Basics of MHD

II) ▷ Towards MHD

- seek system of *one fluid* equations, describing plasma dynamics at low frequencies, large scales i.e. *configurational* equilibrium/stability, etc.

- *simplest possible* plasma model cf: kinetic Alfven wave derivation for simplifications from gyrokinetic theory (full).

proceed via:
-) display and discussion
-) derivation

MHD Equations: *Eulerian Fluid*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{V}) = 0 \qquad \text{(continuity)}$$

$$\rho \left(\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \underline{\nabla} \underline{V} \right) = -\underline{\nabla} p + \frac{\underline{J} \times \underline{B}}{c} + \underline{f}_{body} \qquad \text{(momentum balance)}$$

$$p/\rho^{\gamma} = \text{const} \qquad \text{(eqn. state)}$$

$$E + \frac{\underline{V} \times \underline{B}}{c} = \underline{J}/\sigma \qquad \text{(Ohms Law)}$$

σ → conductivity

$(1/\sigma) \sim$ resistivity

$(\eta \sim 1/\sigma)$

and, from Maxwell Eqns:

$$\underline{\nabla}\cdot\underline{B} = 0$$

$$\underline{\nabla}\times\underline{B} = \frac{4\pi}{c}\underline{J}$$ (low frequency → neglect displacement current)

$$\underline{\nabla}\times\underline{E} = -\frac{1}{c}\frac{\partial\underline{B}}{\partial t}$$

Now, often convenient to write:

$$\frac{\underline{J}\times\underline{B}}{c} = \frac{c}{4\pi}\frac{(\underline{\nabla}\times\underline{B})}{c}\times\underline{B}$$

$$= -\underline{\nabla}\left(\frac{B^2}{8\pi}\right) + \frac{\underline{B}\cdot\underline{\nabla}\underline{B}}{4\pi}$$

⇒

$$\rho\left(\frac{\partial\underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\underline{V}\right) = -\underline{\nabla}\underbrace{\Big(\underset{\text{fluid pressure}}{p} + \underset{\text{magnetic pressure}}{\frac{B^2}{8\pi}}\Big)}_{\text{total pressure}} + \overset{\text{magnetic tension}}{\frac{\underline{B}\cdot\underline{\nabla}\underline{B}}{4\pi}} + \underline{F}_{bdy}$$

Similarly,

$$\underline{E} + \frac{\underline{V}\times\underline{B}}{c} = \underline{J}/\sigma$$

$$\underline{\nabla}\times\underline{E} + \underline{\nabla}\times\frac{(\underline{V}\times\underline{B})}{c} = \underline{\nabla}\times\frac{c}{\sigma 4\pi}(\underline{\nabla}\times\underline{B})$$

$$\frac{1}{c}\frac{\partial\underline{B}}{\partial t} + \frac{\underline{\nabla}\times(\underline{V}\times\underline{B})}{c} = \frac{c}{4\pi\sigma}\left(\underline{\nabla}(\underline{\nabla}\cdot\underline{B}) - \underline{\nabla}^2\underline{B}\right)$$

$$\Rightarrow \quad \frac{\partial \underline{B}}{\partial t} - \eta \nabla^2 \underline{B} = \underline{\nabla} \times (\underline{V} \times \underline{B}) \qquad \eta = \frac{c^2}{4\pi\sigma}$$

<u>induction equation</u>

⇒ ignoring $\underline{f}_{body}$, have:

$$\left.\begin{aligned} \rho\left(\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V}\right) &= -\underline{\nabla}\left(P_{Tot}\right) + \frac{\underline{B}\cdot\underline{\nabla}\,\underline{B}}{4\pi} \\ \frac{\partial \underline{B}}{\partial t} - \eta \nabla^2 \underline{B} &= \underline{\nabla} \times (\underline{V}\times\underline{B}) \end{aligned}\right] \text{dynamical equations for } \underline{V}, \underline{B} \text{ fluids}$$

+ [continuity, eqn state, Maxwell] → <u>given</u>

if incompressible MHD: $(\underline{\nabla}\cdot\underline{V} = 0)$

$$\rho\left(\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V}\right) = -\underline{\nabla}\, P_{Tot} + \frac{\underline{B}\cdot\underline{\nabla}\,\underline{B}}{4\pi}$$

$$\rho\left(\cancel{\frac{\underline{\nabla}\cdot\partial\underline{V}}{\partial t}} + \underline{\nabla}\cdot(\underline{V}\cdot\underline{\nabla}\,\underline{V})\right) = -\nabla^2 P_{tot} + \nabla\cdot\left(\frac{\underline{B}\cdot\underline{\nabla}\,\underline{B}}{4\pi}\right)$$

determines P_{Tot} ⇒ equation of state gone.

also $$\frac{\partial \rho}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\rho = 0$$

a.i) $$\frac{\partial \underline{B}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{B} - \eta\nabla^2\underline{B} = \underline{B}\cdot\underline{\nabla}\,\underline{V} - \underline{B}\,\cancel{\underline{\nabla}\cdot\underline{V}}$$

Impt : $\eta \to 0$

$$\frac{\partial \underline{B}}{\partial t} + \underline{v} \cdot \nabla \underline{B} = \underline{B} \cdot \underline{\nabla} \underline{V} - \underline{B}\, \underline{\nabla} \cdot \underline{V}$$

$$\underline{\nabla} \cdot \underline{V} = -\frac{1}{\rho} \frac{d\rho}{dt}$$

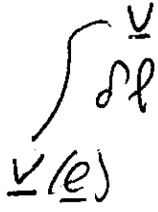
$$\frac{d\underline{B}}{dt} = \underline{B} \cdot \underline{\nabla}\, \underline{V} + \frac{1}{\rho} \frac{d\rho}{dt}$$

$$\boxed{\frac{d}{dt} (\underline{B}/\rho) = \frac{\underline{B}}{\rho} \cdot \underline{\nabla}\, \underline{V}}$$

, understand:

- consider a "fluid line" $\xrightarrow{}$ passive thread – line moving with fluid particles.

