) MHD Stability and Energy Principle

a) Basic Ideas

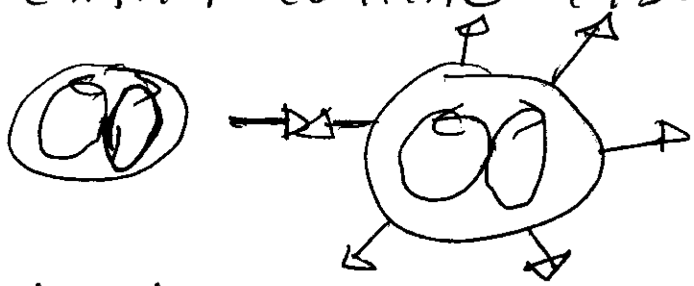
- up till now, concerned with localized instabilities, local theory, etc.

i.e. $\tilde{\phi} \sim \vec{\phi}_{\underline{k}}\, e^{i(\underline{k}\cdot\underline{x} - \omega t)}$, etc.

but

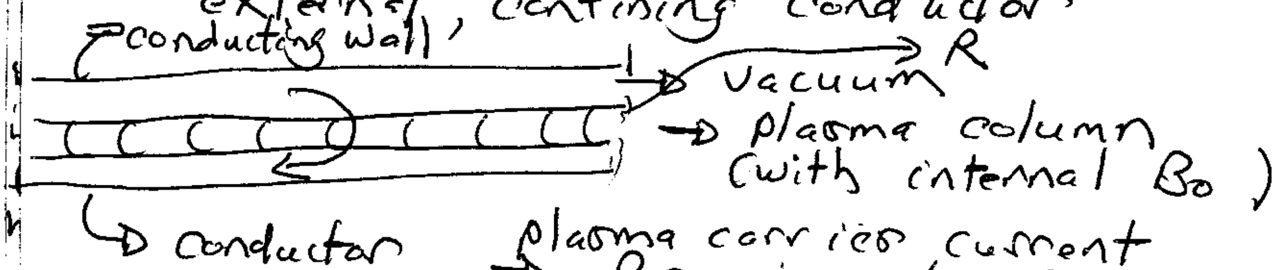
- to understand implications of stability for confinement, need consider effects of boundary conditions, etc.

i.e. recall: Virial theorem ⇒ plasmoid cannot confine itself



∴ natural to consider simple confined plasma, the linear pinch — plasma column with external, confining conductor

Linear Pinch ⇒



plasma carries current ⇒ B_θ in exterior.

N.B. Note effect of B.C.'s:

- plasma - vacuum interface to electrically, thermally isolate plasma, and allow small displacements

- <u>conductor</u> (i.e. wall) introduces electrical "no-slip" b.c.'s

  i.e. $E_{tan}\big|_{wall} = 0$, $B_n\big|_{wall} = 0$

  $\Rightarrow$ restrict perturbation.

- plasma - vacuum interface akin to air - water interface in waves.

idealizations:

- vacuum gap
- B_{0z} in plasma (i.e. B_{0z} pervades gap as well)
- B_θ external only ("thin current")
  $\Rightarrow$ in reality, $B_\theta = B_\theta(r)$ as $J_z(r)$ distributed

still, linear pinch has been very useful idealized venue for stability studies...

$\Rightarrow$ Equilibrium

Now, for equilibrium:

$$\cancel{\rho \frac{d\underline{v}}{dt}} = -\underline{\nabla} p + \underline{J} \times \underline{B}$$

$$= -\underline{\nabla}\left(p + \frac{B^2}{8\pi}\right) + \frac{\underline{B}\cdot\underline{\nabla}\,\underline{B}}{4\pi}$$

note: B_{0z} irrelevant to equilibrium! $\rightarrow$ ignore.

then, noting $J_z = \frac{1}{4\pi}\frac{1}{r}\frac{\partial}{\partial r}(r B_\theta)$

$\Rightarrow$ have eqbrm. condition:

$$\frac{dp}{dr} = \frac{-1}{4\pi} B_\theta \frac{1}{r} \frac{\partial}{\partial r}(rB_\theta)$$

$r^2 \times$ above $\Rightarrow$

$$\int_0^R dr\, r^2 \frac{dp}{dr} = \frac{-1}{4\pi} \int_0^R r^{\cancel{2}} dr\, \frac{B_\theta}{\cancel{r}} \frac{\partial}{\partial r}(rB_\theta)$$

assuming $p(R) = 0$ (limit of confined pressure)

