Physics 218B

I.) Particle Orbits and Adiabatic Invariants

→ seek characterize particle orbits in complex magnetic and electric fields

→ proceed via:

① guiding center concept

② g.c. drifts
  - $\underline{E} \times \underline{B}$
  - polarization expansion
  - ∇B, curvature

③ adiabatic invariants
  - magnetic moment
  - general theory

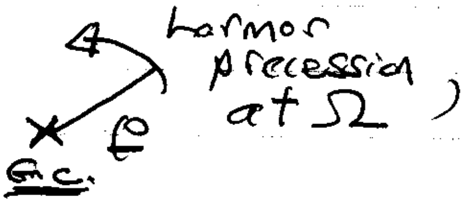
④ particle motion in mirror, tokamak fields.

i.) G.C. Concept

$\underline{B}_0 = B_0 \hat{z}$

Crucial to orbit theory is notion of guiding center, i.e.

orbit =



, $\rho = \dfrac{v_\perp}{\Omega}$

$\Omega = ZB/mc$

$$\therefore \quad \underset{\substack{\downarrow \\ \text{particle position}}}{\underline{x}} = \underset{\substack{\downarrow \\ \text{guiding center position}}}{\underline{X}_{gc}} + \underset{\substack{\downarrow \\ \text{Larmor precession}}}{\underline{\rho}}$$

i.e. $\underline{\rho} = \underline{\rho}_0 e^{i\Omega t}$

Utility: → for $t \gg \Omega^{-1}$, can average out fast cyclotron motion, and keep track of g.c. dynamics alone
⇒ drift, gyrokinetics ↔ kinetic equation for g.c.'s.

→ Can simplify motion of particle in complex field (Electric, curved magnetic, etc) via g.c. dynamics.

ii.) G.C. Drifts

a.) First, take $\underline{B} = B_0 \hat{z}$, with $\underline{E}$

$$m \underline{\dot{v}} = q \underline{E} + q \frac{\underline{v} \times \underline{B}_0}{c}$$

$$\Rightarrow \quad m \dot{v}_{\parallel} = q E_{\parallel} \qquad (\cdot\ \underline{B}_0)$$

$$m \underline{\dot{v}}_{\perp} = q \underline{E}_{\perp} + q \frac{\underline{v}_{\perp} \times \underline{B}_0}{c} \qquad \left(\begin{array}{l} \text{for } \perp \\ \underline{v}_{\perp} = \underline{v} - \underline{v}_{\parallel} \end{array}\right)$$

$$\underline{\dot{v}}_{\perp} = \frac{q \underline{E}_{\perp}}{m} + \underline{v}_{\perp} \times \Omega \hat{z}$$

for g.c. motion $|\dot{V}_\perp|/|V_\perp| \ll \Omega$, so l.o.

l.o. $$0 = \frac{q}{m}\underline{E}_\perp + \underline{v}_\perp \times \Omega\hat{z}$$

$$\Rightarrow 0 = \hat{z}\times\frac{q}{m}\underline{E}_\perp + \hat{z}\times\underline{V}_\perp\times\Omega\hat{z}$$

$$\therefore\ \boxed{\left.\begin{aligned}\underline{V}_\perp &= \frac{c}{B}\underline{E}_\perp\times\hat{z}\\ &= \frac{c}{B_0^2}\underline{E}_\perp\times\underline{B}_0\end{aligned}\right\}\ \underline{E}\times\underline{B}\text{ drift.}}$$

N.B. → ⊥ E, B
→ independent mass, charge
(i.e. all species $\underline{E}\times\underline{B}$ drift the same).

Now, can extend, generalize to time varying field:

$$\dot{\underline{V}}_\perp = \frac{q}{m}\underline{E} + \underline{V}\times\Omega\hat{z}$$

$$\underline{V}_\perp = \underline{V}_\perp^{(0)} + \epsilon\,\underline{V}_\perp^{(1)} + \cdots$$

$$\epsilon \sim O(\omega/\Omega)$$
↳ frequency of variation.

$$\dot{\underline{V}}_{\perp}^{(0)} + \epsilon\, \dot{\underline{V}}_{\perp}^{(1)} = \frac{q}{m}\underline{E} + \left(\underline{V}_{\perp}^{(0)} + \underline{V}_{\perp}^{(1)}\epsilon\right) \times \Omega \hat{z}$$

l.o.: $$0 = \frac{q}{m}\underline{E} + \underline{V}_{\perp}^{(0)} \times \Omega \hat{z}$$

$$\Rightarrow \quad \underline{V}_{\perp}^{(0)} = \frac{c}{B}\underline{E}_{\perp} \times \hat{z}$$