$\delta \underline{l}$ ≡ element of length of line



$$\underline{V}(l + \delta \underline{l}) \simeq \underline{V}(l) + \delta \underline{l} \cdot \underline{\nabla}\, \underline{V}$$

so in dt, change in length = $dt(\delta \underline{l} \cdot \nabla)\underline{V}$

$$\Rightarrow \boxed{\frac{d}{dt} \delta \underline{l} = \delta \underline{l} \cdot \nabla \underline{V}}$$ → same evolution as $\underline{B}/\rho$!

Thus: – if two (infinitely close) fluid elements on same line of force at any time, these will <u>always</u> be on same line of force.

and
- $\underline{B}/\rho$ proportional to distance between elements (i.e. length of line).

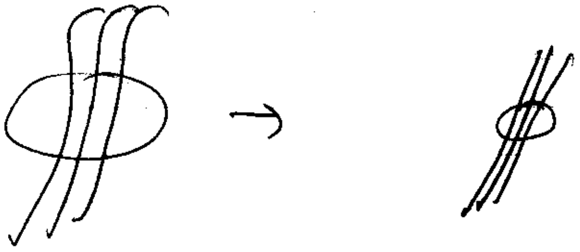
$\Rightarrow$ in ideal MHD ($\eta \to 0$), $\underline{B}/\rho$ frozen onto fluid ("frozen-in law").

For $\underline{\nabla} \cdot \underline{v} = 0$, ρ constant along trajectory $\Rightarrow$ $\underline{B}$ frozen in.

N.B. what does frozen-in law mean?

i.e. consider B-field threading gravitationally collapsing gas

∴ in absence of dissipation/diffusion



B/ρ fixed

⇒ B increases with collapse.

→ other conservation laws in MHD:

- Can immediately explore mass, energy, momentum conservation, i.e.

a) mass

$$\frac{d}{dt}\int_V \rho\, d^3x = \int_V \frac{\partial \rho}{\partial t}\, d^3x = -\int_S \rho \underline{v}\cdot dS$$

→ mass conserved up to inflow/outflow to volume. Mass contained in volume moving with fluid (i.e. Lagrangian) conserved — $V_n = 0$

b.) momentum

total stress tensor

$$\frac{d}{dt}\int_V d^3x\, \rho \underline{v} = -\int d^3x\, \nabla\cdot\left[\rho \underline{v}\,\underline{v} + \left(p + \frac{B^2}{8\pi}\right)\underline{\underline{I}} - \frac{\underline{B}\,\underline{B}}{4\pi}\right]$$

$$= -\int_S \left[\rho \underline{v}\,\underline{v} + \left(p + \frac{B^2}{8\pi}\right)\underline{\underline{I}} - \underline{B}\,\underline{B}\right]\cdot dS$$

So change in momentum of · volume of fluid given by:

$$\frac{d\underline{P}}{dt} = -\int d\underline{s}\cdot \underbrace{\underline{\underline{\tilde{T}}}}_{\text{net stress exerted on surface.}}$$

$$\underline{\underline{\tilde{T}}} = (p + B^2/8\pi)\,\underline{\underline{I}} - \frac{\underline{B}\,\underline{B}}{4\pi}$$

c) Similarly, for energy:

$$\underbrace{\frac{d}{dt}\int\left(\overset{\text{kin}}{\rho\frac{\underline{V}^2}{2}} + \overset{\text{magn.}}{\frac{B^2}{8\pi}} + \overset{\text{th.}}{\frac{p}{\gamma-1}}\right)d^3x}_{\text{rate}}$$

$$= -\int d\underline{s}\cdot\left[\underbrace{\underbrace{\left(\frac{\rho V^2}{2} + \frac{p}{\gamma-1}\right)}_{\text{kin \& Th}}\underline{V}}_{\text{energy flux thru surface}} + \underbrace{p\underline{V}}_{\text{pV work}} + \underbrace{\frac{c}{4\pi}\,\underline{E}\times\underline{B}}_{\text{Poynting flux}}\right]$$

For evolution of energy in co-moving volume:

$$\frac{dE}{dt} = -\int d\underline{s}\cdot\left(\underline{\underline{\tilde{T}}}\cdot\underline{V}\right)$$

where $\underline{\underline{\tilde{T}}} = (p + B^2/8\pi)\,\underline{\underline{I}} - \frac{\underline{B}\,\underline{B}}{4\pi}$

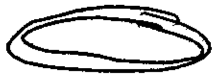
- other conservation laws:

a) helicity $\qquad\qquad \underline{A} \equiv$ vector potential

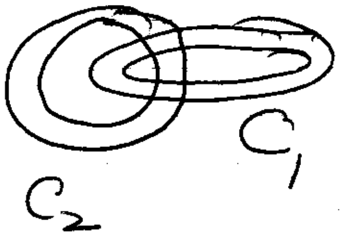
$$H = \int d^3x\ \underline{A}\cdot\underline{B} \qquad \equiv \text{magnetic helicity}$$

H is measure of structural complexity of magnetic field → degree of self-linkage,

i.e. Consider $\qquad\qquad 0 =$



flux tube (untwisted)



linked flux tubes

$$B\, d^3x \equiv \underset{\substack{\downarrow \\ \text{magnetic flux}}}{\phi}\, d\underline{l}$$

$$H = \int d^3x\ \underline{A}\cdot\underline{B} = \phi \int \underline{A}\cdot d\underline{l} = 0$$

with two linked tubes,

$$H = H_1 + H_2$$

For H_1 piece,

$$H_1 = \phi_1 \oint_{C_1} \boxed{\underline{A}\cdot d\underline{l}}$$

but $\oint_{C_1} \underline{A}\cdot d\underline{l} = \phi_2$ (flux thru 2)

$\therefore$ $H_1 = \phi_1\phi_2$

$H_2 = \phi_2\phi_1$

$H = 2\phi_1\phi_2$ → product of linked fluxes.

