

$$\tau(\delta u_{xt} + u_0 \delta u_{xx}) = -\delta u_x + V'(\rho_0)\delta\rho_x - \frac{\nu}{\rho_0}\delta\rho_{xx}$$
$$= -\frac{\tau}{\rho_0}\left(\delta\rho_{tt} + 2u_0\delta\rho_{xt} + u_0^2\delta\rho_{xx}\right)$$

$$\Rightarrow \quad \delta\rho_t + u_0\delta\rho_x + \rho_0 V'(\rho_0)\delta\rho_x - \nu\,\delta\rho_{xx} = -\tau(\partial_t + u_0\partial_x)^2\delta\rho$$

Thus, with

$$c(\rho_0) \equiv c_0 = V(\rho_0) + \rho_0 V'(\rho_0)$$

we have

$$\left(\frac{\partial}{\partial t} + c_0 \frac{\partial}{\partial x}\right)\delta\rho = \left\{\nu \frac{\partial^2}{\partial x^2} - \tau\left(\frac{\partial}{\partial \tau} + u_0 \frac{\partial}{\partial x}\right)^2\right\}\delta\rho$$

This linear PDE is solved by

$$\delta\rho(x,t) = A\, e^{i(kx-\omega t)}$$

provided

$$\tau(\omega - u_0 k)^2 - \nu k^2 + i(\omega - c_0 k) = 0$$

$$\omega = u_0 k - \frac{i}{2\tau} \pm \sqrt{\frac{\nu k^2}{\tau} - \frac{1}{4\tau^2} + i\,\frac{(c_0 - u_0)k}{\tau}}$$

The sol$^{\underline{n}}$ is stable if $\mathrm{Im}(\omega) < 0$ for both roots. This requires

$$\nu > (c_0 - u_0)^2 \tau \iff \text{stability}$$

We can factor the RHS of the linearized wave eqn to obtain

$$(\partial_t + c_0 \partial_x)\delta\rho = -\tau(\partial_t + c_+ \partial_x)(\partial_t + c_- \partial_x)\delta\rho$$

with

$$c_\pm = u_0 \pm \sqrt{\nu/\tau}$$

It turns out that the fastest and slowest propagation speeds are determined by the highest order derivatives. Thus, we can approximate the behavior by

$$(\partial_t + c_0 \partial_x)\delta\rho = 0$$

provided $c_- < c_0 < c_+ \iff |c_0 - u_0| < \sqrt{\nu/\tau}$, as before.

## Shock Structure

Once again, we assume $u = u(\xi)$, $\rho = \rho(\xi)$ with $\xi = x + v_s t$.
Then

$$[(u - v_s)\rho]_x = 0 \quad \Rightarrow \quad (v_s - u)\rho = A$$

We also have

$$\tau \rho (u - v_s) u_x + \nu \rho_x + \rho u - J(\rho) = 0$$

But $u = u(\rho) = v_s - A/\rho$, so

$$\left(\nu - \frac{\tau A^2}{\rho^2}\right)\rho_x = J(\rho) - v_s\rho + A = \overbrace{-\frac{u_0}{\rho_{BB}}(\rho_+ - \rho)(\rho - \rho_-)}^{\text{N.B. } J''(\rho) = -\frac{u_0}{\rho_{BB}}}$$

For $\tau = 0$, we recover an earlier result. Now consider a sol$^{\underline{n}}$ interpolating between $\rho(-\infty) = \rho_-$ and $\rho(+\infty) = \rho_+$. If $\nu - \frac{\tau A^2}{\rho^2}$ remains positive on this interval, then $\rho_x > 0$ and the profile is smooth. If, however, $\nu - \frac{\tau A^2}{\rho^2}$ vanishes, i.e. when $\rho = A^{-1}\sqrt{\nu/\tau} \equiv \rho_*$, the slope $\rho_x$ diverges. In order for $\nu - \frac{\tau A^2}{\rho^2} > 0$, we need

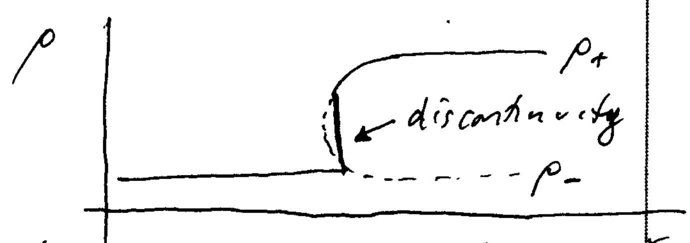
$$\nu > \tau \frac{A^2}{\rho^2} = \tau (v_s - u)^2$$

i.e.