1st order: $$\dot{\underline{V}}_{\perp}^{(0)} = \underline{V}_{\perp}^{(1)} \times \Omega \hat{z}$$

$$\Rightarrow \quad \underline{V}_{\perp}^{(1)} = \frac{1}{\Omega}\left(\hat{z} \times \dot{\underline{V}}_{\perp}^{(0)}\right)$$

$$\therefore \quad \boxed{\left.\begin{aligned} \underline{V}_{\perp}^{(1)} &= \frac{c}{B\Omega}\dot{\underline{E}}_{\perp} \\ &= \frac{c^2}{B^2 q} m \dot{\underline{E}}_{\perp} \end{aligned}\right\} \text{polarization drift}}$$

→ 2nd term in $O(\omega/\Omega)$ expansion (i.e. polarization expansion)

→ unlike $\underline{E} \times \underline{B}$ drift,

$$\underline{V}_{\perp}^{(1)} \sim \frac{m}{q}$$

→ larger for ions

→ direction ⇔ charge.

$$\rightarrow \quad \underline{\nabla}_\perp \cdot \underline{V}_\perp \cong \cancelto{0}{\underline{\nabla}\cdot\left(\frac{c}{B_0}\underline{E}_\perp\times\hat{z}\right)} + \underline{\nabla}\cdot\left(\frac{c}{B\Omega_i}\dot{\underline{E}}_\perp\right) \checkmark$$

$$\underline{\nabla}_\perp\cdot\underline{V}_\perp = \frac{c}{B\Omega_i}\,\underline{\nabla}\cdot\hat{\underline{E}}_\perp$$

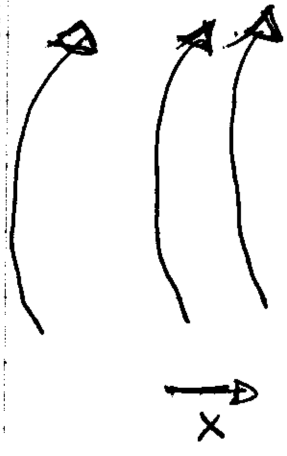
→ compressibility of G.C. drift set by polarization

→ note that time averaged $V_\perp$ $\left(\langle\,\rangle = \int_0^T \frac{dt}{T}\right)$

$$\langle V_\perp\rangle = \frac{c}{B}\,\underline{E}_\perp\times\hat{z}.$$

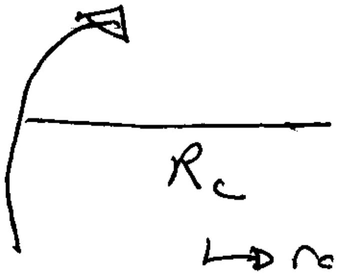
i.e. polarization, et al. are $\frac{d}{dt}$

i) inhomogeneous magnetic field.



→ types of variation
- curvature
- increase of strength
- increase of strength along → mirroring

curvature:



↳ radius of curvature

then if particle/g.c. streams along field line, feels centrifugal force i.e. $\underline{F} = \frac{m V_{\parallel}^2}{R_c^2} \underline{R_c}$

then, if insert body force into Lorentz eqn.:

$$\frac{d\underline{v}}{dt} = \frac{\underline{F}}{m} + \Omega\, \underline{v} \times \hat{n} \qquad \hat{n} = \frac{\underline{B_0}}{|B_0|}$$

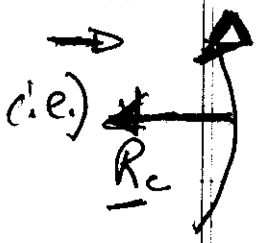
$$= \frac{\cancel{m} V_{\parallel}^2}{\cancel{m} R_c}\underline{R_c} + \Omega\, \underline{v} \times \hat{n}$$

(n.b.: concept of locally strong field)

∴ a la' $\underline{E} \times \underline{B}$ drift

$$\boxed{\underbrace{\underline{V}_d}_{\text{cross field drift}} = \frac{V_{\parallel}^2}{\Omega}\,\frac{\underline{R_c} \times \underline{B}}{B\, R_c^2}}$$ curvature drift

⇒ (i.e.)



$\odot$ B_0 (out) ⇒ ↑ V_d ion

↓ V_d electron

opposite for diff'nt species!

⇒ curvature drift induces charge separation!

→ can re-write using:

$$\boxed{\underline{V_d} = \frac{V_{\parallel}^3}{\Omega}\,\frac{\hat{r}_c \times \hat{n}}{R_c}}$$ → curvature drift.

