Dynamical model of traffic congestion and numerical simulation

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We present a dynamical model of traffic congestion based on the equation of motion of each vehicle. In this model, the legal velocity function is introduced, which is a function of the headway of the preceding vehicle. We investigate this model with both analytic and numerical methods. The stability of traffic flow is analyzed, and the evolution of traffic congestion is observed with the development of time.

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I. INTRODUCTION

Traffic flow problems have been given much attention with considerable interest for decades. Many investigations have been done with different points of view to consider the various aspects of traffic phenomena. There are several ways of studying the traffic problem [1–3]. In the viewpoint of traffic dynamics, each vehicle obeys the common equation of motion. This equation is determined by relation to other vehicles moving in traffic flow [1,4–8].

One of the most impressive problems of traffic dynamics is traffic congestion (traffic jam). Many works have been concerned with the static relations between variables and densities of vehicles in the traffic flow with congestion. Our interest is in the dynamical evolution of congestion. We think that congestion exists which is induced by a small perturbation without any specific origin such as a traffic accident or a traffic signal. This kind of congestion is often observed in a highway or freeway. We can regard this congestion phenomena as the instability and the phase transition of a dynamical system. We shall concentrate our attention here on such congestion. For the studies of this problem, we introduce the dynamical model of traffic flow that induces traffic congestion.

Many authors have adopted the following strategy to build a dynamical model. The equation of motion of each vehicle is based on the assumption that each driver of a vehicle responds to a stimulus from other vehicles in some specific fashion. The response is expressed in terms of acceleration, which is the only direct controllable quantity for a driver. Generally, the stimulus and the sensitivity may be a function of the positions of vehicles, their time derivatives, and so on. This function is decided by supposing that the drivers of vehicles obey postulated traffic regulations at all times in order to avoid traffic accidents.

There are two major types of theories for regulations. The first type is based on the idea that each vehicle must maintain the legal safe distance of the preceding vehicle, which depends on the relative velocity of these two successive vehicles [4]. These theories are called the follow-the-leader theories. The other idea for regulation is that each vehicle has the legal velocity, which depends on the following distance of the preceding vehicle.

Most of the earlier works of traffic dynamics have been done along the former direction. This approach has to take into account the time lag of the driver's response to become a realistic theory [1,4,5].

In this paper we investigate the equation of traffic dynamics based on the latter assumption and find a realistic model of traffic flow. In our model the stimulus is a function of a following distance and the sensitivity is a constant. Moreover we do not introduce the time lag of response.

In the following sections, we introduce our dynamical equations for traffic flow. The instability of the steady state flow of traffic is investigated in an analytic way (Sec. II). Next, we propose two models, a simple model (Sec. II A) and a realistic model (Sec. II B). We shall see that the latter model works very well for our purpose. The numerical simulations of this model are performed. Section IV is devoted to a summary and further discussion.

II. DYNAMICAL MODEL OF TRAFFIC

A. Dynamical equation of traffic

Now, a dynamical model of traffic is presented by the idea mentioned in the Introduction. For simplicity, we ig-

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nore the length of vehicle and consider the case in which all the drivers have common sensitivities. We assume that each vehicle has the legal velocity V and that each driver of a vehicle responds to a stimulus from the vehicle ahead of him. He must control the acceleration by putting on or getting off the accelerator and the brakes in such a way that he can maintain the legal safe velocity according to the motion of the preceding vehicle.

Then the dynamical equation of the system is obtained as,

$$\ddot{x}_n = a\{V(\Delta x_n) - \dot{x}_n\}, \tag{1}$$

where

$$\Delta x_n = x_{n+1} - x_n, \tag{2}$$

for each vehicle number n ($n = 1, 2, \ldots, N$). N is the total number of vehicles, a is a constant representing the driver's sensitivity (which has been assumed to be independent of n), and x_n is the coordinate of the nth vehicle. The dots denote differentiation with respect to time t. We assume here that the legal velocity $V(\Delta x)$ of vehicle number n depends on the following distance of the preceding vehicle number $n + 1$. When the headway becomes short the velocity must be reduced and become small enough to prevent crashing into the preceding vehicle. On the other hand, when the headway becomes longer the vehicle can move with higher velocity, although it does not exceed the maximum velocity. Thus, V is a function having the following properties: (i) a monotonically increasing function, and (ii) $|V(\Delta x)|$ has an upper bound. $V^{max} \equiv V(\Delta x_n \to \infty)$. Further, we set the periodic boundary condition: vehicles move on a circuit with length L and the $(N + 1)$th vehicle is identical to the first vehicle. So far as we consider the case of a large enough number of vehicles N with enough circuit length L, this boundary condition is not essential for both analytical and numerical investigations. The detailed analysis of this effect is given in a subsequent paper.

B. Stability in linearized theory

It is apparent that the following solution of steady state flow satisfies the above dynamical equations of traffic,

$$x_n^{(0)} = b\,n + c\,t \tag{3}$$

with

$$b = L/N\,,\; c = V(b), \tag{4}$$

where b is the constant spacing of two successive vehicles and c is the constant velocity of the steady state of traffic flow. Hereafter we call this solution the "steady state flow" without congestion, in which vehicles are uniformly distributed with identical car spacing and move with the same constant velocity.