$\Rightarrow$

$$+2\int_0^R dr\, r\,p(r) = +\frac{1}{8\pi} \left(r B_\theta(r)\right)^2 \Big|_0^R$$

$$\Rightarrow \boxed{2\int_0^R dr\, r\,p(r) = \frac{1}{8\pi}\left(R B_\theta(R)\right)^2}$$

Take $T_i = T_e = T$ (isothermal), $n_e = n_i$

$\therefore$ $p = 2n k_b T$

$$N = \frac{\#\text{ particles}}{\text{length}} = \int_0^R dr\, 2\pi r n$$

line density

and $I = \frac{1}{2} R B_\theta(R) = \int dr \; 2\pi r J_z(r)$

$\therefore \quad 2 N k_b T = \frac{1}{2} I^2$

i.e. more current $\Rightarrow$ more pressure held!

$$\Rightarrow \quad \boxed{I = 2(N k_b T)^{1/2}}$$

Bennett criterion

$\rightarrow$ Stability

Now, convenient to think in terms of displacements, for linear theory:

i.e. perturbation $\Rightarrow \underline{x} \rightarrow \underline{x} + \underbrace{\underline{\xi}(\underline{x}, t)}_{\text{displacement}}$

so $\underbrace{\delta \underline{V}}_{\text{perturbed velocity}} = \frac{\partial}{\partial t} \underline{\xi}$

then have:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0$$

① $\Rightarrow \boxed{\delta\rho + \nabla\cdot(\rho\underline{\xi}) = 0}$

$P\rho^{-\gamma} = \text{const.}$

$\Rightarrow \frac{d}{dt} P\rho^{-\gamma} = 0 \quad , \quad \frac{d}{dt} = \frac{\partial}{\partial t} + \underline{V}\cdot\underline{\nabla}$

$$\rho^{-\gamma}\frac{\partial \tilde{P}}{\partial t} + (-\gamma)\rho^{-\gamma-1}P\frac{\partial\tilde{\rho}}{\partial t} + \left[\left(\frac{\partial}{\partial t}\tilde{\underline{\xi}}\right)\cdot\underline{\nabla}(P\rho^{-\gamma})\right] = 0$$

$$\Rightarrow \delta P + \underline{\xi}\cdot\underline{\nabla}P = \frac{\gamma P}{\rho}(\delta\rho + \underline{\xi}\cdot\underline{\nabla}\rho)$$

and using ①

② $\boxed{\delta P = -\gamma P\,\underline{\nabla}\cdot\underline{\xi} - \underline{\xi}\cdot\underline{\nabla}P}$

$$\Rightarrow \frac{\partial}{\partial t}\delta\underline{B} = \underline{\nabla}\times\tilde{\underline{V}}\times\underline{B}$$

$$= \underline{\nabla}\times\frac{\partial\underline{\xi}}{\partial t}\times\underline{B}$$

③ $\therefore \boxed{\delta\underline{B} = \underline{\nabla}\times(\underline{\xi}\times\underline{B})}$

and similarly:

④ $\partial \underline{J} = \frac{1}{4\pi} \underline{\nabla} \times \left[\underline{\nabla} \times (\underline{\xi} \times \underline{B}) \right]$

so

$$\rho_0 \frac{\partial \tilde{\underline{V}}}{\partial t} = -\underline{\nabla} \tilde{p} + \widetilde{\underline{J} \times \underline{B}}$$

$$\Rightarrow \boxed{\rho \frac{\partial^2 \underline{\xi}}{\partial t^2} = \underline{F}(\underline{\xi})}$$ where

$$\boxed{\begin{aligned} \underline{F} = \underline{\nabla} \left[\gamma p \, \underline{\nabla} \cdot \underline{\xi} + \underline{\xi} \cdot \underline{\nabla} p \right] \\ + \frac{1}{4\pi} \Big\{ \left[\underline{\nabla} \times \underline{\nabla} \times (\underline{\xi} \times \underline{B}) \right] \times \underline{B} \\ + (\underline{\nabla} \times \underline{B}) \times \left[\underline{\nabla} \times (\underline{\xi} \times \underline{B}) \right] \Big\} \end{aligned}}$$

<u>with</u> boundary conditions restricting ω^2

i.e.

$$-\omega^2 \rho_0 \underline{\xi} = \underline{F}(\underline{\xi}) + \text{B.C.'s}$$

s/t ω^2 quantized, <u>so</u> $\omega^2 \to \omega_n^2$

$\xi \to \xi_n$

eigenfunctions, etc.