Similarly, can consider gradient in field strength

$$\dot{\underline{v}}_\perp = \frac{q}{mc}\,\underline{v} \times \underline{B}_0(x)$$

expand and iterate ⇒

$$\underline{B}_0(x) \cong \underline{B}_0(0) + x\frac{\partial}{\partial x}\underline{B}_0\Big|_0 + \cdots$$

$$\underline{x} = \underline{x}_{gc} + \vec{\rho}$$

$$\dot{\underline{x}}_{gc} \cancel{+ \dot{\vec{\rho}}} = \cancel{\frac{q}{mc}\,\underline{v} \times \underline{B}(0)} + \frac{qB}{mc}\,\underline{v} \times \left(x\,\frac{1}{B}\frac{\partial B_0}{\partial x} \right)$$

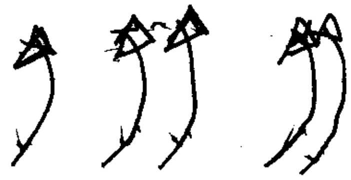
for $\langle \dot{\underline{x}}_{gc} \rangle = \underline{v}_d$, avg ⇒

$$\underline{v}_d = \frac{qB}{mc}\left\langle \underline{v} \times \left(\frac{v_x}{\Omega}\,\frac{1}{B}\,\frac{\partial B_0}{\partial x} \right)\hat{z} \right\rangle_{\Omega^{-1}}$$

$$= \frac{1}{2}\,\frac{v_\perp^2}{\Omega}\,\frac{1}{B}\,\frac{\partial B}{\partial x}\,\hat{y}$$

more generally,

for



$$\boxed{\underline{V}_d = \frac{1}{2}\frac{V_\perp^2}{\Omega}\frac{\underline{B}\times\underline{\nabla}B}{B^2}} \rightarrow \nabla B \text{ drift.}$$

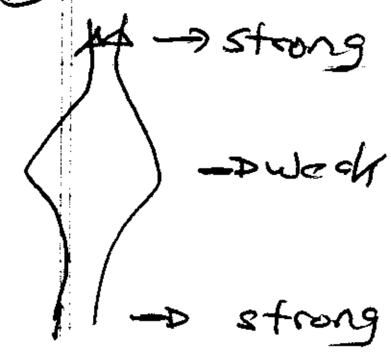
- → particles drift ⊥ to field and direction of field variation
- → opposite for e, i

Note: total drift due magnetic inhomog.

$$\boxed{\underline{V_\perp}_d = \frac{V_\parallel^2}{\Omega}\frac{\underline{R_c}\times\underline{B}}{B R_c^2} + \frac{V_\perp^2}{2\Omega}\frac{\underline{B}\times\underline{\nabla}B}{B^2}}$$

(iii.) Adiabatic Invariants

→ first, what of field intensification along B



obviously, magnetic mirroring will occur

→ to describe, convenient to introduce concept of magnetic moment

$$\mu = (\text{current}) \times (\text{area}) \Big/ c$$

$$= \left(\frac{q\Omega}{2\pi}\right)\left(\frac{\pi \rho_L^2}{c}\right)$$

$$\mu = \frac{m v_\perp^3}{2B}$$

Ignoring $\underline{E}\times\underline{B}$ drift (i.e. $\underline{E} = 0$, at g.c.),

$$\frac{d}{dt}\left(\frac{m v_\perp^3}{2}\right) = q\,\underline{E}\cdot\underline{v}_\perp$$

then, averaging over 1 cyclotron orbit:

$$\delta\left(\frac{m v_\perp^2}{2}\right) = \int_{\Omega^{-1}} dt\, q\,\underline{E}\cdot\underline{v}$$

$$= \oint_{\rho} d\underline{\ell}\cdot\underline{E}\, q$$

$$= \int_{\rho} d\underline{a}\cdot q\, \nabla\times\underline{E}$$

$$= \int_{\rho} d\underline{a}\cdot\left(\frac{+q}{c}\frac{\partial \underline{B}}{\partial t}\right)$$

$$= \pi \rho_L^2 \left(\frac{+q}{c}\right)\frac{\partial \underline{B}}{\partial t}$$

