→ Deriving MHD

→ MHD is derived from 2-fluid equations

- first discuss 2 fluid derivation from Boltzmann
- then, discuss reduction to one-fluid MHD (i.e. approximations / limitations — especially in Ohm's Law)

→ deriving fluid equations

Have, in general, Boltzmann eqn

$$\underset{①}{\frac{\partial f}{\partial t}} + \underset{②}{\underline{v} \cdot \underline{\nabla} f} + \underset{③}{\frac{q}{m}\left(\underline{E} + \frac{\underline{v} \times \underline{B}}{c}\right) \cdot \underline{\nabla} f} = \underset{④}{C(f)}$$

collision operator

and can assign time scale

① ↔ ω → frequency

② ↔ $V_{Th} / L_{\parallel}$

↳ relevant parallel scale

③ $\frac{q}{m}\frac{E}{\Delta V}$ $\Delta V \sim V_{Th} \rightarrow$ non-resonant

$\Delta V \sim \Delta V_{Tr} \rightarrow$ resonant

NL scattering rate (→ small, usually)

④ ν_{eff} – collision frequency.

For "fluid description", need:

→ $\nu_{eff} > V_{Th}/L_{\parallel}$

i.e. short mean free path limit

or

→ $\omega > V_{Th}/L_{\parallel}$ → a/a' gyrokinetic KSAW, where $\nu \rightarrow 0$

i.e. "fluid" ↔ blob / fluid element of particles.

→ what holds blob together? (i.e. prevents dispersal?)

⇒ collisions (i.e. particles collide and scatter prior dispersal)

or ⇒ vibrations in wave.

here, focus on short mean-free path ordering.

For $C(f) \gg \partial f/\partial t$, $\underline{v}\cdot\underline{\nabla} f$, etc.

l.o. $C(f) = 0$

$$\Rightarrow f = f_{Maxwellian}$$

i.e. – collisions drive distribution function to local Maxwellian on time scale short compared all else
– n.b. Maxwellian can be shifted, and have gradients.

1st order:

$$\frac{\partial f^{(0)}}{\partial t} + \underline{v}\cdot\nabla f^{(0)} + \frac{q}{m}\left(\underline{E} + \frac{\underline{v}}{c}\times\underline{B}\right)\cdot\underline{\nabla} f^{(0)} = C(f^{(1)})$$

then integrating:

IBP

0

$$\int d^3v \left[\frac{\partial f^{(0)}}{\partial t} + \nabla\cdot v f^{(0)} + \frac{\partial}{\partial v}\cdot\left(\frac{q}{m}\left(\underline{E} + \frac{\underline{v}}{c}\times\underline{B}\right)\right) f^{(0)}\right]$$

$$= \int d^3v\, C(f^{(1)})$$

Now, $\int d^3v \, C(f_0) = 0$ → collisions conserve #!

so, have:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\underline{V}) = 0$$

i.e. continuity equation

$$\begin{cases} n \equiv \int d^3v \; f \\ \underline{V} \equiv \int d^3v \; \underline{v} f \Big/ n \end{cases}$$

→ basic moments.

→ Now first order moment:

$$\int d^3v\, \underline{v} \left(m\frac{\partial f}{\partial t} + \underline{v}\cdot\underline{\nabla} f m + q\left(\underline{E} + \underline{v}\times\underline{B}\right)\cdot \underline{\nabla} f = C m \right)$$

(1) (2) (3) (4)

$$① = m\frac{\partial}{\partial t}\left(n\underline{V}\right) \qquad \underline{V} = V(\underline{x}, t)$$

$$③ = \int \underline{v} q \left(\underline{E} + \underline{v}\times\underline{B}\right)\cdot\frac{\partial f}{\partial \underline{v}}$$

$$= \int \frac{\partial}{\partial \underline{v}}\left[f\underline{v}\cancel{\left(\underline{E} + \underline{v}\times\underline{B}\right)}\right] d^3v - \int f\underline{v}\, \frac{\partial}{\partial \underline{v}}\cdot \cancel{\left(\underline{E} + \underline{v}\times\underline{B}\right)}$$

$$\to 0 \qquad \to 0 \quad \perp \underline{v}$$

$$- \int f\left(\underline{E} + \underline{v}\times\underline{B}\right)\cdot\frac{\partial \underline{v}}{\partial \underline{v}}$$