N. B. : Will show $\underline{F}(\underline{\xi})$ hermitian
$\Rightarrow \omega^2$ real

in MHD have stable oscillations ($\Rightarrow \omega^2 > 0$) or purely growing modes ($\omega^2 < 0$).

Now

→ For boundary conditions : [plasma-vacuum interface]

refer internal (to plasma) perturbations with i

refer external with e

So, for external fields: (vaccuum region)

$$\delta \underline{E}^{e} = -\frac{1}{c}\frac{\partial}{\partial t}\delta \underline{A}^{e} \qquad \text{(induced only)}$$

$$\delta \underline{B}^{e} = \underline{\nabla} \times (\delta \underline{A}^{e}).$$

and require tangential electric field continuous at interface.

i.e. if $\hat{n} \equiv$ unit normal to plasma,

then :

$$\hat{n} \times \left[\delta \underline{E}^{i} + \frac{1}{c} \delta \underline{V} \times \underline{B}^{i} \right]$$

$$= \hat{n} \times \left[\delta \underline{E}^{e} + \frac{1}{c} \delta \underline{V} \times \underline{B}^{e} \right]$$

but ideal MHD $\Rightarrow$ $\delta \underline{E}_i^{i} + \frac{\delta \underline{V}}{c} \times \underline{B}^{i} = 0$

$\Rightarrow$

$$\hat{n} \times \left[-\frac{1}{c} \frac{\partial}{\partial t} \delta \underline{A}^{(e)} + \frac{1}{c} \frac{\partial \xi}{\partial \cancel{t}} \times B^{e} \right] = 0$$

$\Rightarrow$

$$\hat{n} \times \frac{\partial}{\partial t} \delta \underline{A}^{(e)} = - \left(\hat{n} \cdot \frac{\partial \xi}{\partial t} \right) \underline{B}^{e} - \left(\underline{n} \cdot \underline{B}^{e} \right) \frac{\partial \vec{\xi}}{\partial t}$$

but here (and anything sensible),

$$\hat{n} \cdot \underline{B}_0 = 0$$

$\Rightarrow$ $$\boxed{\hat{n} \times \delta \underline{A}^{(e)} = - (\underline{\hat{n}} \cdot \underline{\xi}) \underline{B}^{(e)}}$$

[electromag. B.C. at interface

Now, at conducting wall

$$\hat{n} \times \delta \underline{A}^{(e)} = 0 \qquad (E_{tan} = 0)$$

$$\hat{n} \cdot (\nabla \times \delta \underline{A}^{(e)}) = 0 \qquad (B_n = 0)$$

and, in vaccum:

$$\boxed{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\delta \underline{A}^{(e)} + \underline{\nabla}\times(\underline{\nabla}\times\delta\underline{A}^{(e)}) = 0}$$

Finally, must have pressure balance at interface:

$$p^i_{tot}(\underline{x}+\underline{\varepsilon}) = p^e_{tot}(\underline{x}+\underline{\varepsilon})$$

$$p_{tot} = p + \frac{B^2}{8\pi}$$

so, to first order:

$$(1+\underline{\varepsilon}\cdot\underline{\nabla})\left[p + \frac{B^{i\,2}}{8\pi} + \delta p + \frac{1}{4\pi}\underline{B}^i\cdot\delta\underline{B}^i\right]$$

$$= (1+\underline{\varepsilon}\cdot\underline{\nabla})\left[\frac{(B^e)^2}{8\pi} + \frac{1}{4\pi}\underline{B}^e\cdot\delta\underline{B}^e\right]$$

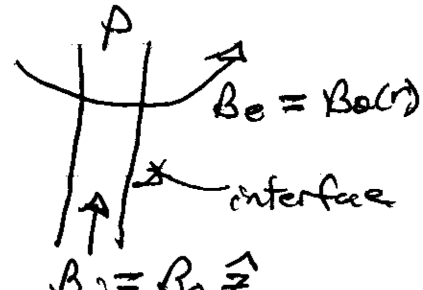
and, using ② for δp $\Rightarrow$

$$\boxed{\begin{aligned}-\gamma p\,\underline{\nabla}\cdot\underline{\varepsilon} + \frac{1}{4\pi}\underline{B}^i\cdot\left[\delta\underline{B}^i + \underline{\tilde{\varepsilon}}\cdot\underline{\nabla}\,\underline{B}^i\right] \\ = \frac{1}{4\pi}\underline{B}^e\cdot\left[\delta\underline{B}^e + \underline{\varepsilon}\cdot\underline{\nabla}(\underline{B}^e)\right]\end{aligned}}$$

Sausage and Kink Modes

More MHD stability → 'Links and kinks'