~~sign → contour reversal particle motion~~

$$\therefore \quad \underset{\substack{\downarrow \\ \text{change in} \\ \text{cyclotron period}}}{\delta}\left(\frac{mv_\perp^2}{2}\right) = +\pi \frac{q}{c} \frac{v_\perp^2}{\frac{q^2B^2}{m^2c^2}} \frac{\partial B}{\partial t}$$

$$= +\frac{mv_\perp^2}{\Omega} \frac{\pi}{B} \frac{\partial B}{\partial t}$$

but, $\underset{\substack{\downarrow \\ \text{change} \\ \text{in one period}}}{\delta B} = +\frac{2\pi}{\Omega} \frac{\partial B}{\partial t} \equiv \delta t \frac{\partial B}{\partial t}$

$$\Rightarrow \quad \delta\left(\frac{mv_\perp^2}{2}\right) = \frac{-mv_\perp^2}{2} \frac{1}{B} \frac{\partial B}{\partial t}$$

$$\therefore \quad \boxed{\delta\left(\frac{mv_\perp^2}{2B}\right) = 0} \longrightarrow$$

magnetic moment invariant on $t \gg \Omega^{-1}$ (adiabatic inv.)

Now:

→ obviously μ not invariant on time scales Ω^{-1} $\Longleftrightarrow$ averaging!

→ re: mirroring, if $B(\ell) \gg B(0)$

$$\Rightarrow v_\perp^2(\ell) \gg v_\perp^2(0)$$

$$\therefore v_\parallel^2(\ell) \ll v_\parallel^2(0)$$

$$V_\parallel^2(0) + V_\perp^2(0) = V_\parallel^2(\ell) + V_\perp^2(\ell)$$

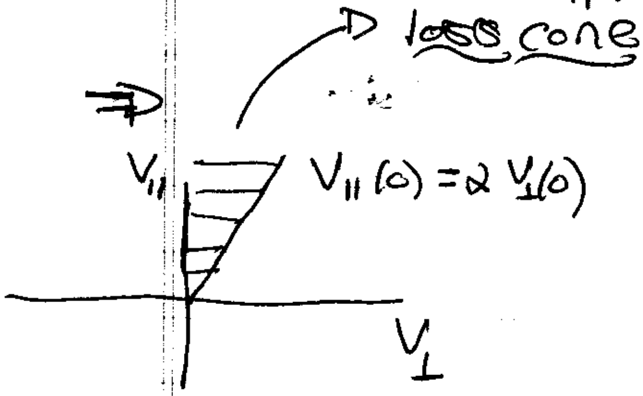
(energy)

$$\frac{m V_\perp^2(0)}{2B(0)} = \frac{m V_\perp^2(\ell)}{2B(\ell)}$$

$$\Rightarrow \quad V_\perp^2(\ell) = \frac{B(\ell)}{B(0)} V_\perp^2(0)$$

$$\Rightarrow \quad V_\perp^2(0)\left(1 - \frac{B(\ell)}{B(0)}\right) + V_\parallel^2(0) = V_\parallel^2(\ell)$$

Confinement $\Rightarrow V_\parallel^2(\ell) = 0$



$$\Rightarrow \quad \frac{V_\parallel^2(0)}{V_\perp^2(0)} \leqq \left(\underbrace{\frac{B(\ell)}{B(0)}}_{\text{mirror ratio}} - 1\right)$$

$$\alpha = \left(\frac{B(\ell)^2}{B(0)^2} - 1\right)^{1/2}$$

<u>i.e.</u>

→ some class of particles always lost

→ distribution of survivors has loss cone 'hole' in it.

→ $V_\parallel = \left[\frac{2}{m}(E - \mu B)\right]^{1/2}$

Adiabatic Invariants, cont'd.

As adiabatic invariants, such as μ, etc., clearly are quite useful, its worthwhile to develop general theory.

Thus consider:

- finite 1D motion
- system/external field characterized by $\lambda(t)$ parameter s/t

$$\frac{1}{\lambda}\frac{d\lambda}{dt} \ll \underset{\text{freq.}}{\underset{\downarrow}{\omega}}$$

$\therefore$ - E not conserved, but

- $\dot{E} \sim \dot{\lambda}$ $\Rightarrow$ suggests some (linear) combination E, λ s/t combo invariant

Now, consider $H = H(p, q, \lambda)$

$$\frac{dE}{dt} = \frac{\partial H}{\partial t} = \frac{\partial H}{\partial \lambda}\frac{d\lambda}{dt}$$

as λ varies slowly, by construction

$$\frac{d\overline{E}}{dt} = \overline{\frac{\partial H}{\partial \lambda}}\,\frac{d\lambda}{dt}$$

where $\bar{\ } \equiv \frac{1}{\tau}\int_0^{\tau} dt$ $\qquad \tau = \frac{2\pi}{\omega}$

Noting $\tau = \oint dq \big/ dq/dt$

path for given particular λ

$$= \oint dq \big/ (\partial H/\partial p)$$

⇒

$$\frac{d\bar{E}}{dt} = \frac{d\lambda}{dt}\,\frac{\oint (\partial H/\partial \lambda)\,dq \big/ (\partial H/\partial p)}{\oint dq \big/ (\partial H/\partial p)}$$

where path at fixed λ.