So

$$= -qn\left(\underline{E} + \underline{V}\times\underline{B}\right)$$

$$④ = \int d^3v\, mC(f)\, \underline{v}$$

$$= \underline{P}_{ij}$$

→ collisional momentum transfer from species i to j

which leaves ②:

$$② = m\int d^3v \; \underline{v}\,(\underline{v}\cdot\underline{\nabla})\, f$$

$$= m\int d^3v \; \underline{\nabla}\cdot(f\,\underline{v}\,\underline{v})$$

$$= \underline{\nabla}\cdot\left[m\int d^3v\, f\,\underline{v}\,\underline{v}\right] = m\,\underline{\nabla}\cdot(n\,\overline{\underline{v}\,\underline{v}})$$

clearly useful to separate $\underline{v}$ into mean and fluctuating pieces

$$\underline{v} = \underline{V} + \underline{w}$$

$$\Rightarrow \underline{\nabla}\cdot(n\,\overline{\underline{v}\,\underline{v}}) = \underline{\nabla}\cdot(n\,\underline{V}\,\underline{V}) + \underline{\nabla}\cdot(n\,\overline{\underline{w}\,\underline{w}})$$

$$+ \underbrace{\underline{\nabla}\cdot n\,(\cancel{\underline{V}\,\overline{\underline{w}} + \overline{\underline{w}}\,\underline{V}})}_{0,\ \text{defn.}}$$

$$\underline{\nabla}\cdot(n\,\underline{V}\,\underline{V}) = \underline{V}\;\underline{\nabla}\cdot(n\,\underline{V}) + n\,(\underline{V}\cdot\underline{\nabla})\,\underline{V}$$

$$n\,\overline{\underline{w}\,\underline{w}} \equiv \underline{p}$$

pressure tensor !.

so, can write for momentum equation

$$m\frac{\partial}{\partial t}(n\underline{V}) + m\underline{\nabla}\,\underline{\nabla}\cdot(n\underline{V}) + mn(\underline{\nabla}\cdot\underline{V})\underline{V}$$
$$+ \underline{\nabla}\cdot\underline{P} - qn(\underline{E} + \underline{V}\times\underline{B}) = P_{ij}$$

and using continuity:

$$\boxed{mn\left[\frac{\partial}{\partial t}\underline{V} + \underline{V}\cdot\underline{\nabla}\,\underline{V}\right] = qn(\underline{E} + \underline{V}\times\underline{B}) - \underline{\nabla}\cdot\underline{P} + \underline{P}_{ij}}$$

Now, for form $\underline{P}$:

$$\underline{\underline{P}} = \int d^3v\; m\; v_i\, v_j\, f$$

in short mean-free-path ordering,

$$f \cong f_{Maxwellian}$$

As mean extracted, symmetry $\Rightarrow$

$$\underline{\underline{P}} = \int d^3v\; v_i\, v_j\, \delta_{ij}\; f$$

so $\underline{\underline{P}} = \begin{bmatrix} P_1 & & 0 \\ 0 & P_2 & \\ & & P_3 \end{bmatrix}$ pressure tensor diagonal

if isotropic : $P_1 = P_2 = P_3$ (fast ∥, ⊥ thermal equilibration)

$\Rightarrow$

$$\underline{\underline{P}} = p \, \underline{\underline{I}}$$

and pressure reduces to scalar, i.e.

$$mn\left[\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V}\right] = qn\left(\underline{E} + \underline{V}\times\underline{B}\right) - \underline{\nabla} p + \underline{R}_{ij}$$