Consider surface current sheet pinch:



$J \sim \delta(r-a)$ → approximation of convenience

- As $dp/dr = -\frac{1}{4\pi} \frac{B_\theta}{r} \frac{d}{dr}(rB_\theta)$ → for pressure field irrelev. to confinement

profile is <u>flat</u> inside $\left(\underline{\nabla}(B_0^2/8\pi) = 0\right)$ (ideal MHD)

- recall derived: → generalized displacement

$$\rho_0 \frac{d^2\underline{\xi}}{dt^2} = F(\underline{\xi})$$

→ force (hermitian operator)

$$= \underline{\nabla}\left[\gamma \rho \underline{\nabla}\cdot\underline{\xi} + (\underline{\xi}\cdot\underline{\nabla})\rho\right]$$

$$+ \frac{1}{4\pi}\Big\{\left[\underline{\nabla}\times\underline{\nabla}\times(\underline{\xi}\times B)\right]\times\underline{B}$$

$$+ (\underline{\nabla}\times\underline{B})\times\left[\underline{\nabla}\times(\underline{\xi}\times\underline{B})\right]\Big\}$$

and $n \times \delta\underline{A}^e = -(\hat{n}\cdot\underline{\xi})\,\underline{B}^e$ at interface

$\hat{n}$ = <u>unit normal to plasma</u>

$$\hat{n}\times\delta\underline{A}^e = \hat{n}\cdot(\underline{\nabla}\times\delta A^e) = 0$$

at wall

[relates external e.m. to displacement]

in gap: $\underbrace{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}}_{\text{no displ., MHD}} \delta A^e + \nabla \times (\nabla \times \delta A^e) = 0$

and, at interface:

$$-\gamma p \nabla \cdot \xi + \frac{1}{4\pi} B^i \cdot [\delta B^i + (\xi \cdot \nabla) B_i]$$
$$= \frac{1}{4\pi} B^e \cdot [\delta B^e + (\xi \cdot \nabla) B^e]$$

perturbed pressure balance

Now, for surface current pinch:

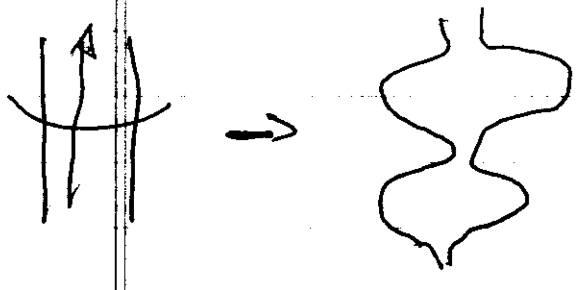
$(\nabla \times B)_{in} = 0 \qquad (\nabla p)_{in} = 0 \quad \Rightarrow$ simplifies.

no internal current, pressure gradient

so

$$F = \nabla [\gamma p \nabla \cdot \xi] + \frac{1}{4\pi} [\nabla \times \nabla \times (\xi \times B)] \times B$$

Now, consider sausage ('kinks'):



$\xi =$

$= (\xi_z, \xi_r, 0)$

$\xi_\theta = 0$ (azimuthally symmetric perturbation)
($\xi_\theta \to$ torsion $\to$ torsional Alfven penalty)

Thus – necking amplifies B_θ, p-inforcing necking $\Rightarrow$ destabilizing.

out – necking compresses, bends B_z $| \to \}$.
stabilizing.

$\Rightarrow$ suggests role of B_z in pinches/tokamak is for stability, and expect B_z min for stability.

Now $\underline{\xi} = \begin{Bmatrix} \xi_r(r) \\ \xi_z(r) \end{Bmatrix} e^{i(m\theta + kz)} e^{-i\omega t}$

for surface current pinch $\Rightarrow$ $J_{in} = 0$, $\nabla P_{in} = 0$

$$-\omega^2 \rho \underline{\xi} = \gamma \rho \underline{\nabla} (\underline{\nabla} \cdot \underline{\xi}) + \frac{1}{4\pi} \left[\underline{\nabla} \times \underline{\nabla} \times (\underline{\xi} \times \underline{B}_i) \right] \times \underline{B}_i$$

taking $m = 0 \Rightarrow$ (azimuthally symmetric perturbation)

$$-\omega^2 \rho \xi_z = ik \gamma \rho \underline{\nabla} \cdot \underline{\xi} \, ; \qquad \text{for } \xi_z$$

$$-\omega^2 \rho \xi_r = \gamma \rho \frac{d}{dr} (\underline{\nabla} \cdot \underline{\xi}) + \frac{1}{4\pi} B_z \left\{ \frac{d}{dr} \frac{B_z}{r} \frac{d}{dr}(r \xi_r) - k^2 B_z \xi_r \right\} \qquad \text{for } \xi_r.$$

then, can re-write:

ξ_z:

$$(c_s^2 k^2 - \omega^2) \xi_z = c_s^2 \frac{ik}{r} \frac{d}{dr} (r \xi_r)$$

and, for ξ_r;

$$(V_A^2 k^2 - \omega^2)\,\xi_r = (c_s^2 + V_A^2)\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r\xi_r)\right] + ikc_s^2\frac{d\xi_z}{dr}$$

Now, eliminating ξ_z:

$$\frac{d^2\xi_r}{dr^2} + \frac{1}{r}\frac{d\xi_r}{dr} - k_\perp^2 \xi_r = 0$$

$$k_\perp^2 = \frac{(c_s^2 k^2 - \omega^2)(V_A^2 k^2 - \omega^2)}{c_s^2 V_A^2 k^2 - (c_s^2 + V_A^2)\omega^2}$$

$$= k^2\left[1 + \frac{(\omega/k)^4}{c_s^2 V_A^2 - (c_s^2 + V_A^2)(\omega/k)^2}\right].$$

Now, ξ_r eqn. $\Rightarrow$ Bessel eqn.

Clearly $\xi_z\big|_{r=0} \to 0$; so

$$\xi_r = \xi_{r0}\, I_0(k_\perp r)$$

Can use equations of motion to relate ξ_z to ξ_r:

$$\hat{\mathcal{E}}_z = -\frac{ik_z}{k}\left[\frac{C_s^2(V_A^2k^2-\omega^2)-V_A^2\omega^2}{C_s^2(V_A^2k^2-\omega^2)}\right] I_0'(k_z r)$$

Now, applying B.C.'s:

- for perturbed pressure balance of boundary:

$$-\gamma \rho \underline{\nabla}\cdot\hat{\underline{\mathcal{E}}} + \frac{1}{4\pi}\underline{B}^i\cdot\delta\underline{A}^i = \frac{1}{4\pi}\underline{B}^e\cdot\left[\delta\underline{B}^e + \hat{\underline{\mathcal{E}}}\cdot\underline{\nabla}\,\underline{B}^e\right]$$

but $\nabla\times\mathcal{E}\times\underline{B} = \delta\underline{B} \Rightarrow$

$$\delta B_z = \frac{B_0}{r}\frac{d}{dr}(r\mathcal{E}_r)$$

(**)

so

$$\boxed{-\rho\underline{\nabla}\cdot\hat{\underline{\mathcal{E}}} - \frac{B_0^2}{4\pi r}\frac{d}{dr}(r\hat{\mathcal{E}}_r) = \frac{1}{4\pi}\underline{B}_e\cdot\left(\nabla\times\delta\underline{A}^e + \hat{\mathcal{E}}_r\frac{d\underline{B}^e}{dr}\right)}$$

- Now, for tangential e-field continuity $\Rightarrow$

$$\hat{n}\times\delta\underline{A}^e = -(n\cdot\underline{\mathcal{E}})B_\phi \qquad \text{(forces form } \delta\underline{A}^e)$$

$$\Rightarrow \quad \delta\underline{A}^{(e)} = (\mathcal{E}_r B_\phi,\ \delta A_r^e,\ 0)$$

$z,\ r,\ \phi$

form for $\delta\underline{A}^{(e)}$

Now, in the gap:

slow ⇒ neglect displ. current in MHD

$$\underline{\nabla} \times \underline{\nabla} \times \delta \underline{A}^{(e)} - \cancel{\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \delta \underline{A}^{e}} = 0$$

$$\Rightarrow \frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial}{\partial z} \delta A_r^{e} - \frac{\partial}{\partial r} \left(B_\theta \, \varepsilon_r \right) \right] = 0$$

$$- \frac{\partial}{\partial z} \left[\frac{\partial}{\partial z} \left(\delta A_r^{e} \right) - \frac{\partial}{\partial r} \left(B_\theta \, \varepsilon_r \right) \right] = 0$$

Now, as $B_\theta = B_\theta(r)$ and $\varepsilon_r = \varepsilon_r(r)$, only:

$$\Rightarrow \quad \frac{\partial}{\partial z} \delta A_r^{e} = f(r)$$

$$\frac{\partial}{\partial z} \delta A_r^{e} - \frac{\partial}{\partial r} \left(B_\theta \, \varepsilon_r \right) = g(z)$$

now, as both ↑ ↑ are functions of r, only, $g(z)$ must be constant.