Now, at such at path, H fixed and E constant. Thus, on such a path

$$p = p(q;\ \underbrace{E, \lambda})$$

↳ indep, const param

and $\frac{d}{d\lambda}\left(H(p,q;\lambda) = E\right)$

$$\Rightarrow \quad \frac{\partial H}{\partial \lambda} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial \lambda} = 0$$

so $$\frac{\partial H/\partial \lambda}{\partial H/\partial p} = -\frac{\partial p}{\partial \lambda}$$

and plugging into $\dot{E}$ ⇒

$$\frac{dE}{dt} = \frac{d\lambda}{dt}\,\frac{\oint dq\,(-\partial p/\partial \lambda)}{\oint dq\,(\partial p/\partial E)}$$

⇒

$$\oint dq\left(\frac{\partial p}{\partial E}\frac{dE}{dt} + \frac{\partial p}{\partial \lambda}\frac{d\lambda}{dt}\right) = 0$$

for $p = p(E, \lambda)$ ⇒

$$\frac{d}{dt}\oint dq\, p = 0$$

Thus: - $\underset{(J)}{I} = \oint dq\, p$, for fixed $\begin{pmatrix} E \\ \lambda \end{pmatrix}$ is invariant (corresponds action variable)

- as E, λ 'fixed' ⇒ I invariant for $t \gg 2\pi/\omega$ s/t $\frac{1}{\lambda}\frac{d\lambda}{dt} \ll \omega$

Useful to note:

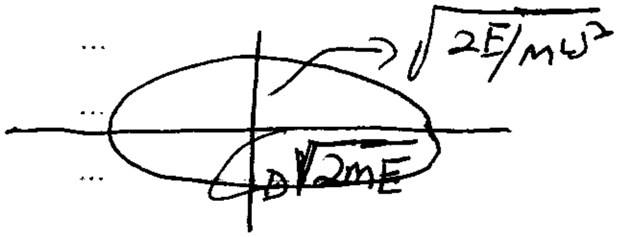
$$\rightarrow \quad I = \oint dq\, p$$

$$= \iint dp\, dq / 2\pi$$

$\Rightarrow$ $\left[\right.$ I is <u>area</u> enclosed by phase path of closed p, q curve

e.g. oscillator:

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2 = E$$



$$A = \pi a b$$

$$\therefore \quad I = \frac{A}{2\pi} = E/\omega \quad \rightarrow \underline{\text{action}}$$

$\downarrow$
pendulum example.

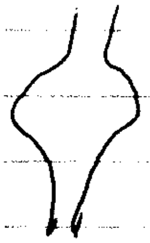
i.e. $E = I\omega$

or $\dot{E} = I \dot{\omega} \quad \Rightarrow \quad \dot{E} = I \frac{\partial \omega}{\partial \lambda} \dot{\lambda}$

where recover invariance of linear proportionality.

For waves
$$\begin{cases} N = \mathcal{E}/\omega \\ \mathcal{E} = \mathcal{E}(\underline{k}, \underline{x}, t) \end{cases}$$
is useful adiabatic invariant.

Now, returning to mirror problem



→ for μ, first invariant:

$$J = \oint \underline{p} \cdot d\underline{q} \quad , \quad \text{for cyclotron motion}$$

Now, in B field $\quad \underline{p} = m\underline{v} + \frac{q\underline{A}}{c}$

so

$$J_1 = \oint \left(m\underline{v} + \frac{q\underline{A}}{c}\right) \cdot d\underline{q}$$

now, $\quad A(\underline{r}) = A(\underline{x}_{gc}) + \underline{\rho} \cdot \underline{\nabla} A + \ldots\ldots$

as orbit is cyclotron

$$d\underline{q} = \frac{\partial \underline{\rho}}{\partial \alpha}\, d\alpha$$

$$\underline{\rho} = \frac{v_\perp}{\Omega}$$

$$\Rightarrow \quad J_1 = \oint \frac{d\alpha}{\Omega}\, \underline{v}_\perp \cdot \left(m\underline{v}_\perp + \frac{q}{c}\underline{A}(\underline{B}) + \underline{\rho}\cdot\underline{\nabla}\,\underline{A} + \cdots \right)$$