$$u(\rho) - \sqrt{\nu/\tau} < v_s < u(\rho) + \sqrt{\nu/\tau}$$

Note that the asymptotic sol$^{\underline{ns}}$ $\rho = \rho_\pm$ are stable if

$$u(\rho_\pm) - \sqrt{\nu/\tau} < c(\rho_\pm) < u(\rho_\pm) + \sqrt{\nu/\tau}$$


The diverging of $\rho_x$ is cured by another discontinuity.

## Car-Following Theories

Consider single-lane traffic flow, and let $x_n(t)$ be the trajectory of the $n^{th}$ vehicle. We wish to model the behavior of $x_n(t)$ via an ODE. There are three classes of models generally considered in the literature:

- Traditional car-following model (CFM):

$$\ddot{x}_n(t+\tau) = \alpha\{\dot{x}_{n+1}(t) - \dot{x}_n(t)\}$$

Here, $\tau$ is the delay time. Idea: try to keep up, plus reaction lag.

- Optimal velocity model (OVM):

$$\ddot{x}_n = \alpha\{V(x_{n+1}-x_n) - \dot{x}_n\}$$

Here, $V(\Delta x)$ is the optimal velocity function.
Idea: optimal velocity set by distance; $\alpha = \frac{1}{\tau}$ = reaction rate.

- Optimal headway model (OHM):

$$\ddot{x}_n = \alpha\{H(\dot{x}_n) - (x_{n+1} - x_n)\}$$

Here,

$$H(V(\Delta x)) = \Delta x \quad , \text{ i.e. } H = V^{-1}$$

is the headway function (optimal).

## Burgers' Equation

Burgers' equation is the simplest model containing nonlinear wave propagation and diffusion:

$$c_t + c c_x = \nu c_{xx}$$

This arises, as we've seen, from

$$\left.\begin{array}{l} \rho_t + j_x = 0 \\ j = J(\rho) - \nu \rho_x \end{array}\right\} \quad \rho_t + \overset{\overset{J'(\rho)}{\shortparallel}}{c(\rho)} \rho_x = \nu \rho_{xx}$$

if $J'''(\rho) = 0$, i.e. $c(\rho) = 2\alpha\rho + \beta$. When $c'' \neq 0$, we have

$$c_t + c c_x = \nu c_{xx} - \nu c''(\rho) \rho_x^2$$

When $c'' \rho_x^2 / c_{xx} \ll 1$, the last term presumably can be neglected.

We've already obtained a wave sol$^{\underline{n}}$, writing $c = c(\xi)$, with $\xi \equiv x - v_s t$, so

$$(c - v_s) c_\xi = \nu c_{\xi\xi}$$

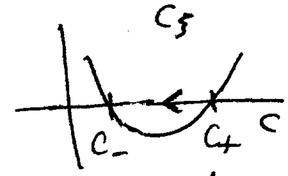
$$\Rightarrow \quad \frac{1}{2} c^2 - v_s c + A = \nu c_\xi$$

Now

$$\frac{1}{2} c^2 - v_s c + A = \frac{1}{2} (c - c_+)(c - c_-)$$

$$c_\pm = v_s \pm \sqrt{v_s^2 - 2A}$$

$$c_\xi = \frac{1}{2\nu}(c-c_+)(c-c_-) :$$




$A < \frac{1}{2} v_s^2$

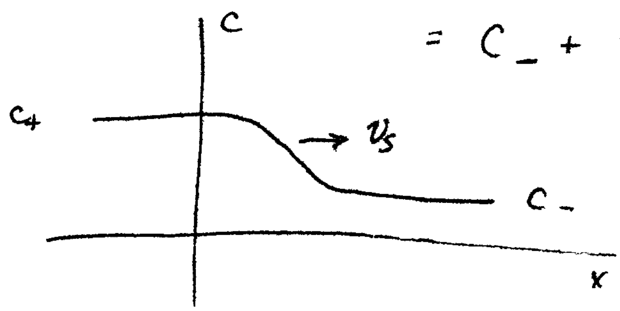
$$d\xi = \frac{2\nu \, dc}{(c-c_+)(c-c_-)}$$