In general, for N twisted and linked flux tubes:

$$H = \sum_i \underset{\substack{\downarrow \\ \text{\# twists of} \\ \text{tub } i}}{T_i}\, \phi_i^2 + \sum_{i,j} \underset{\substack{\downarrow \\ \text{number of cross-linkages}}}{L_{ij}}\, \phi_i \phi_j$$

(N.B. $H \Leftrightarrow$ mutual inductance)

- Now, to show helicity invariant:
- choose gauge s/t $\phi = 0$, so:

$$H = \int_V d^3x\ \underline{A}\cdot\underline{B}$$

$$\frac{d}{dt}H = \int_V \left(\partial_t\underline{A}\cdot\underline{B} + \underline{A}\cdot\partial_t\underline{B}\right) d^3x$$

$$-\frac{1}{c}\partial_t\underline{A} = \cancel{\nabla\phi} - \frac{\underline{v}}{c}\times\underline{B} \quad ; \quad \partial_t\underline{B} = \nabla\times(\underline{v}\times\underline{B})$$

<u>So</u>

$$\frac{dH}{dt} = \int_V d^3x \quad A \cdot \nabla \times \left(\frac{\underline{V} \times \underline{B}}{c}\right)$$

$$= \int d^3x \quad \underline{A} \cdot \underline{\nabla} \times \left(\frac{\underline{V} \times \underline{B}}{c}\right)$$

$$= \int d\underline{s} \cdot \quad (\underline{A} \cdot \underline{V} \; \underline{B} - \underline{A} \cdot \underline{B} \; \underline{V})$$

$$= 0 \quad \text{if} \quad V_n = B_n = 0$$

$\therefore$ — helicity conserved if no field / flow crosses boundary ($V_n = B_n = 0$, at bndry).

— H not uniquely defined if $B_n \neq 0$.

— for $\underline{\nabla} \cdot \underline{V} = 0$, $\rho = \text{const.}$

$$H_c = \int \underline{V} \cdot \underline{B} \, d^3x \qquad \text{is also invariant}$$

$\downarrow$

<u>cross helicity</u>

can show (HW)

$$\frac{dH_c}{dt} = -\int d\underline{s} \cdot \left(\underline{V} \cdot \underline{B} \, \underline{V} - \frac{V^2 \underline{B}}{8\pi} + P \, \underline{B}\right)$$

$$= 0, \text{ for } V_n = B_n = 0 \quad , \text{ at bndry.}$$

- why is helicity interesting?
- magnetic helicity conserved in ideal MHD
- H is a measure of topological complexity of magnetic field - 'topological invariant'

∴ H is natural constraint on relaxation / minimization of magnetic energy.

i.e. interesting to approach question of magnetic configuration via magnetic energy minimization subject to conserved helicity !?

$B(\underline{x})$ s/t

$$\delta\left[\underbrace{\int d^3x\, B^2}_{\text{energy}} - \overset{\text{Lagrange multiplier}}{\mu}\int d^3x\, \underline{A}\cdot\underline{B}\right] = 0$$

⇒

$$\int d^3x\left[2\underline{B}\cdot\delta\underline{B} - \mu\,\underline{A}\cdot\delta\underline{B} - \mu\,\underline{B}\cdot\delta\underline{A}\right] = 0$$

and, with standard vector manipulations, can re-write as:

$$\int d^3x\Big[2\underline{\nabla}\cdot(\delta A\times\underline{B}) + 2(\underline{\nabla}\times\underline{B})\cdot\delta\underline{A} - 2\mu\,\underline{B}\cdot\delta\underline{A} - \mu\,\nabla\cdot(\delta\underline{A}\times\underline{A})\Big] = 0$$

and can re-group divergence, non divergence terms as:

$$\int d^3x \; \nabla \cdot \left[2(\delta \underline{A} \times \underline{B}) - \mu (\delta \underline{A} \times \underline{A}) \right]$$
$$+ \int d^3x \, 2\delta \underline{A} \cdot \left[\underline{\nabla} \times \underline{B} - \mu \, \underline{B} \right] = 0$$

$\Rightarrow$

$$\underset{\substack{\downarrow \\ \text{integral over bounding surface.}}}{\int d\underline{S}} \cdot \left[2\delta \underline{A} \times \underline{B} - \mu (\delta \underline{A} \times \underline{A}) \right] + 2 \int d^3x \, \delta \underline{A} \cdot \left[\underline{\nabla} \times \underline{B} - \mu \underline{B} \right] = 0$$

Now, for ideal plasma $(\Phi = 0$, with gauge s/t:

$$\frac{1}{c} \frac{\partial A}{\partial t} = \underline{v} \times \underline{B} \quad \rightarrow \quad \text{n.b.: explicit use ideal MHD.}$$

write $\underline{v} = \dfrac{\partial \underline{\xi}}{\partial t}$ → <u>displacement</u>

$$\therefore \quad \frac{\delta \underline{A}}{c} = \underline{\xi} \times \underline{B}$$

Now, if:
- bndry S fixed
- $B_n = 0$ (B contained) (i.e. confined/closed config.)

$\therefore$ $\delta \underline{A}$ parallel to $\hat{n}$
↓
normal to surface.

$\underline{\underline{\text{so}}}$ $\quad \underline{\xi} \times \underline{B} \parallel \hat{n}$

∴ − $\delta\underline{A} \parallel \hat{n}$ (shown)

− $\underline{B} \perp \hat{n}$ (assumption)

⇒ $\delta\underline{A} \times \underline{A} = 0$

$\delta\underline{A} \times \underline{B} \perp \hat{n}$

$$\underbrace{\cancel{\int d\underline{S}\cdot\left[2\delta\underline{A}\times\underline{B}\right]}}_{(\hat{n}\cdot\mu\hat{n})} - \underbrace{\cancel{\int d\underline{S}\cdot\left(\delta\underline{A}\times\underline{A}\right)}}_{\Rightarrow\, 0} = 0$$

∴

$$\delta\left[\int d^3x\, B^2 - \mu\int d^3x\, \underline{A}\cdot\underline{B}\right] = 2\int d^3x\, \delta\underline{A}\left[\nabla\times\underline{B} - \mu\,\underline{B}\right]$$

$$= 0$$

⇒ requires:

$$\boxed{\begin{gathered}\nabla\times\underline{B} = \mu\,\underline{B}\\ \underline{J} = \mu\,\underline{B}\\ (\text{i.e. } \underline{J}\times\underline{B} = 0).\end{gathered}}$$

(linear − μ)

⇒ 'force free' state

i.e current flows along field lines

→ Woltjer's Thm: Configuration which minimizes energy at constant helicity is linear force − free state.