n.b. ∥ motion, drifts etc. vanish on integration

$$\therefore \quad J_1 = 2\pi m \frac{v_\perp^2}{\Omega} + 2\pi \frac{q}{c\Omega} \left\langle \underline{v}_\perp \cdot (\underline{\rho}\cdot\nabla\underline{A}) \right\rangle$$

$$= 2\pi \frac{m v_\perp^2}{\Omega} - \frac{2\pi q}{c\Omega}\,\frac{v_\perp^2}{2\Omega}\,\underline{b}\cdot\underline{\nabla}\times\underline{A}$$

$$= 2\pi \frac{mc}{q}\left(\frac{m v_\perp^2}{B}\right) - 2\pi \frac{mc}{q}\left(\frac{m v_\perp^2}{B}\right)$$

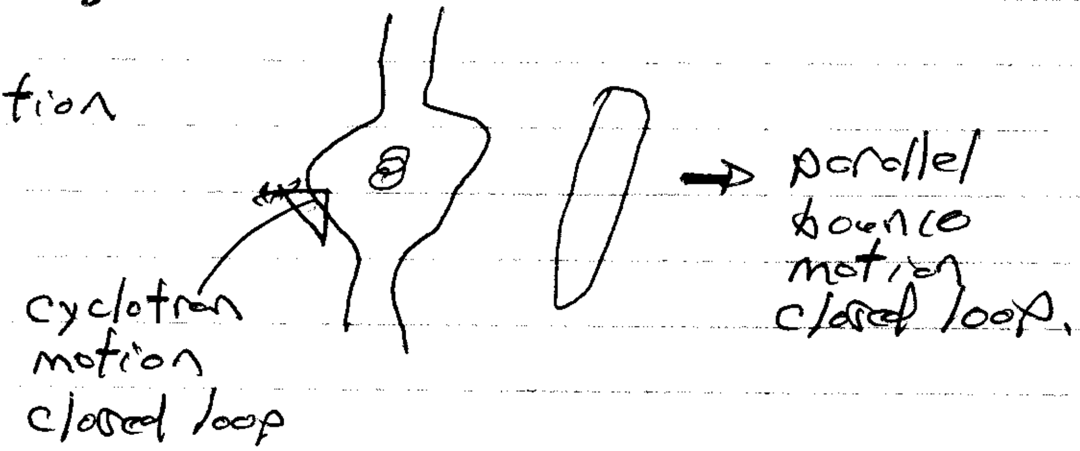
$$\boxed{J_1 = \frac{2\pi mc}{q}\left(\frac{m v_\perp^2}{2B}\right)}$$

$$\therefore \quad J_1 = (\text{const.}) \times \mu .$$

recovers J_1 ✓

Now, for $\overline{J}_2$:

consider motion



So $$\overline{J}_2 = \oint p_{\parallel}\, dS_{\parallel}$$

traditional notation.

$$v_\parallel^2(0) + v_\perp^2(0) = v_\parallel^2(\ell) + v_\perp^2(\ell)$$

$$\frac{v_\perp^2(0)}{B(0)} = \frac{v_\perp^2(\ell)}{B(\ell)} = \mu(2m)$$

$$\Rightarrow \quad v_\parallel^2(\ell) = v_\parallel^2(0) + v_\perp^2(0) - \mu B(\ell)$$
$$= 2m(E - \mu B(\ell))$$

$$\therefore \quad \boxed{J_2 = \oint d\ell\, \left(2m(E - \mu B(\ell))\right)^{1/2}}$$

= 'bounce' invariant

= note $\nabla \cdot B = 0$ forces $B_r \neq 0$

i.e.

$$\nabla_r B_r + \underset{\neq 0}{\partial_z B_z} = 0$$

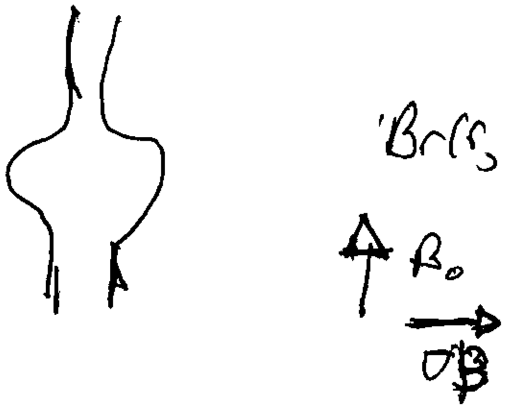
$\Rightarrow B_r \neq 0$

∴ - J_2 conservation $\Leftrightarrow$ conservation of magnetic flux thru G.C. bounce orbit

- for motion on field line, conservation of zero!

for J_3: what is 3rd loop?

i.e.