$$= \frac{2\nu}{c_+-c_-}\left(\frac{dc}{c-c_+} - \frac{dc}{c-c_-}\right)$$

$$= \frac{2\nu}{c_+-c_-}\, d\ln\left(\frac{c_+-c}{c-c_-}\right)$$

$$\Rightarrow \quad c(\xi) = \frac{c_- + c_+ \exp\{-(c_+-c_-)\xi/2\nu\}}{1+\exp\{-(c_+-c_-)\xi/2\nu\}} = \begin{cases} c_+ & \xi \to -\infty \\ c_- & \xi \to +\infty \end{cases}$$

$$= c_- + \frac{c_+-c_-}{\exp\left((c_+-c_-)\xi/2\nu\right)+1}$$




$$\left.\begin{aligned} \tfrac{1}{2}c_+^2 - v_s c_+ + A &= 0 \\ \tfrac{1}{2}c_-^2 - v_s c_- + A &= 0 \end{aligned}\right\} \Rightarrow \quad v_s = \frac{1}{2}(c_+ + c_-)$$

<u>Cole-Hopf Transformation</u>

Consider the nonlinear transformation

$$c = -2\nu \frac{\varphi_x}{\varphi} = \frac{\partial}{\partial x}\left(-2\nu \ln\varphi\right)$$

First, let $c = \psi_x$, so

$$\psi_{xt} + \psi_x \psi_{xx} = \nu \psi_{xxx}$$

$$\Rightarrow \quad \psi_t + \frac{1}{2}\psi_x^2 = \nu \psi_{xx}$$

Next, define $\psi \equiv -2\nu \ln \varphi$, so

$$\psi_t = -2\nu \frac{\varphi_t}{\varphi}$$

$$\psi_x = -2\nu \frac{\varphi_x}{\varphi}$$

$$\psi_{xx} = -2\nu \frac{\varphi_{xx}}{\varphi} + 2\nu \frac{\varphi_x^2}{\varphi^2}$$

so that

$$-2\nu \frac{\varphi_t}{\varphi} + 2\nu^2 \frac{\varphi_x^2}{\varphi^2} = -2\nu^2 \frac{\varphi_{xx}}{\varphi} + 2\nu^2 \frac{\varphi_x^2}{\varphi^2}$$

i.e.

$$\varphi_t = \nu \varphi_{xx}$$

which is the heat eqn, a <u>linear</u> PDE!

Suppose

$$\varphi(x, t=0) = \Phi(x)$$

Then for $t > 0$, we have sol$^{\underline{n}}$ by Laplace transform:

$$\varphi_L(x, \beta) = \int_0^\infty dt\; e^{-\beta t} \varphi(x,t)$$

$$\Rightarrow \quad \varphi(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} dx'\; e^{-(x-x')^2/4\nu t}\, \Phi(x')$$

Thus, if $c(x, t=0) = g(x)$, then the sol$^{\underline{n}}$ for subsequent times is

$$c(x,t) = \frac{\frac{1}{t}\int_{-\infty}^{\infty} dx'\, (x-x')\, e^{-H(x')/2\nu}}{\int_{-\infty}^{\infty} dx'\, e^{-H(x')/2\nu}}$$

where

$$H(x'; x, t) = \int_0^{x'} dx''\, g(x'') + \frac{(x-x')^2}{2t}$$

## $\nu \to 0$ behavior

As $\nu \to 0$, the integrals are dominated by saddles.

$$\frac{\partial H}{\partial x'} = g(x') - \frac{x - x'}{t} = 0$$

Let $\xi = \xi(x,t)$ be a sol$^{\underline{n}}$ for $x'$:

$$g(\xi) = \frac{x - \xi}{t}$$

To do the integrals, write $x' = \xi + s$ and expand in $s$:

$$\int_{-\infty}^{\infty} dx' \, F(x') \, e^{-H(x')/2\nu} \approx \int_{-\infty}^{\infty} ds \, F(\xi) \, e^{-H(\xi)/2\nu} \, e^{-H''(\xi)s^2/4\nu} + \dots$$

$$\approx \sqrt{\frac{4\pi\nu}{H''(\xi)}} \, e^{-H(\xi)/2\nu} \cdot F(\xi)$$

Thus,

$$c \sim \frac{x - \xi}{t}$$

i.e.

$$c = g(\xi)$$

$$x = \xi + g(\xi) t$$

which is precisely what we found for $c_t + c c_x = 0$!