Let us investigate the stability of the state of this solution by linearizing the original nonlinear system. Let y_n be a small deviation from the steady state flow of identical car spacing, $x_n^{(0)}$,

$$x_n = x_n^{(0)} + y_n\,, \quad |y_n| \ll 1. \tag{5}$$

After neglecting higher terms of y_n, the linearized equation is obtained as

$$\ddot{y}_n = a\{f\Delta y_n - \dot{y}_n\}, \tag{6}$$

where f is the derivative of V at b,

$$f = V'(b). \tag{7}$$

The solution of this equation (6) is obtained by expanding the Fourier series with $e^{i\alpha_k n}$ as an orthonormal set,

$$y_k(n,t) = \exp\{i\alpha_k n + zt\}, \tag{8}$$

$$\alpha_k = \frac{2\pi}{N}k \;\; (k = 0, 1, 2, \ldots, N-1), \tag{9}$$

where $z = u + iv$ (u and v are real) satisfies

$$z^2 + az - af(e^{i\alpha_k} - 1) = 0\,. \tag{10}$$

The discreteness of α_k comes from the periodicity of the indexes of vehicles, $y(n,t) = y(n + N, t)$. Each α_k corresponds to the eigenoscillation mode $y_k(n,t)$, which describes the "density wave" of car distribution. The u and v are determined as the solutions of Eq. (10) for each α_k. Thus, for a given f of Eq. (7) we have a complete set of the eigenmodes, $\{\alpha_k\}$. This set of solutions can be presented in the (f, α) polar coordinate plane. Each solution for α_k ($k = 0, 1, 2, \ldots, N-1$) is on the circle of radius f in this plane, which is shown in Fig. 1.

Let us discuss the stability of the steady state flow $x_n^{(0)}$. Since $y_k(n,t)$ is the deviation from the steady state flow,

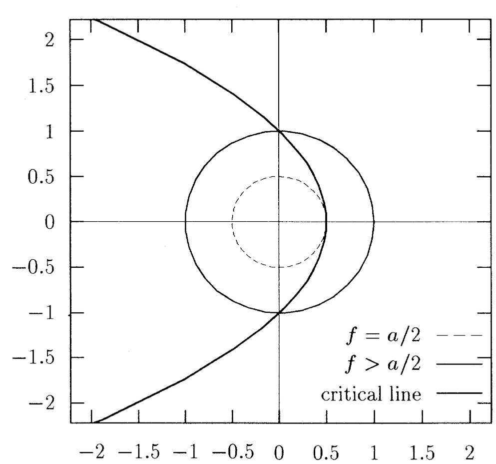

FIG. 1. The region of stability criteria in the (f, α) polar coordinate plane. The thick line is the critical curve. Two examples of the sets of solutions for α_k; $f = a/2$ (the circle is attached on the critical curve: the dashed line) and $f > a/2$ (the thin line).

the state is unstable if the amplitude of $y_k(n,t)$ blows up with time evolution. So, the criterion is the signature of u in $y_k(n,t)$. If $u > 0$, the oscillation of this mode is enhanced with time evolution and the state becomes unstable. On the other hand, if $u < 0$, the oscillation of this mode shrinks and the state tends to initial steady flow. The (f,α) plane is separated into stable $(u < 0)$ and unstable $(u > 0)$ regions by the critical curve $u(\alpha, f) = 0$,

$$f = \frac{a}{2\cos^2\frac{\alpha}{2}}, \tag{11}$$

which is also shown by a thick line in Fig. 1. In order for the state to be stable, every u corresponding to the complete set of α_k $(k = 0, 1, 2, \ldots, N-1)$ must be negative. Even one positive mode $(u > 0)$ is enough to make the steady state flow solution $x_n^{(0)}$ unstable. As we can easily see, if the circle (on which the set of $\{\alpha_k\}$ for a given f is located) intersects with the critical curve, then there exists at least one positive $u(\alpha_k)$ solution and so the state becomes unstable.

Thus, the stability criteria for this steady state are summarized as follows. (a) $f < \frac{a}{2}$; the state is stable, because $u < 0$ for all modes α_k; (b) $f = \frac{a}{2}$; the state is marginal; and (c) $f > \frac{a}{2}$; the state is unstable, because at least one $u > 0$ mode solution exists.