Take const → 0, i.e. as only quantifies ~~absolute~~ level external field.

Then, two eqns. above ⇒

$$\delta \underline{B}^e \equiv \underline{\nabla} \times \delta \underline{A}^e = 0 \quad \Rightarrow \quad$$ i.e. no perturbation on external field

Then, perturbed pressure balance boundary condition (**)
$\Rightarrow$

$$\boxed{-\gamma \rho \, \underline{\nabla} \cdot \underline{\hat{\mathcal{E}}} - \frac{B_0^2}{4\pi r} \frac{d}{dr} \left(r \hat{\mathcal{E}}_r \right) = -\frac{1}{4\pi r} B_e^2 \hat{\mathcal{E}}_r} \quad \text{at } r = R$$

at $r = R$. Here used $B_\phi = I / 2\pi r$.

Using $\mathcal{E}_r$, $\mathcal{E}_z$ eqn. (eqn. motion) $\Rightarrow$

$$\left[\frac{\gamma^2 \omega^2 \rho}{ik C_s^2} - \frac{B_0^2 (C_s^2 k^2 - \omega^2)}{4\pi i k C_s^2} \right] \mathcal{E}_z = -\frac{B_e^2}{4\pi r} \mathcal{E}_r$$

at $r = R$

Finally, plugging in for $\mathcal{E}_z(R)$, $\mathcal{E}_r(R)$ etc.

$$\Rightarrow \boxed{\omega^2 = k^2 V_A^2 - \frac{B_{ext}^2}{B_0^2} V_A^2 \frac{K}{R} \frac{I_0'(KR)}{I_0(KR)}}$$

$$K^2 = k^2 \left[1 + \frac{(\omega/k)^4}{C_s^2 V_A^2 - (C_s^2 + V_A^2)(\omega/k)^2} \right].$$ (V_A with B_0)

- nominally, trancendental equation ⟹ numerics
- but ω^2 takes on only real values

(hermiticity of $\underline{F}$) ⟹

$$\left\{ B_0^2 \leq \frac{B_{ext}^2}{(k_z R)^2} (k_z R) \frac{I_0'(k_z R)}{I_0(k_z R)} \right.$$

condition for instability

⟹ The message: Toroidal field necessary for pinch stability to $m=0$ 'sausage' mode.

⟹ physically, toroidal field introduces bending penalty: $(\sim k^2 V_A^2)$ for sausage perturbations.

⟹ N.B. Note similarity:

$$\omega^2 = k^2 c_s^2 - 4\pi G \rho_0 \quad \text{Jeans}$$

(negative compression)

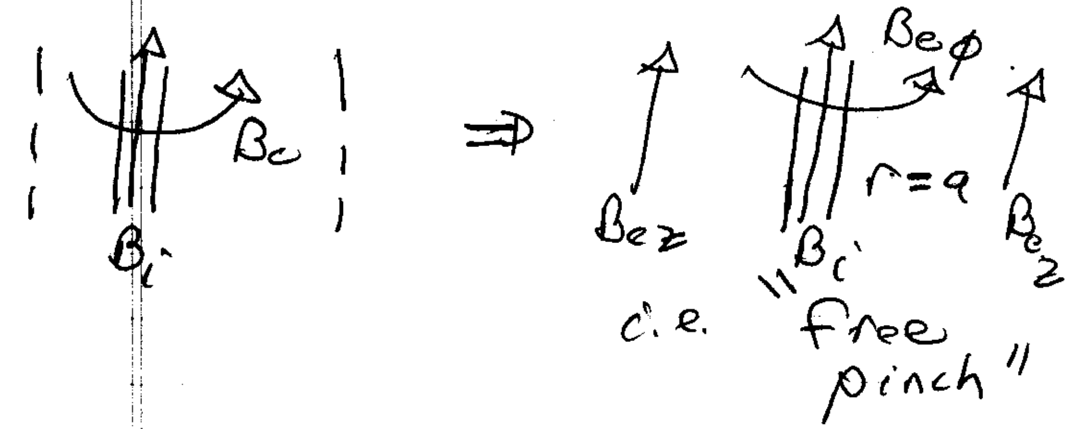
$$\omega^2 = k^2 V_A^2 - \frac{B_{ext}^2}{B_0^2} V_A^2 \frac{k_z}{R} \frac{I_0'}{I_0} \quad \text{Sausage}$$

(negative tension)

Links and Kinks, cont'd.