What about multivalued-ness? This is obviated by the presence of another saddle point sol$^{\underline{n}}$. I.e. beyond some critical time, we have a discontinuous change of saddles as a f$^{\underline{n}}$ of $x$:

$$g(\xi_\pm) = \frac{x - \xi_\pm}{t} \longrightarrow \xi_\pm(x,t)$$

We then have

$$c \sim \frac{1}{t} \, \frac{(x-\xi_-)\left(H''(\xi_-)\right)^{-1/2} e^{-H(\xi_-)/2\nu} + (x-\xi_+)\left(H''(\xi_+)\right)^{-1/2} e^{-H(\xi_+)/2\nu}}{\left(H''(\xi_-)\right)^{-1/2} e^{-H(\xi_-)/2\nu} + \left(H''(\xi_+)\right)^{-1/2} e^{-H(\xi_+)/2\nu}}$$

Thus,

$$H(\xi_+) > H(\xi_-) \quad \Rightarrow \quad c \sim \frac{x-\xi_-}{t}$$

$$H(\xi_-) > H(\xi_+) \quad \Rightarrow \quad c \sim \frac{x-\xi_+}{t}$$

At the shock, these sol$^{\underline{ns}}$ are degenerate :

$$H(\xi_+) = H(\xi_-)$$

$$\Rightarrow \quad \frac{1}{2}(\xi_+ - \xi_-)\left(g(\xi_+) - g(\xi_-)\right) = \int_{\xi_-}^{\xi_+} d\xi \, g(\xi)$$

again exactly as before.

What does it mean for $\nu$ to be small? We need an appropriate dimensional quantity with dim$^{\underline{s}}$ $L^2/T$ :

$$A = \int_{-\infty}^{\infty} dx \left(g(x) - c_0\right)$$

Now we can define

$$R \equiv \frac{A}{2\nu} \quad ; \quad [R] = 1 \qquad (\text{"Reynolds number"})$$

$R$ measures ratio of nonlinear term $(c - c_0)\, c_x$ to diffusion term $\nu c_{xx}$, in regions where $c$ varies on the scale of a hump width.

Diffusion of an initial step

Take
$$g(x) = \begin{cases} c_2 & x > 0 \\ c_1 & x < 0 \end{cases} \qquad (c_1 > c_2)$$

and set $c(x, t=0) = g(x)$. Then

$$c(x,t) = c_2 + \frac{c_1 - c_2}{1 + h(x,t) \exp\{(c_1-c_2)(x-v_s t)/2\nu\}}$$

with

$$v_s = \tfrac{1}{2}(c_1 + c_2)$$

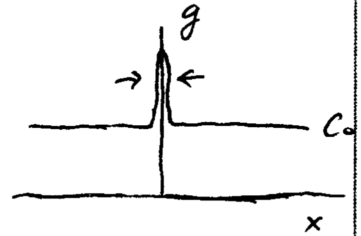
$$h(x,t) = \frac{\operatorname{erfc}\left(-\frac{x - c_2 t}{\sqrt{4\nu t}}\right)}{\operatorname{erfc}\left(\frac{x - c_1 t}{\sqrt{4\nu t}}\right)}$$

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty ds\, e^{-s^2} = \begin{cases} 0 & z = +\infty \\ 1 & z = -\infty \end{cases}$$

Hump

Let

$$g(x) = c_0 + A\,\delta(x)$$




and define

$$c \equiv c_0 + \tilde{c} \qquad\qquad \frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - c_0 \frac{\partial}{\partial \tilde{x}}$$
$$x \equiv c_0 t + \tilde{x}$$

Then

$$\tilde{c}_t + (c_0 + \tilde{c})\tilde{c}_x = \tilde{c}_t + \tilde{c}\,\tilde{c}_{\tilde{x}} = 0$$