$\rightarrow$ For second order moment $\rightarrow$ energy (closure $\Leftrightarrow$ energy flux!) $\Rightarrow$ eqn. state

2 species $\Rightarrow$ $p/\rho^{\gamma} = \text{const.}$

→ Single Fluid (→ MHD)

Can define single fluid variables:

$$\rho = n_i M + n_e m \cong n M \qquad \rightarrow \text{density}$$

mass velocity:

mean velocity

$$\underline{V} = \frac{1}{\rho}\left(n_i M \underline{V}_i + n_e m_e \underline{V}_e\right)$$

$$\approx \left[\frac{M \underline{V}_i + m_e \underline{V}_e}{M+m}\right] \cong \underline{V}_i$$

Current density:

relative velocity

$$\underline{J} = q\,(n_i \underline{V}_i - n_e \underline{V}_e)$$

$$\approx n q\,(\underline{V}_i - \underline{V}_e) \quad , \text{ using QN.}$$

Upshot:

- continuity for ions ⇒ single fluid continuity

$$\therefore \quad \boxed{\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{V}) = 0}$$

– adding electron and ion momentum eqns:

$$M n \left(\frac{\partial \underline{V}_i}{\partial t} + \underline{V}_i \cdot \underline{\nabla}\, \underline{V}_i \right) = q n \left(\underline{E} + \underline{V}_i \times \underline{B} \right) - \underline{\nabla} p_i + \underline{P}_{i,e}$$

$$\oplus$$

$$m_e n \left(\frac{\partial \underline{V}_e}{\partial t} + \underline{V}_e \cdot \underline{\nabla}\, \underline{V}_e \right) = -q n \left(\underline{E} + \underline{V}_e \times \underline{B} \right) - \underline{\nabla} p_e + \underline{P}_{e,i}$$

$\Rightarrow$

$$n \left(\frac{\partial}{\partial t} (M \underline{V}_i + m \underline{V}_e) + M (\underline{V}_i \cdot \underline{\nabla}) \underline{V}_i + m_e (\underline{V}_e \cdot \underline{\nabla}) \underline{V}_e \right)$$
$$= q n (\underline{V}_i - \underline{V}_e) \times \underline{B} - \underline{\nabla} (p_i + p_e)$$
$$+ \underbrace{\cancel{\underline{P}_{e,i} + \underline{P}_{i,e}}}_{\text{momentum cons.}}$$

as: $m_e \ll M$

$\underline{V}$ defn.

$\rho = \rho_e + \rho_i$

$$\Rightarrow \boxed{\rho\left(\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right) = \underline{J}\times\underline{B} - \nabla p + \underset{\substack{\downarrow \\ \text{any additional} \\ \text{body force.}}}{F_{body}}}$$

Momentum balance.

→ Now, only re-maining non-trivial MHD equation is Ohm's Law.

→ where the bodies are buried....

Consider, $\big[m_e * (\text{ion momentum eqn}) - M * (\text{electron momentum eqn.})\big]$

$$\Rightarrow M m_e n\left(\frac{\partial}{\partial t}(\underline{V}_i - \underline{V}_e) + \underline{V}_i\cdot\nabla\underline{V}_i - \underline{V}_e\cdot\nabla\underline{V}_e\right)$$

$$= qn(M+m_e)\underline{E} + qn(m\underline{V}_i + M\underline{V}_e)\times\underline{B}$$

$$- m\nabla p_i + M\nabla p_e - (M+m)\underline{P}_{ei}$$

Now, ① $\underline{P}_{ei}$ = electron-ion momentum transfer

$= -M n q \eta \underline{J}$

② $M \gg m_e$

③ neglecting advective derivatives

$\Rightarrow$

$$\frac{M m_e n}{Z} \frac{\partial}{\partial t}\left(\frac{\underline{J}}{n}\right) = Z \rho \underline{E} - M n q \eta \underline{J} + M \nabla P_e + Z n (m \underline{V}_i + M \underline{V}_e) \times \underline{B}$$