$B_r(r,z) \Rightarrow \nabla B$ drift!

$\Rightarrow$ azimuthal drift around machine/mirror ✓

i.e. $J_3 = \oint \underline{P_d} \cdot \underline{x_d}$

N.B. time scale hierarchy clear.

→ Systematics ?

- How determine <u>most</u> <u>general</u> adiabatic invariant ?

- clues – additive

  <u>i.e.</u> $$J = \oint P_{cyc} \cdot dx_{cyc} + \underbrace{\oint P_{\parallel} dl_{\parallel} + \oint \underline{P_d} \cdot \underline{x_d}}_{G.C.}$$

  – time scale hierarchy :

  $J_1 \rightarrow \tau_{cycl.}$

  $J_2 \rightarrow \tau_{bounce}$

  $J_3 \rightarrow \tau_{drift}$

  ⇒ suggests J_2 corrects J_1, etc.

also ⇒ why care so much ?, ?

Now, most general adiabatic invariant is invariant along particle orbits

<u>i.e.</u> $$\frac{dI}{dt} = 0$$

in absence of E fields

$$\underline{v}\cdot\underline{\nabla} I + \frac{q}{mc}(\underline{v}\times\underline{B})\cdot\nabla I = 0$$ (Vlasov eqn.)

$$I = I_0 + \epsilon I_1 + \epsilon^2 I_2 + \ldots$$

Now – $\underline{v} = v_\parallel \hat{n} + v_\perp \cos\phi\, \underline{e} + v_\perp \sin\phi\, \underline{e}'$

$$v_\parallel = \left(\frac{2}{m}(E-\mu B)\right)^{1/2}$$

$$v_\perp = (2\mu B/m)^{1/2}$$

– take $\underline{B} = \underline{\nabla}\alpha \times \nabla\underline{\beta}$

i.e. $l \rightarrow$ along B

$\alpha, \beta \rightarrow \perp$

Now, ① can see ② in these coordinates:

$$\Omega \frac{\partial I}{\partial \phi} = \underset{\substack{\downarrow \\ \text{complicated mess}}}{\mathcal{D}} I \qquad \frac{②}{①} \sim O(\epsilon)$$

$$I = I_0 + \epsilon I_1 + \epsilon^2 I_2 + \ldots$$

$$\oint d\phi\, \mathcal{D} \equiv \langle \mathcal{D} \rangle = v_\parallel \frac{\partial}{\partial l}$$

so ⇒ l.o. $\Omega \frac{\partial I_0}{\partial \phi} = 0$

$$\Rightarrow I_0 = I_0(\underbrace{\mu, E,}_{\text{ident. as } J \text{ l.o. ad. inv.}} \underbrace{\alpha, \beta, l}_{\text{location in field structure}}) \rightarrow \text{any fnctn const on } \Omega^{-1}$$

1st order:

$$\Omega \frac{\partial I_1}{\partial \phi} = \mathcal{D} I_0$$

and $\oint d\phi$ ⇒

$$\langle \mathcal{D} I_0 \rangle = 0$$

$$\Rightarrow v_\parallel \frac{\partial I_0}{\partial l} = 0$$

$$\Rightarrow I_0 = I_0(\mu, E, \alpha, \beta)$$

and for I_1:

$$I_1 = \frac{1}{\Omega}\int^{\phi} d\phi\, \mathcal{D} I_0 + \bar{I}_1(E,\mu,\alpha,\beta,\ell)$$

⇒ Next order:

$$\Omega \frac{\partial I_2}{\partial \phi} = \mathcal{D} I_1$$

$$\Rightarrow \langle \mathcal{D} I_1 \rangle = \langle \mathcal{D} \bar{I}_1 \rangle + \langle \mathcal{D} \frac{1}{\Omega}\int \mathcal{D} I_0 \rangle = 0$$

evaluting I_0 (messy) ⇒

$$\langle \mathcal{D} I_1 \rangle = v_{\parallel} \frac{\partial}{\partial \ell} \bar{I}_1 + (V_d \cdot \nabla I_0) = 0$$

so now can integrate in ℓ such:

$$\boxed{\oint \frac{d\ell}{v_{\parallel}} \underline{V}_d \cdot \nabla I_0 = 0}$$

etc.

⇒ what does above mean?