Taylor's Conjecture: For plasma with modest resistivity, helicity is approximate invariant, so minimum energy state also linear, force free.

→ Virial Theorem

Recall Virial Thm ↔ force 'lever arms'.

Now, for body force on plasma/fluid element in MHD

$$\underline{F} = -\underline{\nabla} p + \frac{\underline{J} \times \underline{B}}{c}$$

$$\therefore \quad F = \underset{\substack{\downarrow \\ \text{stress tensor}}}{\frac{\partial}{\partial x_s} T_{rs}}$$

$$T_{rs} = -p\delta_{rs} + \frac{B_r B_s}{4\pi} - \frac{B^2}{8\pi}\delta_{rs}$$

useful to define V (virial)

$$V = \int d^3x \; x_r F_r = \int d^3x \, x_r \frac{\partial}{\partial x_s} T_{rs}$$

↳ moment arm of body force.

integrating by parts ⇒

$$V = -\int d^3x\, \delta_{r,s} T_{r,s} + \int dS\, \hat{n}_r x_r T_{r,s}$$

Now, for finite system, self confined

① $B \sim 1/r^3$ (dipole), or faster

② S finite/bnded

$$\Rightarrow \underset{r^2}{\int dS}\, n_r \underset{r}{x_r} \underset{\substack{\sim \\ \frac{1}{r^6}}}{T_{r,s}} \to 0$$

∴

$$V = -\int d^3x\, T_{r,r}$$

Interesting Application: Can a finite plasma configuration (i.e. plasmoid) be self confined?

i.e. Can

currents produce field s/t confine gas pressure of system !?

For self confinement, must have equilibrium, $\nabla P = \underline{J}\times\underline{B}/c$

or $$\underline{F} = -\underline{\nabla} p + \frac{\underline{J}\times\underline{B}}{c} = 0 \text{, everywhere}$$

$$\Rightarrow \quad V = \int d^3x \; x_r F_r = 0$$

<u>but</u>, have, with similar assumptions, shown

$$V = -\int d^3x \; T_{r,r}$$

$$= -\left[\int d^3x \left[-p\,\delta_{r,s} + \frac{B_rB_s}{4\pi} - \frac{B^2\delta_{r,s}}{8\pi}\right]\delta_{r,s}\right.$$

$$= \int d^3x \left[3p - \frac{B^2}{4\pi} + \frac{3}{8\pi}B^2\right]$$

$$= \int d^3x \left[3p + \frac{B^2}{8\pi}\right] > 0$$

$\Rightarrow \quad V > 0$ (positive definite)

$\therefore$ $\underline{F} \neq 0$, everywhere, so <u>can't</u> have self-confining plasmoid in MHD.

N.B.: What it Means?

- cannot have self-confining MHD system

  i.e. need $V=0$, but have shown $\overline{V} > 0$

- external fields, etc. needed to confine plasma in equilibrium

  i.e.

for toroid, vertical field necessary to balance hoop force.

N.B.: why Care?

- Virial theorem gives simple, 'lumped parameter' description of plasma equilibrium, stability
- useful for simple confinement estimates.

→ More and greater Virial Theorem:

- Can write equation of motion as

$$\frac{\partial}{\partial t}(\rho V_i) = \frac{\partial T_{ij}}{\partial x_j}$$

where $T_{ij} = -\rho V_i V_j - p\delta_{ij} + \frac{1}{4\pi} B_i B_j - \frac{B^2}{8\pi}\delta_{ij}$

Now, can write zero in complicated way as:

$$V_{ij} = -\int d^3x \left\{ x_i \left(\frac{\partial}{\partial t}(\rho V_j) - \frac{\partial}{\partial x_t} T_{jt} \right) + x_j \left(\frac{\partial}{\partial t}(\rho V_i) - \frac{\partial}{\partial x_t} T_{it} \right) \right\}$$

(follows from eqn. of motion).

- can also write moment of inertia (N.B. V_{ij} is integrated lever arm of force density) as:

$$I_{ij} = \int d^3x \, \rho \, x_i x_j,$$

then $\frac{d}{dt} I_{ij} = \int d^3x \, \frac{\partial \rho}{\partial t} x_i x_j$

and, using continuity ⇒

$$\frac{d}{dt} I_{ij} = -\int d^3x \, \frac{\partial}{\partial x_t}(\rho V_t)\, x_i x_j$$

integrating by parts, assuming ρ compact $\Rightarrow$

$$\frac{d}{dt} I_{ij} = \int d^3x \left[\rho x_i V_j + \rho x_j V_i\right]$$

$\Rightarrow$

$$\frac{d^2 I_{ij}}{dt^2} = \int d^3x \left[x_i \frac{\partial}{\partial t}(\rho V_j) + x_j \frac{\partial}{\partial t}(\rho V_i)\right]$$

and, plugging in from eqn. of motion:

$$\frac{d^2 I_{ij}}{dt^2} = \int d^3x \left[x_i \frac{\partial}{\partial x_t} T_{j,t} + x_j \frac{\partial}{\partial x_t} T_{it}\right]$$

finally, integrating by parts 'one more time' gives:

$$\frac{1}{2}\frac{d^2 I_{ij}}{dt^2} = -\int d^3x \, T_{ij} \qquad (T_{ij} \text{ symmetric})$$

Now, contracting the tensor:

$$I \equiv I_{ii} = \int d^3x \, \rho \underline{x}^2$$

one obtains:

$$\frac{1}{2}\frac{d^2I}{dt^2} = +\int d^3x\left[\rho V^2 + 3p - \frac{B^2}{4\pi} + \frac{3B^2}{8\pi}\right]$$

$$= \int d^3x\left[\rho V^2 + 3p + \frac{B^2}{8\pi}\right]$$

∴ have

$$\boxed{\frac{1}{2}\frac{d^2I}{dt^2} = \int d^3x\left[\rho V^2 + 3p + \frac{B^2}{8\pi}\right]}$$

Thus:

$\rightarrow$ RHS $> 0 \Rightarrow \frac{d^2I}{dt^2} > 0$

∴ plasma configuration, in absence of self-gravitation, will necessarily expand.