Note that the above criteria is valid in models with any legal velocity function V. The stability is determined by the values of the sensitivity constant a of a driver and the derivative of the legal velocity function $V(\Delta x)$ at constant car spacing b of steady state flow $x_n^{(0)}$:

$$f = V'(b) < \frac{a}{2}. \tag{12}$$

C. Stability and Fourier analysis

We have discussed the stability of the steady flow state within the framework of the linearized theory. Actually, if one wants to study the system far off the perturbative region, we have to investigate numerically. In such cases, it is convenient to define the amplitude of Fourier mode α,

$$C_{\alpha_k} + iS_{\alpha_k} = \sum_{n=1}^{N} y_k(n,t)\exp(-i\alpha_k n), \tag{13}$$

$$\mathcal{A}_{\alpha_k} = \sqrt{C_{\alpha_k}^2 + S_{\alpha_k}^2}, \tag{14}$$

$$\alpha_k = \frac{2\pi}{N}k, \quad (k = 0, 1, 2, \ldots, N-1), \tag{15}$$

where C_{α_k} and S_{α_k} are, of course, time dependent functions. Note that those Fourier modes are introduced just in a parallel way to Sec. II B. If the corresponding solutions u are all negative, then all their amplitudes shrink and so the system is stable. On the other hand, if the modes exist in which the corresponding u is positive, then in the early stage the relevant amplitudes are enhanced whereas the other amplitudes (with negative u) shrink. As the time evolves, however, the nonlinear effect due to those enhanced amplitudes necessarily becomes appreciable. The evolution of each amplitude is then governed by nonlinear equations which we cannot study analytically and we can study only by numerical simulation.

III. MODELS AND NUMERICAL SIMULATION

A. Simple model

First, we simply take the function V as

$$V(\Delta x) = \tanh(\Delta x). \tag{16}$$

This legal velocity function has the properties given in Sec. II A. Without any loss of generality, we can set $a = 1$. For the technical reason of numerical simulation, we set the car number $N = 100$. The average density $b^{-1} = N/L$ is then dependent on L. In the following we analyze the stability by setting the initial small disturbance in such a way that vehicles move according to solution (3) except one vehicle which shifts with 0.1 unit length ahead from this solution:

$$x_1(0) = x_1^{(0)} + 0.1, \tag{17}$$

$$x_n(0) = x_n^{(0)} \text{ for } n \neq 1, \tag{18}$$

$$\dot{x}_n(0) = 0. \tag{19}$$

The numerical studies are made by taking the typical stable and unstable cases.

(i) Stable case: $L = 200, N = 100$,

$$b = \frac{L}{N} = 2\ , \tag{20}$$

$$f = V'(b) = 1 - \tanh^2(b)\ ,$$

$$= 0.077 < \frac{a}{2} = \frac{1}{2}\ . \tag{21}$$

(ii)Unstable case: $L = 50, N = 100$,

$$b = \frac{L}{N} = 0.5\ , \tag{22}$$

$$f = 0.786 > \frac{a}{2} = \frac{1}{2}\ . \tag{23}$$

Let us check the behavior of the Fourier modes defined in Eqs. (13)–(15). Figures 2(a) and 2(b) show the time evolutions of several typical modes for the above stable and unstable cases. In Fig. 2(a) (stable case), we can see that all the amplitudes monotonically shrink with t. On the other hand, in the unstable case [Fig. 2(b)] the amplitude of the positive u mode increases, while the others decrease.

Figure 3 shows the time dependence of the traffic distance of the 50th vehicle for both cases (stable and unstable) and examples of typical behavior of vehicle movement. The curves stand for the traveling distance of the vehicle with time development.

In the stable case (dotted line), the vehicle moves with constant velocity, i.e., the distance increases linearly. On the other hand, in the unstable case (solid line) we observe a vehicle moving backward. This always happens whenever the solution of this model is in the unstable region [see Eq. (12)]. As long as we take the models of a single lane, this means a crush of two successive vehicles. The above behavior indicates that, instead of congestion, such traffic accidents occur everywhere. Then, by choosing an appropriate legal velocity function, we modify the model so that a vehicle never moves backward (see Fig. 4).

B. Realistic model

Now let us choose the following function for the legal velocity as a more realistic model:

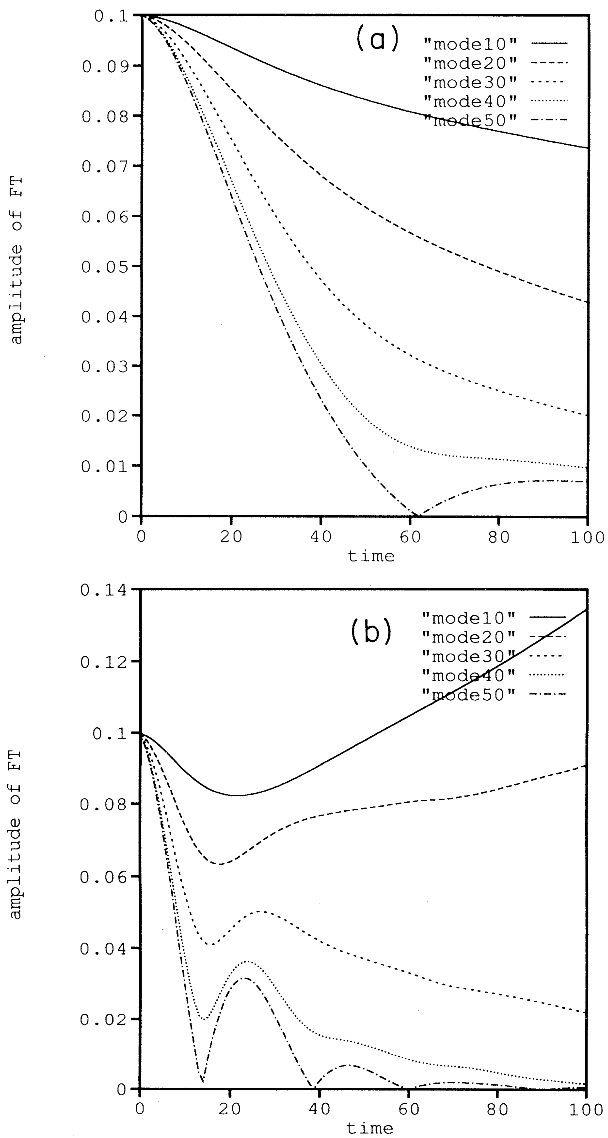

FIG. 2. (a) Fourier mode evolution of the simple model in the stable case defined in Eqs. (20)–(21). The amplitudes of five typical modes for α_k ; $k = 10, 20, 30, 40$, and 50. (b) Fourier mode evolution of the simple model in the unstable case defined in Eqs. (22)–(23). The amplitudes of five typical modes for α_k; $k = 10, 20, 30, 40$, and 50.

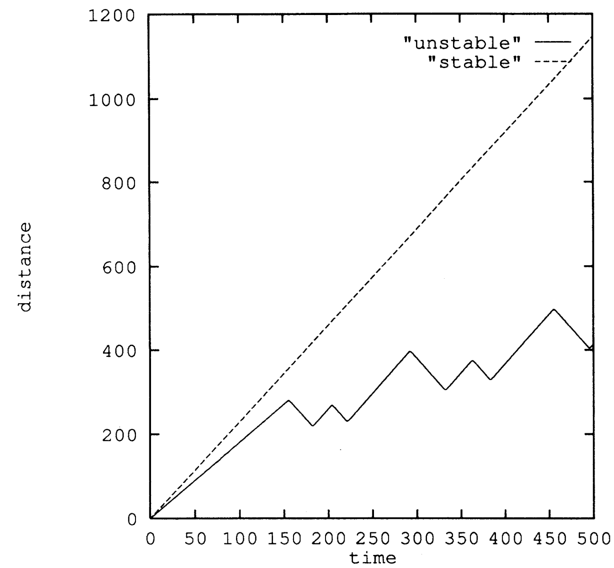

FIG. 3. Trajectories of a vehicle (the 50th vehicle) in two typical cases. The stable case defined in Eqs. (20)–(21) (the dotted line) and the unstable case defined in Eqs. (22)–(23) (the solid line).

$$V(\Delta x) = \tanh(\Delta x - 2) + \tanh 2\ . \tag{24}$$

In this case, a driver controls the vehicle by gradually accelerating or braking in such a way that it never passes the preceding vehicle. We shall see that this model gives the expected behavior of traffic flow, and the congestion phenomena appears instead of accidents.

We present a typical result of traffic congestion induced from this realistic model. Let us take the parameters, $N = 100$ and $L = 200$. We set the same initial disturbance as the previous model [Eqs. (17)–(19)]. Numerical

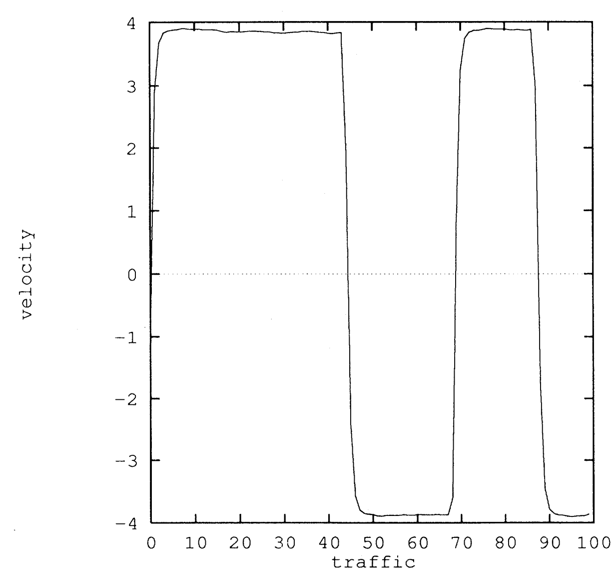

FIG. 4. The snapshot of velocity configuration of all vehicles at $t = 300$ in the unstable case of the simple model.

simulation is carried out for 1000 steps with the development of time.

Figures 5(a)–5(c) present the snapshots of velocity configuration of all vehicles at $t = 100$, 300, and 1000. We clearly see in the figures that negative velocity never appears. We also observe that vehicle velocities tend to locate on the two typical values as time develops; one value is almost zero and the other is nearly the maximum velocity V^{max}.