→ Surface Layer Pinch, Again (Simplified)

Now → remove outer wall to ∞.



ρ_0 flat
B_i const.
$B_\theta = I/2\pi r$
$J \sim \delta(r-a)$

- cylindrical symmetry

$$\hat{\underline{\xi}} = \underline{\xi}(r)e^{i(m\theta+kz)}$$

- for $k \neq 0$, can always arrange ξ_z so $\underline{\nabla}\cdot\underline{\xi} = 0$

⇒ no loss of generality to take incompressible.

- for plasma dynamics:

$$\delta\underline{B} = \underline{\nabla}\times(\underline{\xi}\times\underline{B}_0) \quad B_i \text{ const.}$$

$$= \underline{B}_0\cdot\underline{\nabla}\,\hat{\underline{\xi}} - \hat{\underline{\xi}}\cdot\cancel{\underline{\nabla}\,\underline{B}_0} - \underline{B}_0\cancelto{0}{\underline{\nabla}\cdot\hat{\underline{\xi}}}$$

so

$$\rho_0 \frac{\partial \tilde{\underline{V}}}{\partial t} = -\underline{\nabla}\left(\overbrace{\tilde{p} + \frac{\underline{B}_0 \cdot \tilde{\underline{B}}}{4\pi}}^{\tilde{p}_*}\right) + \frac{\underline{B}_0 \cdot \underline{\nabla}}{4\pi} \underbrace{\tilde{\underline{B}}}_{\sim \underline{B}_0 \cdot \underline{\nabla}\, \hat{\underline{\xi}}}$$

$$+ \frac{\cancel{\tilde{\underline{B}} \cdot \underline{\nabla}\, \underline{B}_0}}{4\pi}$$

$$= -\nabla\left(\tilde{p}^*\right) + \frac{(\underline{B}_0 \cdot \underline{\nabla})(\underline{B}_0 \cdot \underline{\nabla})}{4\pi} \hat{\underline{\xi}}$$

$\Rightarrow$

$$\left(-\omega^2 \rho_0 + \frac{k^2 \overbrace{B_0}^{B_0}}{4\pi}\right) \hat{\underline{\xi}} = -\underline{\nabla} p^*$$

now, $\underline{\nabla} \cdot \hat{\underline{\xi}} = 0 \Rightarrow \nabla^2 \tilde{p}^* = 0$

$$\therefore \left(\nabla_r^2 - \frac{m^2}{r^2} - k^2\right) \tilde{p}^* = 0$$

p well behaved as $r \to 0$

$$\Rightarrow \tilde{p}^*(r) = \tilde{p}^*(a) \frac{I_m(kr)}{I_m(ka)}$$

$$\tilde{p}^* = \sum_m \left[a I_m(kr) + \cancel{b K_m(kr)}\right]$$

(to match B.C.'s)

Thus, for displacement on boundary;

$$\left(-\omega^2\rho_0 + \frac{k^2B_e^2}{4\pi}\right)\hat{\xi}_r(a) = -k\,\tilde{p}^*(a)\,\frac{I_m'(ka)}{I_m(ka)}$$

$$\Rightarrow \boxed{\hat{\xi}_r(a) = \left(\frac{4\pi k\ \tilde{p}^*(a)}{4\pi\rho_0\omega^2 - B_i^2k^2}\right)\frac{I_m'(ka)}{I_m(ka)}}$$

· Exterior

$$\nabla\times\tilde{B} = 0 \qquad B = \nabla\tilde{\psi}$$

$$\nabla\cdot\tilde{B} = 0 \qquad \nabla^2\tilde{\psi} = 0$$

$$\Rightarrow \left(\nabla_r^2 - \frac{m^2}{r^2} - k^2\right)\tilde{\psi} = 0$$

$$\Rightarrow \hat{\psi} = C\,\frac{K_m(kr)}{K_m(ka)}$$, $\left\{\begin{array}{l}\text{as } K_m(kr) \\ \text{well behaved} \\ \text{at } r\to\infty.\end{array}\right.$

→ B.C.'s

Observe have 3 constants $\left\{ \begin{matrix} C \\ \tilde{p}(a) \\ \tilde{\xi}_r(a) \end{matrix} \right\}$