Now, can simplify Virial Theorem by noting:

$$\Theta = \int d^3x \, \frac{3}{2} NkT = \frac{3}{2}\int d^3x \, p$$

$\downarrow$ total thermal energy (taking plasma as ideal gas)

Similarly,

$K = \int d^3x \, \frac{1}{2}\rho V^2$ → kinetic energy

$M = \int d^3x \, \frac{B^2}{8\pi}$ → magnetic energy

Thus, can write:

$$\boxed{\frac{1}{2}\frac{d^2 I}{dt^2} = 2K + 2\Theta + M}$$

Virial Thm in terms energies

- clearly $\ddot{I} > 0 \Rightarrow$ expansion;

So natural to explore gravitation, i.e.
$\mapsto$ gravitational potential

$$F_{gravity_r} = -\rho \frac{\partial \psi}{\partial x_r}$$

Now, $\nabla^2 \psi = 4\pi G \rho$ Poisson's

$$V_{grv} = -\int d^3x \, x_r \Big[\underbrace{\rho \frac{\partial \psi}{\partial x_r}}_{(F_G)_r}\Big]$$

gravitational contribution to virial expression

As $\psi(x) = -G \int d^3x' \dfrac{\rho(x')}{|\underline{x}-\underline{x}'|}$

$\Rightarrow$

$$V_{grav_r} = -G \int d^3x \, \rho(x) \, x_r \int d^3x' \left\{ \frac{(x_r - x_r') \, \rho(x')}{|\underline{x}-\underline{x}'|^3} \right\}$$

(symmetrizing)

Thus, can write:

$$V_g = -\frac{1}{2} G \int d^3x \int d^3x' \frac{\rho(x)\rho(x')}{|\underline{x}-\underline{x}'|^3} \Big[x_r (x_r - x_r') + x_r' (x_r' - x_r) \Big]$$

write: $$V_g = -\frac{G}{2} \int d^3x \int d^3x' \frac{\rho(x)\rho(x')}{|\underline{x}-\underline{x}'|}$$

but $$E_{grav} = \frac{1}{2} \int d^3x \, \rho(\underline{x}) \, \Psi(\underline{x})$$

grav. potential energy

$$= \frac{1}{2} \int d^3x \int d^3x' \frac{\rho(\underline{x})\rho(\underline{x}')}{|\underline{x}-\underline{x}'|}$$

can see $V_g = -E_{grav} = -\Omega$

∴ now, including gravitational piece:

$$\boxed{\frac{1}{2}\frac{d^2 I}{dt^2} = 2K + 2\Theta + M - \Omega}$$

As $\Omega > 0$, with gravity, it is possible to form static system in equilibrium and stable → i.e. magnetized stars.

→ Virial Theorem ⇔ Stellar Stability

Now, have shown

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2K + 2\Theta + M - \Omega$$

so equilibrium condition is:

$$2K + 2\Theta + M - \Omega = 0$$

Thus, for star supported by thermal and magnetic pressure vs. gravity:

$$2\Theta + M - \Omega = 0$$ (i.e. $V = 0$ at eqbrm.)

For stability, consider uniform expansion perturbation

$$\underline{x} \rightarrow (1+\epsilon)\,\underline{x} \equiv \underline{x}^*$$

So, to conserve mass

$$\rho \rightarrow (1+\epsilon)^{-3}\rho \equiv \rho^*$$

- $$\begin{cases} p \sim \rho^{\gamma} \\ \Theta = \frac{1}{(\gamma-1)}\int d^3x \;\; p \;\; \cong \frac{1}{(\gamma-1)}\int d^3x \; P_0 \rho^{\gamma} \end{cases}$$

(i.e. for ideal gas $\gamma = 5/3$ so $\Theta = \frac{3}{2}\int d^3x \, p$).

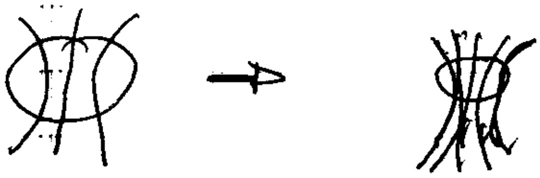
Thus, $\Theta \rightarrow \Theta \, (1+\epsilon)^3 (1+\epsilon)^{-3\gamma}$

$$\rightarrow \Theta \, (1+\epsilon)^{-3(\gamma-1)} \equiv \Theta_*$$

similarly, for magnetic pressure:

$$M = \int d^3x \;\; B^2/8\pi$$

now, in ideal MHD, flux conserved!



i.e. $BA \sim \text{const.}$

As $A \rightarrow A(1+\epsilon)^2 \Rightarrow B \rightarrow B/(1+\epsilon)^2$

$\therefore \; B \rightarrow B\,(1+\epsilon)^{-2} \equiv B^*$

$$\Omega = \frac{1}{2}\int d^3x \int d^3x' \frac{\rho(\underline{x})\rho(\underline{x}')}{|\underline{x}-\underline{x}'|} \rightarrow \frac{(1+\epsilon)^6(1+\epsilon)^{-6}}{(1+\epsilon)}\Omega = \frac{\Omega}{(1+\epsilon)}$$