Then we have $\tilde{g}(x) = A\,\delta(\tilde{x})$, and

$$H(\zeta) = \int_0^{\zeta} d\zeta'\, g(\zeta') + \frac{(\tilde{x}-\zeta)^2}{2t} = \begin{cases} \frac{1}{2}A + \frac{(\tilde{x}-\zeta)^2}{2t}, & \zeta > 0 \\ -\frac{1}{2}A + \frac{(\tilde{x}-\zeta)^2}{2t}, & \zeta < 0 \end{cases}$$

Then

$$\tilde{c}(\tilde{x},t) = \sqrt{\frac{\nu}{\pi t}} \cdot \frac{(e^R-1)\, e^{-\tilde{x}^2/4\nu t}}{1 + \frac{1}{2}(e^R-1)\,\mathrm{erfc}(\tilde{x}/\sqrt{4\nu t})}$$

$$= \sqrt{\frac{\nu}{t}}\, f\Big(\underbrace{\frac{\tilde{x}}{\sqrt{\nu t}}\,;\,\frac{A}{\nu}}_{\text{dimensionless}}\Big) \qquad R \equiv \frac{A}{2\nu}$$

As $R \to 0$, diffusion wins out over nonlinearity, and indeed

$$\tilde{c}(\tilde{x},t) \sim \frac{A}{\sqrt{4\pi\nu t}}\, e^{-\tilde{x}^2/4\nu t}$$

which solves $\tilde{c}_t = \nu\, \tilde{c}_{\tilde{x}\tilde{x}}$.

For large $R$, write

$$\frac{\tilde{x}}{\sqrt{4\nu t}} = \sqrt{\frac{A}{2\nu}} \cdot \frac{\tilde{x}}{\sqrt{2At}} \equiv \sqrt{R}\cdot z \quad ; \quad z = \frac{\tilde{x}}{\sqrt{2At}}$$

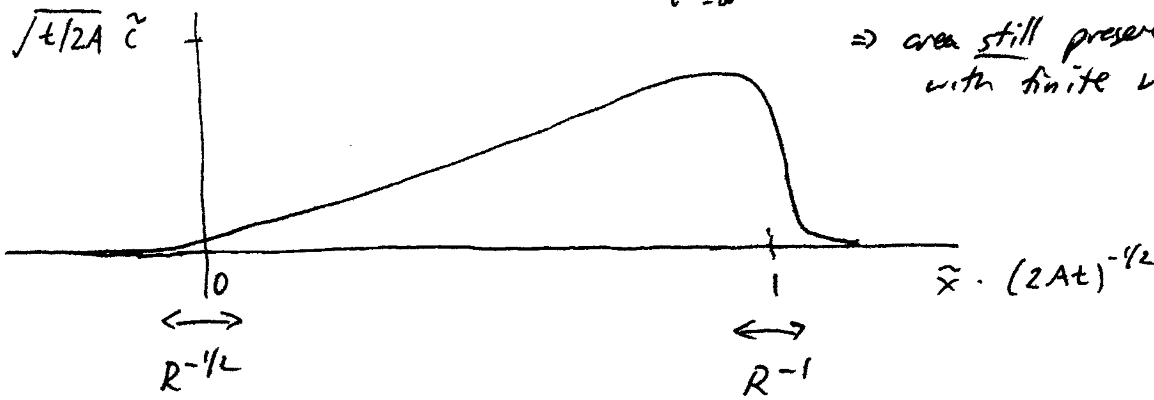
so that

$$\tilde{c}(\tilde{x},t) = \sqrt{\frac{A}{2\pi R t}}\, \frac{(e^R-1)\, e^{-Rz^2}}{1 + \frac{1}{2}(e^R-1)\,\mathrm{erfc}(\sqrt{R}\, z)}$$

$$\simeq \begin{cases} \frac{x}{t}, & 0 < x < \sqrt{2At} \quad (0 < z < 1) \\ 0, & \text{otherwise} \end{cases}$$

We therefore have
$$\frac{d}{dt}\int_{-\infty}^{\infty} dx\, c = \Big[\nu c_x - \tfrac{1}{2}c^2\Big]_{-\infty}^{\infty} = 0$$

$\Rightarrow$ area still preserved with finite $\nu$

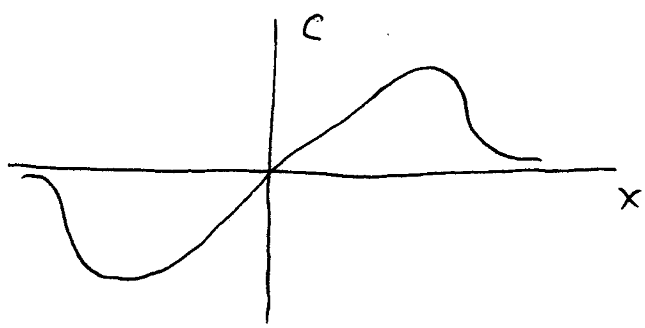



Note there are two transition regions. Survival of front is exceptional.