One may expect that this suggests the generation of congestion. Indeed, we can clearly observe the growth of congestion by plotting the positions of all vehicles on the circuit with time development; (x_n, t) (Fig. 6). From this figure, we find that five clusters of congestion are formed even from a very tiny disturbance existing in the almost uniform vehicle distribution. We can also observe the process of how the congestion structure evolves. Moreover, Fig. 6 implies that all clusters of

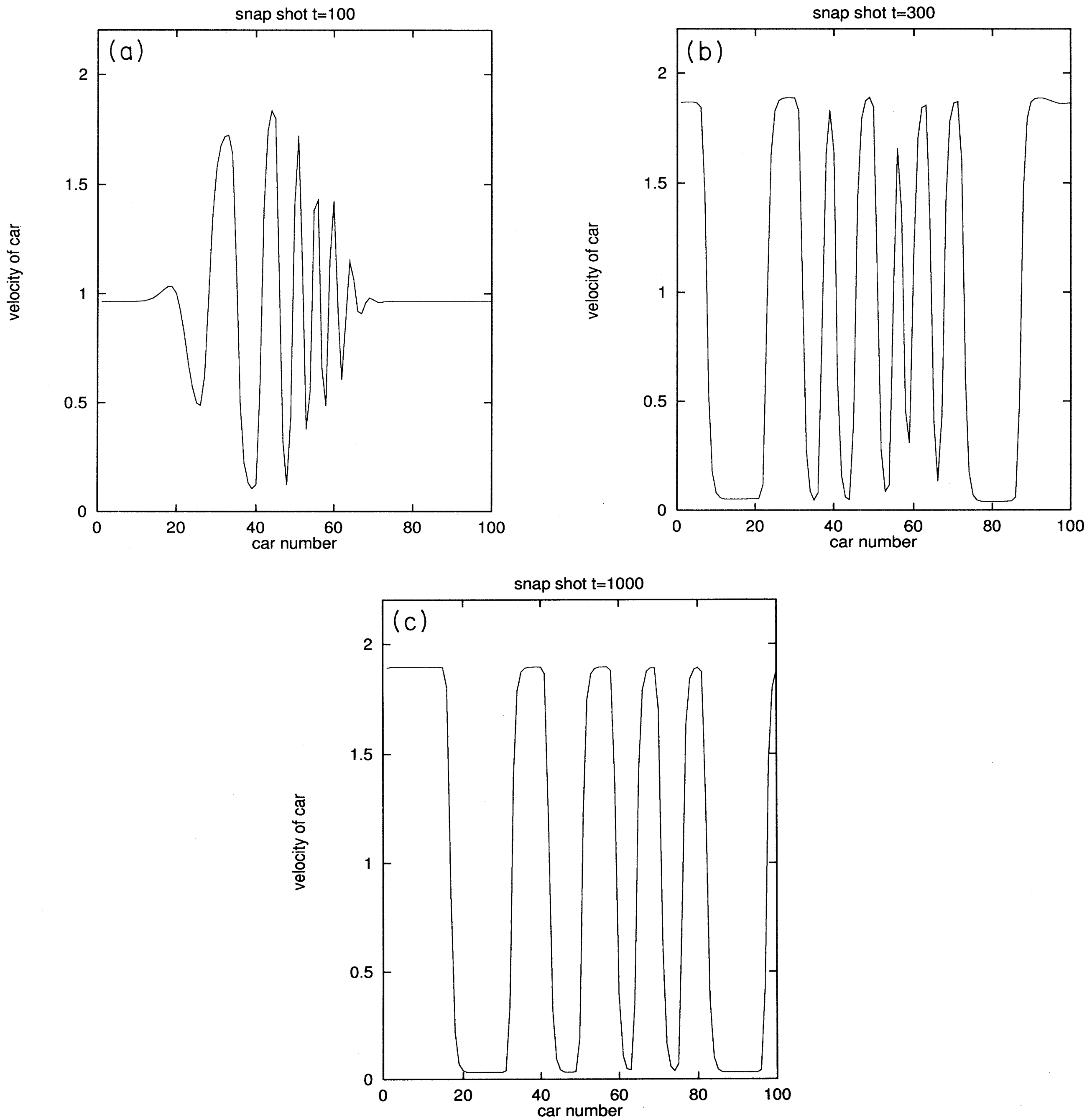

FIG. 5. The snapshots of velocity configuration of all vehicles at $t = 100$ (a), 300 (b), and 1000 (c) of the realistic model in the unstable condition; $N = 100$, $L = 200$.

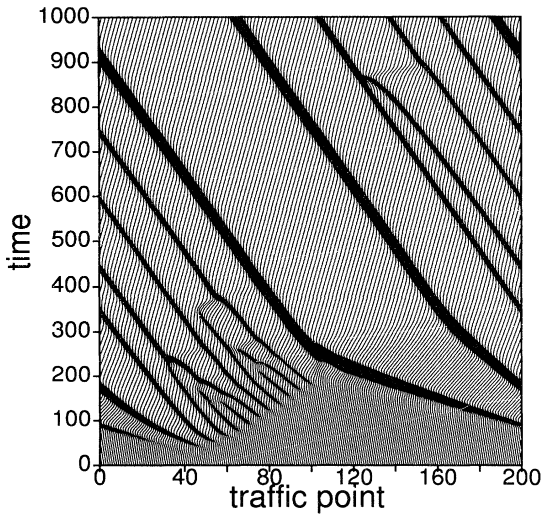

FIG. 6. Plots of the positions of all vehicles on the circuit with time development (x_n, t).