$\underline{\text{so}}$

$$M = \int d^3x \; B^2/8\pi$$

$$\Rightarrow \frac{(1+\epsilon)^3}{(1+\epsilon)^4} M = (1+\epsilon)^{-1} M \equiv M^*$$

and $\mathbf{I}$, finally:

$$I \rightarrow I^* = (1+\epsilon)^2 I$$

Plugging into

$$\frac{1}{2}\frac{d^2I}{dt^2} = 2\Theta + M - \Omega$$

$$\Rightarrow \quad \frac{1}{2}\frac{d^2}{dt^2}(1+\epsilon)^2 I = 2\Theta(1+\epsilon)^{-3(\gamma-1)} + \frac{M}{(1+\epsilon)} - \frac{\Omega}{(1+\epsilon)}$$

$$\therefore \quad \boxed{\begin{aligned} I\ddot{\epsilon} &= 2\left[1 - 3(\gamma-1)\epsilon\right]\Theta \\ &\quad + (1-\epsilon)M - (1-\epsilon)\Omega \end{aligned}}$$

where retained ϵ^0, ϵ^1, only.

$$I\ddot{\epsilon} = \underbrace{\cancel{(2\Theta + M - \Omega)}}_{\text{from equilibrium}} + \epsilon\left(\Omega - M - 6(\gamma-1)\Theta\right)$$

and noting $\Theta = \frac{(\Omega - M)}{2}$

$$I\ddot{\epsilon} = \epsilon\left(2\Theta - 6(\gamma-1)\Theta\right)$$

$$= \epsilon\left(8\Theta - 6\gamma\Theta\right)$$

$\Rightarrow$

$$\boxed{I\ddot{\epsilon} = 6\epsilon\Theta\left(\frac{4}{3} - \gamma\right)}$$

Thus, $\gamma < 4/3$ → instability

$\gamma > 4/3$ → stability

Point:

→ simple, classical stellar stability is consequence of <u>equation</u> <u>of</u> <u>state</u>

<u>i.e.</u> ideal, degenerate gases different

i.e. ideal gas, $\gamma = 5/3$

$\Rightarrow$ stable star/object, supported against gravity by thermal, magnetic pressure, is possible.

but, if degenerate gas ... $\gamma_{eff} = 4/3$

$\Rightarrow$ Marginal $\Rightarrow$ Chandrasekhar Limit on Mass.

see Ps.5

Physics 218B

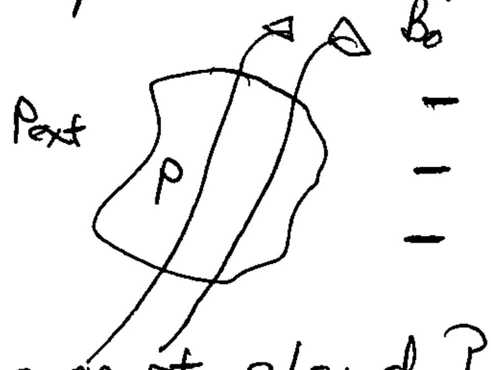
P. Diamond

MHD

→ Equilibrium : Magnetic Virial Theorem

⟷ Star Formation

i.e. Consider plasma cloud :



- mass M, radius R
- threaded by B_0
- pressure P

P_{ext} to support cloud ?, Will cloud collapse to star?

Problem is one of finding conditions for equilibrium

i.e.

$$\boxed{\rho \underbrace{\cancel{\frac{d\mathbf{V}}{dt}}}_{0 \text{ eqlbrm}} = -\rho \underbrace{\nabla \phi}_{\text{gravitational potential}} - \underbrace{\nabla P}_{\text{pressure}} + \underbrace{\frac{\mathbf{J} \times \mathbf{B}}{c}}_{\text{Magnetic force}}}$$

$$\nabla^2 \phi = 4\pi G \rho$$

$$\frac{\mathbf{J} \times \mathbf{B}}{c} = -\nabla\left(\frac{B^2}{8\pi}\right) + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi}$$

$$= \nabla \cdot \underline{\underline{T}} = \nabla_k \left[\frac{B_i B_k}{4\pi} - \frac{B^2}{8\pi} \delta_{ik} \right]$$

↑ Maxwell stress tensor

$$\Rightarrow \boxed{0 = -\rho \frac{\partial \phi}{\partial x_i} - \frac{\partial P}{\partial x_i} + \frac{\partial T_{ik}}{\partial x_k}}$$

→ equilibrium condition

To proceed:

Approach ① → solve equilibrium equations

but info. often incomplete, i.e.

- T; from luminosity
- B_0; from synchrotron emission

etc

→ often unable to resolve cloud, spatially

⇒

Approach ② → Bulk parametrization and Virial theorem $\left\{ \begin{array}{c} M \\ R \\ \rho \\ \underset{\substack{\uparrow \\ \text{flux} \\ \text{threading}}}{\Phi} \end{array} \right\}$

Where: Virial theorem extends moment equation program

i.e.

$$\begin{array}{ccccc} & \text{(velocity moments)} & & \text{(spatial moment)} & \\ f(\underline{x},\underline{v},t) & \xrightarrow{\int d^3\underline{v}\, \underline{v}^n} & \begin{array}{c} n(\underline{x},t) \\ V(\underline{x},t) \end{array} & \xrightarrow{\int d^3\underline{x}\, x^n} & \begin{array}{c} M,\ \Phi,\ \rho \\ \text{etc.} \end{array} \\ \text{phase space} & \rightarrow & \text{configuration space} & \rightarrow & \text{parameter} \end{array}$$

Of special interest is <u>first</u> virial moment

i.e. $\int d^3x \; x_i \times$ (Force Balance)