N - Wave

$$\varphi = 1 + \sqrt{\frac{a}{t}}\; e^{-x^2/4\nu t}$$

$$c = -\frac{2\nu\varphi_x}{\varphi} = \frac{x}{t}\cdot\frac{\sqrt{a/t}\; e^{-x^2/4\nu t}}{1+\sqrt{a/t}\; e^{-x^2/\nu t}}$$




total area $=0$ preserved
"positive" and "negative" areas not preserved, and shrink to zero as $t\to\infty$

As $t\to\infty$,

$$c \sim \frac{\sqrt{a}\; x}{t^{3/2}}\; e^{-x^2/4\nu t}$$

i.e. diffusion wins. Shocks emerge for fixed $t$ and $\nu\to 0$.

# Confluence of Shocks

We may write for a single shock

$$\varphi = \varphi_1 + \varphi_2$$

$$\varphi_j(x,t) = e^{-c_j x/2\nu} e^{+c_j^2 t/4\nu} e^{-b_j}$$

$$\Rightarrow \quad c = \frac{-2\nu\varphi_x}{\varphi} = \frac{c_1\varphi_1 + c_2\varphi_2}{\varphi_1 + \varphi_2}$$

For $c_2 > c_1$, $\varphi_1$ dominates as $x \to \infty$, so $c(\infty) = c_1$; $\varphi_2$ dominates as $x \to -\infty$, so $c(-\infty) = c_2$. Shock centre is at $\varphi_1 = \varphi_2 \Rightarrow x = \frac{1}{2}(c_1 + c_2)t$.

Suppose now we take

$$\varphi = \varphi_1 + \varphi_2 + \varphi_3$$

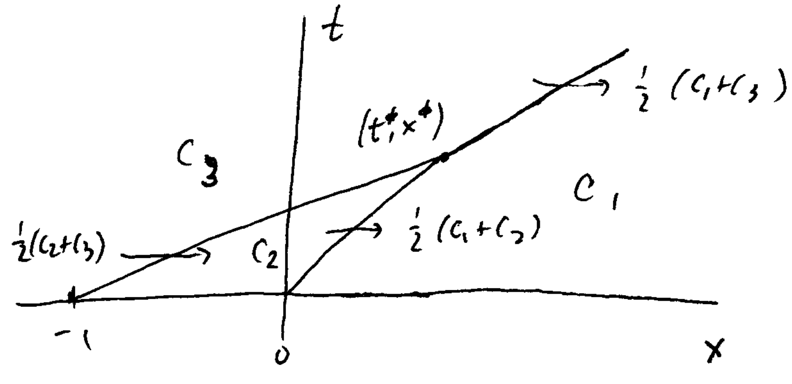
$$\Rightarrow \quad c = \frac{c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3}{\varphi_1 + \varphi_2 + \varphi_3} \quad ; \quad 0 < c_1 < c_2 < c_3$$

We take $b_1 = b_2 = 0$, $b_3 = \frac{c_3 - c_2}{2\nu}$. Which $\varphi_j$ dominates in which region? For $t = 0$, we have

$x < -1$ : $\varphi_3$ dominates

$-1 < x < 0$ : $\varphi_2$ dominates

$x > 0$ : $\varphi_1$ dominates



The trailing shock overtakes the leading shock at time $t^*$:

$$x_{trail} = -1 + \frac{1}{2}(c_2 + c_3)t$$

$$x_{lead} = \frac{1}{2}(c_1 + c_2)t$$

$$\Rightarrow \quad t^* = \frac{2}{c_3 - c_1} \quad , \quad x^* = \frac{c_1 + c_2}{c_3 - c_1}$$

For $t > t^*$, $\varphi_2$ never dominates. The shocks merge.