and can further simplify:

$$m \underline{V}_i + M \underline{V}_e = M \underline{V}_i + m \underline{V}_e - (M-m)(\underline{V}_i - \underline{V}_e)$$

$$\cong \frac{\rho \underline{V}}{n} - \frac{M}{nq} \underline{J}$$

Finally, re-arrangeing $\Rightarrow$

$$\boxed{\frac{m_e}{nq^2}\frac{\partial \underline{J}}{\partial t} = (\underline{E} + \underline{V} \times \underline{B}) - \eta \underline{J} - \frac{1}{nq}(\underline{J} \times \underline{B}) + \frac{1}{nq}\nabla P_e}$$

Now, have generalized Ohm's Law:

$$\underset{①}{\frac{m_e}{nq^2}\frac{\partial \underline{J}}{\partial t}} = \underset{②}{\left(\underline{E} + \underset{⑤}{\underline{V}\times\underline{B}}\right)} - \underset{③}{\eta \underline{J}} - \underset{④}{\frac{(\underline{J}\times\underline{B})}{nq}} + \frac{\nabla p_e}{nq}$$

② ⟹ ideal MHD Ohm's Law

③ ⟹ collisional resistivity

$\Big\}$ resistive MHD

'ning in ④ : Hall Term

⟹ Hall MHD

bring in ⑤ : Electron thermal force / pressure

⟹ diamagnetic / finite electron ω_* MHD

i.e. Boltzmann response : $\underline{E}$ vs $\frac{\nabla p_e}{nq}$

① : Electron inertia term ($\sim m_e$)

⟹ EMHD, electron inertially modified MHD.
($\omega\, m_e/nq^2 > \eta$)

For low frequency, strong collisionality, etc.

$\Rightarrow \quad \underline{E} + \underline{V} \times \underline{B} = \eta \underline{J}$ } Resistive MHD.

N.B. :- Ohm's Law is most sensitive part of MHD structure → need care.

- high ω → electron inertia
  tok. μ-inst → thermal force term.
  $\lambda \sim c/\omega_{pi}$ → Hall term.

→ Reduced MHD

Note: ① full MHD : 3 $\underline{V}$ components
2 $\underline{B}$ " " $(\nabla \cdot \underline{B} = 0)$
ρ, p

⇒ 7 components

② if $\nabla \cdot \underline{V} = 0$ ⇒ 4 components
(ρ = const, p from $\nabla \cdot \underline{V} = 0$)

③ strongly magnetized system ⇒ Reduced MHD
⇒ scalar equations for ϕ, ψ (2 scalar fields)

Now:
- assume strong B_z (strong magnetization → gyrokinetics)

"strong" ⟷ $\rho v^2 \sim p \ll B_z^2/8\pi$

so motion strongly anisotropic, and small scales generated in ⊥ direction only, as strong B_z inhibits line bending. (energy to perturb strong, high energy density field).

⇒ Order : $B_z \sim \nabla_\perp \sim 1$

$B_\perp \sim \partial_z \sim O(\epsilon)$

Take $\rho \sim 1$, as $\underline{\nabla} \cdot \underline{V} = 0$ enforced by strong B_z:

$$V_\perp^2 \sim p \sim B_\perp^2$$ (i.e. equipartition of energy) (springiness)

$$\Rightarrow \quad V_\perp \sim \epsilon, \quad p \sim \epsilon^2, \quad \partial_t \sim \underline{V_\perp} \cdot \underline{\nabla_\perp} \sim \epsilon$$

and pressure balance ($\underline{\nabla} \cdot \underline{V} = 0$ and incompressibility)

$$\delta(B_z^2) \sim 2 B_z \delta(B_z) \sim p \qquad \text{(eqbm)}$$

$$\Rightarrow \delta B_z \sim \epsilon^2.$$

$\therefore$ to lowest order $\Rightarrow B_z = \text{const}$.