$\Rightarrow$

$$0=\int_V d^3x\left[-x_i\rho\frac{\partial\phi}{\partial x_i}-x_i\frac{\partial P}{\partial x_i}+x_i\frac{\partial T_{ik}}{\partial x_k}\right]$$

$$=\int d^3x\left[-\frac{\partial}{\partial x_i}(x_iP)+\frac{\partial x_i}{\partial x_i}P\right.$$

$$+\frac{\partial}{\partial x_k}(x_iT_{ik})-\frac{\partial x_i}{\partial x_k}T_{ik}$$

$$\left.-x_i\rho\frac{\partial\phi}{\partial x_i}\right]$$

Now; $\frac{\partial x_i}{\partial x_i}=\delta_{ii}=3$

divergence term → surface

$$\underline{\underline{\text{So}}}\begin{cases}-\int_V d^3x\; x_i\frac{\partial P}{\partial x_i}=2U-\overset{\downarrow}{\int}dA\; P\underline{x}\cdot\vec{n}\\ U=\frac{3}{2}\int d^3x\; P \quad\to\text{thermal energy content in volume of cloud}\end{cases}$$

Similarly, $\frac{\partial x_i}{\partial x_k} = \delta_{ik}$

so

$$x_i \frac{\partial T_{ik}}{\partial x_k} = \frac{\partial}{\partial x_k}(x_i T_{ik}) - \delta_{ik} T_{ik}$$

$$= \underline{\nabla} \cdot (\underline{x} \cdot \underline{\underline{T}}) + \frac{B^2}{8\pi}$$

as $T_{ii} = -B^2/8\pi$.

$$\left\{ \begin{aligned} &\text{i.e. } \delta_{ii} = 3 \\ &\frac{B_i B_i}{4\pi} = B^2/4\pi \end{aligned} \right.$$

$\Rightarrow$

$$\left\{ \begin{aligned} &\int x_i \frac{\partial T_{ik}}{\partial x_k} d^3x = \underset{\substack{\uparrow \\ \text{magnetic energy content of cloud}}}{M} + \int dA\ \underline{x} \cdot \underline{\underline{T}} \cdot \hat{n} \\ &M = \int_V d^3x\ \frac{B^2}{8\pi} \end{aligned} \right.$$

and for $-\int x_i \rho \frac{\partial \phi}{\partial x_i} d^3x$:

$$-\frac{\partial \phi}{\partial x_i} = -G \int_V \frac{\rho(\underline{x}')(x_i - x_i')}{|x-x'|^3} d^3x'$$

$$\overset{\omega}{=} -\int x_i \rho \frac{\partial \phi}{\partial x_i} d^3x = -G \int_V d^3x \int d^3x' \frac{\rho(x)\rho(x')\, x_i (x_i - x_i')}{|x-x'|^3}$$

Symmetrizing:

$$-\int \rho x_i \frac{\partial \phi}{\partial x_i} d^3x = \overset{\substack{\text{gravitational energy content of cloud}\\ \uparrow}}{W} = -\frac{G}{2}\int_V d^3x \int d^3x' \frac{\rho(\underline{x}')\rho(\underline{x})(x_i - x_i')^2}{|\underline{x}-\underline{x}'|^3}$$

$$= -\frac{G}{2}\int d^3x \int d^3x' \frac{\rho(\underline{x}')\rho(\underline{x})}{|\underline{x}-\underline{x}'|}$$

(1/2 for dbl. counting)

So finally

$$0 = 2U - \int dA\, P\underline{x}\cdot\vec{n} + W + M + \int dA\, \underline{x}\cdot\underline{\underline{T}}\cdot\vec{n}$$

$\Rightarrow$

$$\boxed{2U + W + M = \int dA\, P\underline{x}\cdot\vec{n} - \int dA\, \underline{x}\cdot\underline{\underline{T}}\cdot\vec{n}}$$

Further, assume surface pressure const ~P_{ext}, so:

$$\boxed{\underset{\substack{\downarrow\\ \text{thermal}\\ \text{energy}}}{2U} + \underset{\substack{\downarrow\\ \text{grav.}\\ \text{energy}}}{W} + \underset{\substack{\downarrow\\ \text{magn.}\\ \text{energy}}}{M} = \int dA\, \underset{\substack{\downarrow\\ \text{external}\\ \text{pressure}}}{P_{ext}}\, \underline{x}\cdot\vec{n} - \int dA\, \underline{x}\cdot\underline{\underline{T}}\cdot\vec{n}}$$

To evaluate integrals, assume <u>spherical</u> <u>cloud</u> with radius R

$$\int \underline{x}\cdot\hat{n}\, dA = \int \partial_i x_i \, dV = \cancel{3}\, \frac{4\pi R^3}{\cancel{3}}$$

so

$$W \cong -\underset{\substack{\downarrow \\ \text{form factor} \\ \text{(shape)}}}{\alpha} \frac{GM^2}{R}$$

$$M = \int \rho\, dV$$

(total mass)

$$M + \int \underline{x}\cdot \underline{\underline{T}}\cdot\hat{n}\, dA \cong \beta \frac{\Phi^2}{R}$$

$$\Phi \sim \pi R^2 B$$

(magnetic flux)

and

$$U \cong \frac{3}{2} c_s^2 M$$

$$\Rightarrow \quad 4\pi R^3 P_{ext} = \beta\frac{\Phi^2}{R} - \alpha \frac{GM^2}{R} + \frac{3}{2} c_s^2 M$$

$$\boxed{P_{ext} = \frac{1}{4\pi}\left(\beta\frac{\Phi^2}{R^4} - \frac{\alpha G M^2}{R^4} + \frac{3}{2}\frac{c_s^2 M}{R^3}\right)}$$

Scalar virial theorem for cloud

Interpreting the Result:

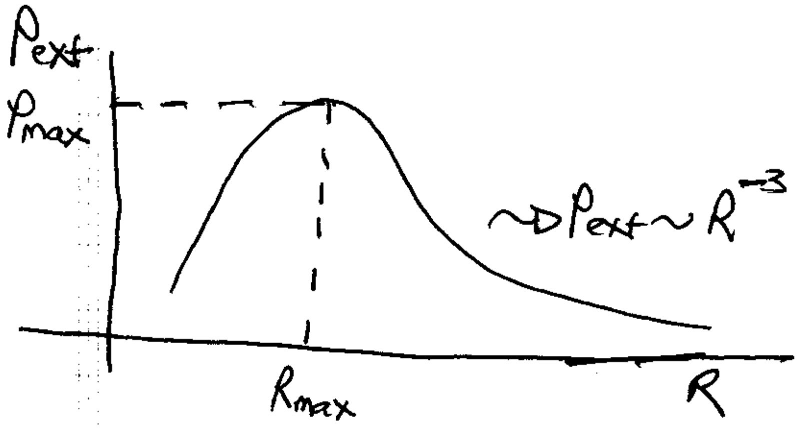
i) if Φ, $G \to 0$

$$P_{ext} = \frac{1}{4\pi} 3 \frac{c_s^2 M}{R^3}$$

$$P_{ext} V_{cloud} = m c_s^2 \frac{M}{m} = N k_b T$$ $\rightarrow$ $P_{ext} = P_{int}$ for confinement

ii) if $\Phi = 0$

$$P_{ext} = \overbrace{-\frac{\alpha G M^2}{4\pi R^4}}^{\text{(self-grav.)}} + \overbrace{\frac{3 c_s^2 M}{4\pi R^3}}^{\text{(thermal)}}$$



- no eqbrm for $P > P_{max}$
- $R < R_{max}$, P_{ext} must decrease to maintain eqbrm.
  $\Rightarrow$ instability to gravitational collapse

$$R_{max} \sim \frac{GM}{c_s^2} \quad \text{if} \quad R_m \sim \frac{G\rho_0 R_m^3}{c_s^2}$$

$$\Rightarrow R_m^2 \sim c_s^2 / G\rho_0 \sim R_{Jeans}^2$$

iii.) $\Phi \neq 0$

Note: Magnetic pressure term scales like self gravity, so;

$$P_{ext} = \frac{1}{4\pi}\left[\frac{1}{R^4}\left(\beta\Phi^2 - \alpha G M^2\right) + \frac{3}{2}\frac{c_s^2 M}{R^3}\right]$$

Net sign ( ) from M^2 vs. $\frac{\beta\Phi^2}{\alpha G} = \underset{\text{(magnetic virial mass)}}{\underset{\downarrow}{M_\Phi^2}}$

$$\boxed{M_\Phi = \sqrt{\frac{\beta}{\alpha}}\,\Phi/G^{1/2}}$$

$$P_{ext} = \frac{1}{4\pi}\left[\frac{\alpha G}{R^4}\left(M_\Phi^2 - M^2\right) + \frac{3}{2}\frac{c_s^2 M}{R^3}\right]$$

$M < M_\Phi$ → magnetically subcritical mass for gravitational collapse

$M > M_\Phi$ → magnetically supercritical mass for gravitational collapse.

i.e. $M < M_\Phi$ → no amount of external compression can induce indefinite contraction if magnetic flux remains frozen in.

$(M_\Phi^2 - M^2 > 0)$

i.e. cloud maintained/supported by magnetic field

$M > M_\Phi$ → sufficient external compression can induce gravitational collapse even if flux frozen in.

$(M^2 - M_\Phi^2 < 0)$

note: NL MHD waves (Alfven waves) can support via kinetic energy contribution

→ Alternative Perspective:

$P = P_0 (\rho/\rho_0)^\gamma$ $\gamma = 4/3$ instead 5/3

- recall White Dwarfs → degenerate Fermi gas (exert P, even for T = 0 due exclusion)

$M > M_{chandra}$ → collapse

$M < M_{chandra}$ → no collapse.

- observe: if flux conserved

$$\Phi \sim BR^2 \quad \underline{\underline{so}} \quad B \sim R^{-2}$$

$$\rho \sim M/R^3 \quad \underline{\underline{so}} \quad \rho \sim R^{-3}$$

$\underline{\underline{so}}$ $B^2 \sim \rho^{4/3}$ i.e. $P_{magn} \sim \rho^{4/3}$

⇝ Magnetic field gas behaves like degenerate Fermi gas so $M_\Phi \sim$ magnetic Chandrasekhar limit.

Aside: Chandrasekhar Limit — Simple Derivation
(c.f.: Shapiro, Teukolsky)

→ suppose: N Fermions in star of radius R

∴ $n_{\text{Fermion}} \sim N/R^3$

∴ Vol./Fermion $\sim 1/n$ (Pauli exclusion)

$$\underset{\substack{\downarrow \\ \text{Fermion Momentum}}}{p} \sim \hbar/\Delta x \sim \hbar n^{1/3}$$ (Heisenberg Uncertainty)

⇒ Fermion energy (per Fermion): $E_F = pc \sim \hbar c \dfrac{N^{1/3}}{R}$ (replaces: i.e. thermal energy)

Gravitational Energy (per Fermion): $E_{Grav} \sim -\dfrac{GM\overset{\text{Baryon mass}}{m_b}}{R}$

$M \sim N \; m_B$

Pressure → electron
Mass → Baryon

$$\therefore \; E = E_F + E_G$$

$$= \frac{\hbar c N^{1/3}}{R} - \frac{G N m_B^2}{R}$$

Note: $E = E_F + E_G$

$$= \frac{\hbar c N^{1/3}}{R} - \frac{G N m_B^2}{R}$$

$E > 0 \Rightarrow$ decrease E, E_F by increasing R.

but as $E_F \downarrow$, electrons non-relativistic,
∴ $E_F \sim 1/R^2 \rightarrow$ eqbrm.

$E < 0 \Rightarrow$ decrease E without bound by decreasing $R \Rightarrow$ collapse

∴ eqbrm: $\hbar c N^{1/3} = G N m_B^2$

$$N_{Max} = \left(\hbar c / G m_B^2\right)^{3/2} \sim 2 \times 10^{57} \quad \text{(proton)}$$

$$\therefore \quad M_{Chandra} = N_{max} m_B \sim 1.5 M_{\odot}$$