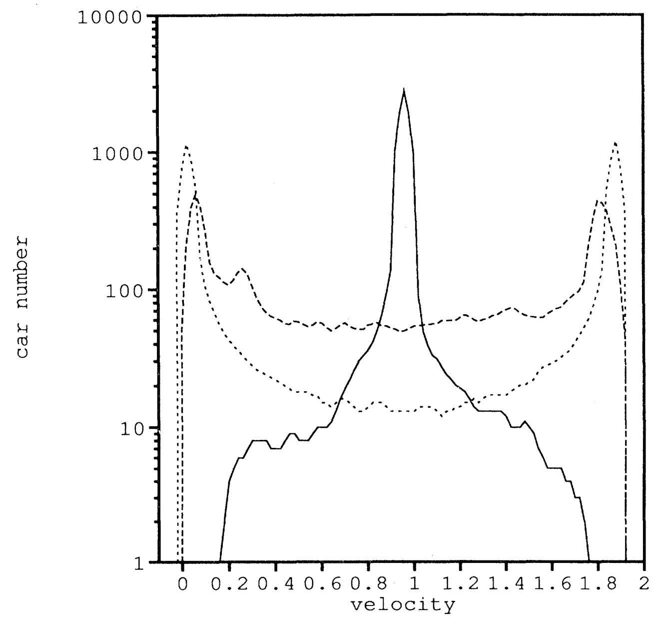

FIG. 8. The velocity distribution accumulated over 100 steps. The distributions of the first 100 steps (the solid line), the third 100 steps (the dashed line), and the final 100 steps (the dotted line).

congestion move backward slowly with the same velocity. The velocity v_c is easily estimated: $v_c[V(\Delta x_{\max})\Delta x_{\min} - V(\Delta x_{\min})\Delta x_{\max}]/(\Delta x_{\max} - \Delta x_{\min})$.

Figures 7 and 8 show the headway and velocity distributions accumulated over 100 steps, respectively. We can read off from the final 100 steps distribution (the dotted line) that the headway in a high concentration area (in a cluster of congestion), Δx_{min}, is about 0.32 and the corresponding velocity $V(\Delta x_{min})$ is about 0.03. Those values are common in all five clusters. On the other hand, the headway Δx_{max} in the low concentration area is about 3.68, in which area vehicles move smoothly with velocity $V(\Delta x_{max})$ about 1.88.

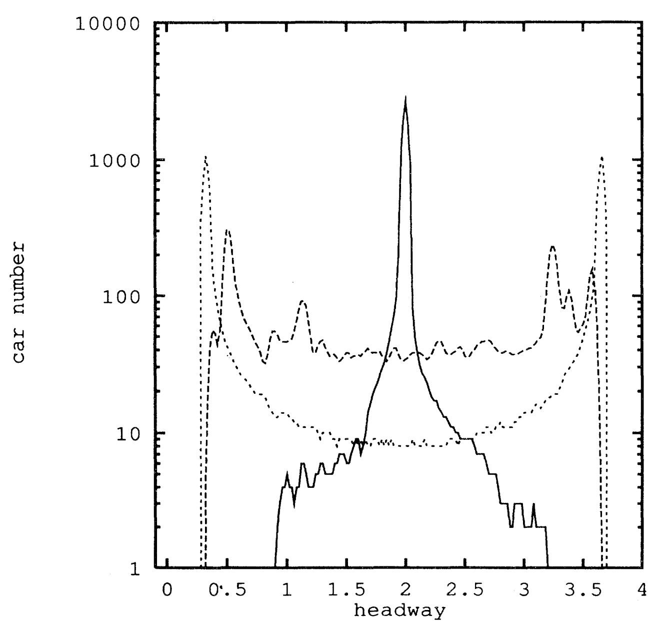

FIG. 7. The headway distribution accumulated over 100 steps within 1000 iteration time steps. The distributions of the first 100 steps (the solid line), the third 100 steps (the dashed line), and the final 100 steps (the dotted line). The vertical axis displays the number of vehicles scaled in logarithm.

If we define the congestion region (the low concentration region) as the one in which the headway of vehicles is shorter (longer) than 2 unit lengths, then the total number of vehicles in the congestion region is 50, half the total vehicles.

Analysis of Fourier mode evolution teaches us the instability of the solution of uniform flow with headway $b = L/N$ and constant velocity $c = V(b)$, which is shown in Fig. 9(a). We further pursue the evolution of this oscillation mode with time iterations of 1000 steps, Fig. 9(b). From this result, we observe an interesting fact that the whole time interval of our simulation can be divided into three stages. As has been already seen in Sec. II B, the perturbative analysis is justified in the first stage (0th to 50th step). In the second stage (50th to 400th step), the amplitude of each mode looks "chaotically" oscillating due to nonlinear effects. In the last stage (after the 400th step), the amplitude of each Fourier mode changes slowly and seems to approach its constant value. Around the 850th step, there is a sudden change of amplitude. This is caused by the combination of two congestions (see Fig. 6) not by the change of the property of congestions, such as Δx_{min} and $V(\Delta x_{min})$. After the congestion is formed completely (900th step), the amplitudes of Fourier modes become stable. In fact, we observe that each amplitude does not change by continuing the simulation for more than 200 steps. This indicates strongly that the congestion maintains its structure from that time forward.