Now then: $(\underline{\nabla} \cdot \underline{B} = 0)$

$$\boxed{\begin{aligned} \underline{B} &= \hat{z} \times \underline{\nabla} \psi + B_z \hat{z} \\ &= \nabla A_\parallel \times \hat{z} + B_z \hat{z} \qquad \psi = -A_\parallel \end{aligned}}$$

B rep. by single scalar potential

parallel comp. of vector pot.

$$\underline{\nabla} \cdot \underline{B} = \partial_z \tilde{B}_z = \epsilon^3 \rightarrow 0.$$

Similarly,

$$\partial_z p \sim O(\epsilon^3), \quad \Rightarrow \quad V_z \ll V_\perp$$
$$J_\perp B_\perp \sim \epsilon^3$$

neglect V_z.

Now, $\underline{E} = -\frac{1}{c}\frac{\partial \underline{A}}{\partial t} - \underline{\nabla}\phi = -\frac{\underline{V}\times\underline{B}}{c}$

$$\Rightarrow \quad +\frac{1}{c}\frac{\partial \underline{A}}{\partial t} = \frac{\underline{V}\times\underline{B}}{c} - \underline{\nabla}\phi \qquad (*)$$

$$B_z = (\underline{\nabla}_\perp \times \underline{A}_\perp)\cdot\hat{z}$$

so $\partial_t A_\perp \sim \epsilon^3$ (a la $\partial_z \rho_z$)

$\therefore$ $\nabla_\perp \phi \cong \left(\frac{\underline{V}\times\underline{B}}{c}\right)_\perp$, in $(*)$

$$\Rightarrow \boxed{\underline{V}_\perp = \frac{c\,\hat{z}\times\underline{\nabla}\phi}{B_z}}$$

$\perp$ velocity

$\rightarrow$ motion $\perp$ is $\underline{E}\times\underline{B}$.

Now, taking parallel component of $(*)$.

(units!)

$$\Rightarrow \quad \frac{\partial}{\partial t}\psi + \underline{V}\cdot\underline{\nabla}\psi = B_z\partial_z\phi$$

so have (vector potential / flux) equation:

$$\boxed{\frac{\partial\psi}{\partial t} + \underline{V}\cdot\underline{\nabla}\psi = B_z\,\partial_z\phi}$$

$$= B_z \hat{z} + \hat{z} \times \nabla \psi$$

or, alternatively,

$$\boxed{\frac{\partial \psi}{\partial t} - \underline{B} \cdot \nabla \phi = 0}\ .$$

Finally, for ϕ, write:

$\perp$ motion

cells of $E \times B$ drift, ('spin up' rate?)

$$\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} = -\frac{\nabla p}{\rho_0} + \frac{\underline{J} \times \underline{B}}{c}$$

$(\nabla \times) \cdot \hat{z} \Rightarrow$ vorticity component ($\parallel \hat{z}$) evolution

$$\frac{\partial}{\partial t} \omega_z + \underline{V}_\perp \cdot \nabla \omega_z = \cancelto{0}{-\nabla \times \frac{\nabla p}{\rho_0}} + \hat{z} \cdot \nabla \times \frac{(\underline{J} \times \underline{B})}{c}$$

$$= \underline{B} \cdot \nabla J_z \cancelto{0}{- \underline{J} \cdot \nabla B_z} \qquad \delta B_z \sim \epsilon^3$$

$$\cong \underline{B} \cdot \nabla J_z$$

$$\therefore \boxed{\frac{\partial}{\partial t} \omega_z + \underline{V} \cdot \nabla \omega_z = \underline{B} \cdot \nabla J}$$

but:

$$\omega_z = \hat{z} \cdot \nabla \times \underline{V} = \nabla^2 \phi$$

$$J_z = \hat{z} \cdot (\nabla \times \underline{B}) \frac{c}{4\pi} = \nabla^2 \psi$$

so finally have:

$$\boxed{\frac{\partial}{\partial t}\nabla^2\phi + \underline{V}\cdot\underline{\nabla}\,\nabla^2\phi = B_z\frac{\partial}{\partial z}\nabla^2\psi + \underline{\tilde{B}}\cdot\underline{\nabla}\,\nabla^2\psi}$$