Now, ① first: perturbed pressure balance ($\hat{n}$ as before)

$$\tilde{p} + \frac{\underline{B}_i \cdot \tilde{\underline{B}}_i}{4\pi}\Bigg|_a = \frac{\underline{B}_e \cdot \tilde{\underline{B}}_e}{4\pi} + \frac{\tilde{\xi}_n}{8\pi}\left(\frac{\partial}{\partial n} B_{0e}^2 - \frac{\partial}{\partial n} B_{0i}^2\right)$$

$$= \frac{\underline{B}_e \cdot \underline{\nabla}\tilde{\psi}}{4\pi}\Bigg|_a + \frac{\tilde{\xi}_r(a)}{8\pi}\left(\frac{\partial}{\partial r} B_{0e}^2 - \cancel{\frac{\partial}{\partial r} B_{0i}^2}\right)\Bigg|_a$$

$$B_{0\theta} = I/2\pi r \Rightarrow \frac{\partial}{\partial r}\left(B_{0e}^2\right)\Bigg|_a = \frac{-2B_{0e}^2}{a}$$

$$\Rightarrow \boxed{\tilde{p}^*(a) = \frac{i}{4\pi}\left(\frac{m}{a} B_\theta + k B_{ze}\right) C - \frac{B_\theta^2}{4\pi a}\xi_r(a)}$$

② $\left.\left(\hat{\underline{E}} + \frac{\hat{\underline{V}}\times\underline{B}_{0\theta}}{c}\right)_t = 0 \right|_a$

$\nabla\times \Rightarrow \left.\left(-\frac{1}{c}\frac{\partial\hat{\underline{B}}}{\partial t} + \underline{\nabla}\times\frac{\hat{\underline{V}}}{c}\times\underline{B}_{0\theta}\right)_t = 0\right|_a$

$\Rightarrow \hat{n}\cdot\hat{B}_0\Big|_a = \hat{n}\cdot\left(\nabla\times\hat{\mathcal{E}}\times B_{0\theta}\right)\Big|_a$

$= \hat{n}\cdot\left(\underline{B}_{0\theta}\cdot\underline{\nabla}\hat{\mathcal{E}}\right)\Big|_a$

n.b. $\left(\hat{r}\cdot\underline{\mathcal{E}}\cdot\underline{\nabla}\underline{B}\right) = 0$

$\hat{r}\cdot\underline{\nabla}\psi\Big|_a = \left(\underline{B}_{0\theta}\cdot\underline{\nabla}\right)\hat{\mathcal{E}}_r\Big|_a$

$$\Rightarrow \boxed{i\left(kB_{0z} + \frac{m}{a}B_\theta\right)\hat{\mathcal{E}}_r(a) = ck\frac{K_m'(ka)}{K_m(ka)}}$$

③ $$\boxed{\tilde{\mathcal{E}}(a) = \left(\frac{4\pi k\,p^*(a)}{4\pi\rho_0\omega^2 - B_i^2k^2}\right)\frac{I_m'(ka)}{I_m(ka)}}$$

Now ①, ②, ③ ⇒ 3 eqns.

Solvability ⇒ dispersion relation:

$$\boxed{4\pi\rho_0\omega^2 = \overset{①}{B_{0z,i}^2 k^2} - \overset{②}{\left(k B_{0ze} + \frac{m}{a}B_\theta\right)^2 \frac{I_m'(ka)\,K_m(ka)}{I_m(ka)\,K_m'(ka)}} \underset{③}{- \frac{B_\theta^2 k}{a}\frac{I_m'(ka)}{I_m(ka)}}}$$

① ⇒ internal field line bending

② ⇒ noting $K_m'/K_m < 0$, ② > 0

⇒ external field line bending

⇒ ② $\sim \left.(\underline{k}\cdot\underline{B}_{0e})^2\right|_{r=a}$

so will vanish when $(\underline{k}\cdot\underline{B}_{0e})^2 = 0$

i.e. perturbation aligns with field.

Now note: if $\underline{k}\cdot\underline{B}_{0e}=0$, $\underline{k}\cdot\underline{B}_{0i}\neq 0$

usually $\Rightarrow$ can't dodge

both bending terms, due

shear.

③ $\rightarrow$ negative definite! $\Rightarrow$ origin of instability!

<u>N.B.</u> observe term ③ from $\frac{\partial}{\partial r}\left(B_{0\theta}^2\right)$,

and that $\frac{\partial}{\partial r} B_{0\theta}^2 < 0 \Rightarrow$

field decreases with increasing distance from boundary!

Consider 2 cases

1.) $B_{0ze}=0$

$$4\pi\rho_0\omega^2 = B_{0zi}^2 k^2 + \left(\frac{m B_\theta}{a}\right)^2 \frac{I_m'}{I_m}\left|\frac{K_m}{K_m'}\right|$$

$$-B_\theta^2 \frac{k}{a}\frac{I_m'(ka)}{I_m(ka)}$$