In a subsequent paper, we will develop extensive numerical simulations and investigate the feature of the organization of congestion as well as the structure stability of congestion in more detail [12].

IV. SUMMARY AND DISCUSSIONS

First we want to comment on the transportation of vehicles. We can make the comparison of the transport for the flow of congestion with that for the steady state flow. The capacity of transportation may be defined as the number of vehicles passing through a point in unit time. This value is roughly estimated as N/T, where N is the total number of vehicles on the circuit, and T is the period in which a vehicle starts at a certain point and arrives at the same point after running through the circuit. In the typical case of congestion, $N = 100, L = 200$,

$$N/T \simeq \frac{100}{208} \simeq 0.48 \ . \tag{25}$$

On the other hand, the flow with no congestion denotes a steady flow of uniform distribution of vehicles, moving with the common velocity $V(b)$, where $b = L/N$. The capacity of transportation in this situation is N/T', where $T' = L/V(b)$. In the case of $N = 100, L = 200$,

$$N/T' \simeq \frac{N}{L}\left\{\tanh\left(\frac{L}{N}-2\right)+\tanh 2\right\} \simeq 0.48 \ . \tag{26}$$

It seems strange that the existence of congestion does not affect the capacity of transportation. A further discussion about this problem is given in a subsequent paper [12].

Next, it would be instructive to answer the question of why the simplest model does not generate congestion. In the solution of the steady state flow of uniform car distribution, once we fix the headway b $(= L/N)$, the corresponding vehicle velocity $V(b)$ is uniquely determined [Eq. (3)]. And the slope of this function $f = V'(b)$ determines the stability of the steady state flow. It must be remarked that the initial steady state flow is stable or unstable depending on in which region this b exists. If b is on the stable region, the state is not affected by a small perturbation, and the structure of the traffic flow remains unchanged. On the other hand, if b is on the unstable region, in which cases we have investigated in Sec. II, each vehicle is driven to shift to either in the left and right sides of stable regions.

For the case of the realistic model (Fig. 10), as in Sec. III B, 50 among 100 vehicles approach $\Delta x_{min} = 0.32$,

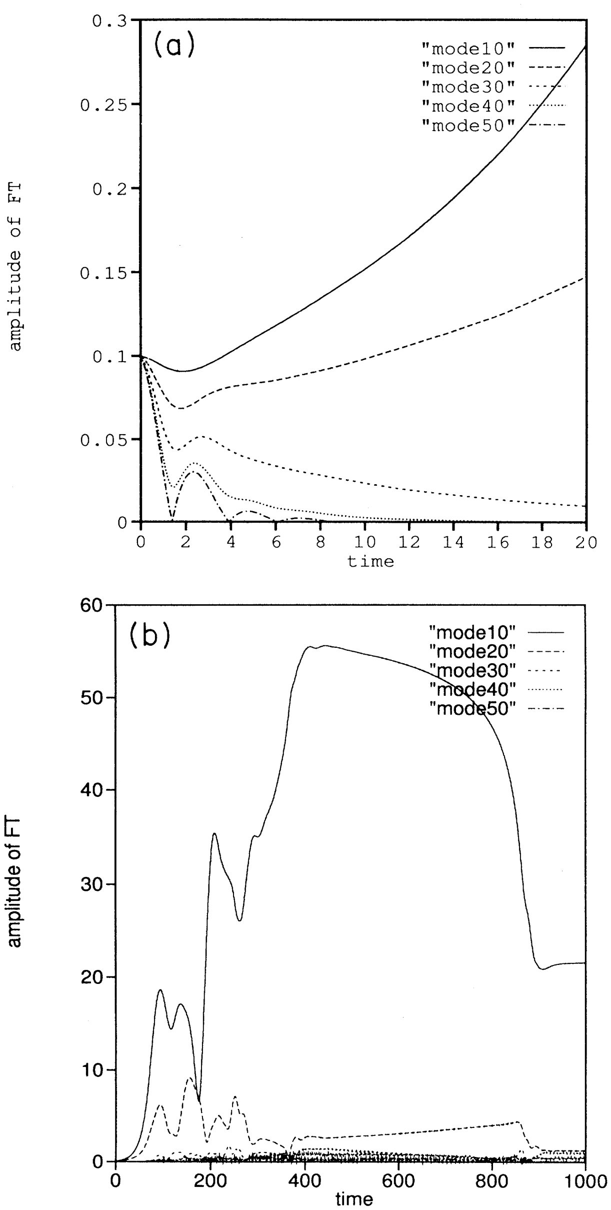

FIG. 9. (a) Fourier mode evolution of the realistic model in the first 20 time steps. The amplitudes of five typical modes for α_k; $k = 10,\ 20,\ 30,\ 40$, and 50. (b) Fourier mode evolution of the realistic model in the whole 1000 steps. The amplitudes of five typical modes for α_k; $k = 10,\ 20,\ 30,\ 40$, and 50.