Finally, have reduced MHD equation:

$$\boxed{\begin{aligned}&\frac{\partial\psi}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\psi = B_z\,\partial_z\phi + \eta\nabla^2\psi\\ &\frac{\partial}{\partial t}\nabla^2\phi + \underline{V}\cdot\underline{\nabla}\,\nabla^2\phi - \nu\nabla^2\nabla^2\phi\\ &\qquad = \underline{\tilde{B}}\cdot\underline{\nabla}\,\nabla^2\psi + B_z\frac{\partial}{\partial z}\nabla^2\psi\end{aligned}}$$

- note have reduced MHD to <u>2</u> <u>scalar</u> <u>evolution equations</u>

- does this look familiar?

consider linearization in straight field

$$-i\omega\hat{\psi} = ik_z B_z\hat{\phi}$$

$$+i\omega k_\perp^2\hat{\phi} = -ik_z k_\perp^2\psi$$

but recall for KSAW, with $k_\perp^2 \rho_s^2 \to 0$

$$\hat{\phi} - \frac{\omega}{c}\frac{\hat{A}_\parallel}{k_\parallel} = 0 \qquad \text{— QN. } \hat{n}/n_0 \text{ computed}$$

$$\nabla_\parallel \hat{J}_\parallel / n_0 |e| = \rho_s^2 \frac{\partial}{\partial t} \nabla_\perp^2 \left(\frac{|e|\hat{\phi}}{T_e}\right) \qquad \text{— adding GKE moments}$$

∴ clearly identical !

⇒ { Reduced MHD equivalent to one-fluid theory based on gyro-kinetic equations !!

- indicates $\underline{2}$ routes to reduced MHD

ⓐ Boltzmann Eqn. → 2 Fluid Eqns → 1 fluid eqns

→ 'strong field' ordering → RMHD

<u>n.b.</u> 'strong field' ordering at <u>macroscopic</u> level

ⓑ Boltzmann Eqn. → Gyrokinetic Eqn. → moment eqns.

→ reduced MHD.

<u>n.b.</u> 'strong field' ordering at <u>microscopic</u> level.

- for $\underline{\underline{2D}}$ MHD:

$$\frac{\partial}{\partial t}\nabla^2\phi + \underline{V}\cdot\underline{\nabla}\,\nabla^2\phi = \underline{B}\cdot\underline{\nabla}\,\nabla^2\psi + \nu\nabla^2\nabla^2\phi$$

$$\frac{\partial\psi}{\partial t} + \underline{V}\cdot\underline{\nabla}\psi = \eta\nabla^2\psi$$

- ① Conservation Laws, etc. (HW)

$$\frac{d}{dt}E \underset{(\text{to }\eta,\,\nu)}{=} 0, \quad E = \int d^3x\left[\frac{(\nabla\phi)^2}{2} + \frac{(\nabla\psi)^2}{2}\right]$$

② $\mathcal{H} = A\cdot B \cong \underset{\substack{\downarrow\\ \text{const.}}}{B_z}\psi$

$$\Rightarrow H = \int d^3x\, B_z\psi, \quad \frac{dH}{dt} = 0, \text{ to } O(\eta)$$

Ohm's Law (flux advection) is simple statement of helicity conservation.

? $\frac{\partial}{\partial t}\psi + \nabla\cdot\Gamma_\psi = \eta\nabla^2\psi$ form Γ_ψ s/t $\begin{cases} H \text{ conserved} \\ E_M \text{ dissipated}\end{cases}$

③ $K = \int d^3x\, \underline{V}\cdot\underline{B} = \int d^3x\,(\nabla\phi\cdot\underline{\nabla}\psi)$

also conserved, to dissipation.