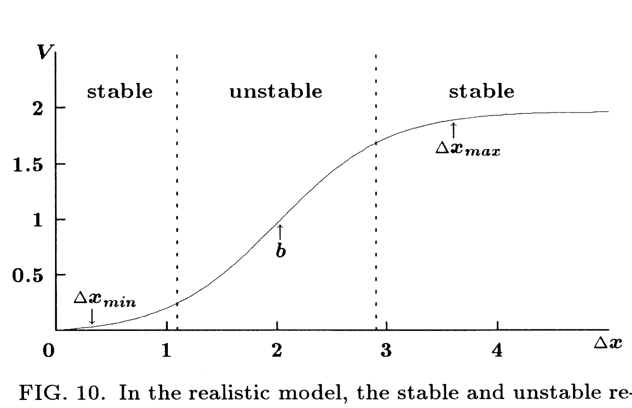

FIG. 10. In the realistic model, the stable and unstable regions are sketched on the figure of the legal velocity function; $V(\Delta x) = \tanh(\Delta x - 2) + \tanh 2$. The initial car spacing b, the headway Δx_{min} of high concentration, and that of low concentration Δx_{max} in the flow with congestion, are indicated by arrows.

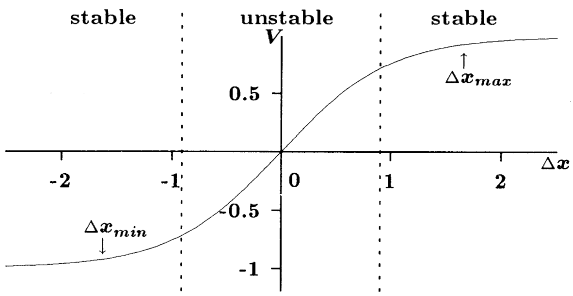

FIG. 11. In the simple model, the stable and unstable regions are sketched on the figure of the legal velocity function; $V(\Delta x) = \tanh(\Delta x)$. In comparison with the realistic model, the corresponding headway Δx_{min} and Δx_{max} are indicated by arrows.

the others $\Delta x_{max} = 3.68$ as indicated by the arrows in Fig. 10. It is just the existence of the stable region around $\Delta x = 0.32$ (≈ 0) that guarantees the congestion formation. On the contrary, the simple model (Fig. 11) does not provide the stable region around $\Delta x \approx 0$. Instead the stable region is around $\Delta x_{min} = -|\Delta x_{max}| < 0$, where the vehicles move with negative velocity. This is why the model necessarily yields traffic accidents. In the simple model, this seems the essential reason why there occurs no "spontaneous" congestion formation.

We have presented a possibly simple but realistic model of traffic flow, which induces the traffic congestion spontaneously. We regard the dynamics of traffic flow as the collective motion problem. The evolution of traffic congestion is an appearance of this substantial property. This phenomena may be regarded as some kind of phase transition induced by the nonlinear effect of dynamical equations of motion. We have seen that our toy model of Sec. III B provides us with an excellent example of a dynamical model of congestion. This model has the desired properties which guarantee that the congestion is generated spontaneously and remains stable.

In earlier works of traffic dynamics, the attention of many investigators has been focused on the time lag of the driver's response to the stimulus from other vehicles. As mentioned in the Introduction, it has long been a common understanding that there is a high correlation between the response of a driver and the relative velocity. The stimulus was, therefore, taken as this relative velocity. The response (acceleration) is controlled by this stimulus (relative velocity). So, most models of traffic flow have been essentially the first order differential equations with respect to time. However, the solution of this differential equation shows far from the realistic behavior of vehicles. People were then led to think that the equation should be changed to the differential-difference equation. In practice, there exists the time lag of the driver's response and so they introduced the time lag in their equations of motion. On the contrary, our model accounts the effect of time lag through the second order differential equations based on the equation of motion in physics. This is the main characteristic feature of our model.

We can consider various ways of modifications of our simple realistic model for further studies. For example, the present model assumes that the sensitivity of drivers are identical and has no dependence of velocity, headway, or the relative velocity of the preceding vehicle. One could adopt the model in which a depends on each driver,

$$\ddot{x}_n = a_n \left\{ V(\Delta x_n) - \dot{x}_n \right\}. \qquad (27)$$

The legal velocity function can also be dependent on drivers, with different maximum speed or the slope of this curve.

Another aspect of further studies of our model are some physical and mathematical characteristics; chaotic structure of nonlinear equations, properties of clusters of congestion, etc.

Note added. After finishing this work, we found several works with respect to the spontaneous formation of congestion from different viewpoints. These studies are based on hydrodynamical or cellular automaton models [9–11].

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