$B = \underline{\nabla \psi}$ s/t $\nabla\alpha \cdot \nabla\psi = \nabla\beta \cdot \nabla\psi = 0$

and

$$\frac{R_c}{R_c} = \frac{\nabla_\perp B}{B}$$

plugging in:

$$\oint \frac{dx}{B^2}\, \underline{B} \times \nabla\left(\frac{V_\parallel}{B}\right) \cdot \nabla I_0 = 0$$

$$B = \underline{\nabla\alpha} \times \underline{\nabla}B$$

$$\underline{\nabla} = \nabla\alpha \frac{\partial}{\alpha} + \underline{\nabla}\beta \frac{\partial}{\partial\beta} + \nabla\chi \frac{\partial}{\partial\chi}$$

$$\Rightarrow \frac{\partial I_0}{\partial\alpha} \frac{\partial}{\partial\beta} J - \frac{\partial I_0}{\partial\beta} \frac{\partial J}{\partial\alpha} = 0$$, where

$$J = \oint d\chi \frac{V_\parallel}{B} = \oint V_\parallel dl$$

so $I_0 = I_0(\mu, E, \underset{\hookrightarrow J_2}{J})$.

Now if J_2 exists; $\Rightarrow$

$$J_2 = J_2(E, \mu, \alpha, \beta)$$

so for long times, particle **must** drift on surface described by:

$J(\mu, E, \alpha, \beta) = \text{const.}$

Thus, adiabatic invariants give insight into motion without need to integrate eqns. of motion.

Drift/Gyrokinetics

- now, Vlasov eqn ⇒ phase space density conserved along particle orbits

i.e. $$\frac{\partial f}{\partial t} + \underbrace{\underline{v}\cdot\underline{\nabla} f}_{\frac{dx}{dt}} + \overbrace{\frac{q}{m}\left(\underline{E} + \frac{\underline{v}\times\underline{B}}{c}\right)}^{\frac{d\underline{v}}{dt}}\cdot\underline{\nabla} f = 0$$

- but in real plasmas,

  orbits complex ⇒ drifts, etc.

∴

- convenient to work with phase space density of <u>guiding centers</u> ⇒ drift kinetics.

  ⇒ useful for phenomena with $\omega \ll \Omega$, $k\rho \ll 1$.

∴ can, in a sense, 'write down' DKE for particles in $B_0 = B_0\hat{z}$ and electrostatic fluctns using drift eqns as characteristics.

i.e. $\frac{dz}{dt} = v_z$ $\qquad \frac{d\underline{x}_\perp}{dt} = \frac{c}{B} \underline{E} \times \hat{z}$

$\frac{dv_z}{dt} = \frac{q}{m} E_z$

$$\therefore \frac{\partial f}{\partial t} + \frac{\partial}{\partial z}\left(\frac{dz}{dt} f\right) + \frac{\partial}{\partial \underline{x}_\perp} \cdot \left(\left[\frac{c}{B_0} \underline{E} \times \hat{z}\right] f\right)$$

$$+ \frac{\partial}{\partial v_z}\left(\frac{q}{m} E_z f\right) = 0$$

$\underline{\nabla}_\perp \cdot \underline{V}_{E\times B} = 0 \Rightarrow$

$$\boxed{\frac{\partial f}{\partial t} + v_z \frac{\partial f}{\partial z} + \frac{c}{B} \underline{E} \times \hat{z} \cdot \underline{\nabla} f + \frac{q}{m} E_z \frac{\partial f}{\partial v_z} = 0}$$

- drift kinetic equation

- useful, with additions in describing electron ($k_\perp \rho_e \to 0$) behavior in MHD, low frequency modes.

→ Now, for $\omega \ll \Omega$ but $k_\perp \rho$ small/finite but non-zero

⇒ gyro-kinetic equation.

i.e.

- modify DKE for finite $k_\perp \rho$
- derive systematically, via time averaging,

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Linearized gyro-kinetics

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Abstract—Finite gyroradius effects are retained in a far simpler manner than previous treatments by transforming to the guiding center variables and gyro-averaging *before* introducing magnetic coordinates.

MANY INSTABILITIES of interest in present and future magnetic confinement devices depend sensitively on finite gyroradius effects. As a result, it is important to develop techniques which *simply* retain finite gyroradius effects in complicated magnetic fields. When the time variation of the waves is slow compared to the gyrofrequency and the gyroradius small compared to unperturbed scale lengths, then gyro-kinetic techniques may be employed to retain gyro-effects for arbitrary values of the gyroradius over the perpendicular wavelength. Unlike drift kinetic descriptions, gyro-kinetic techniques retain finite gyroradius effects to lowest order in the equation for the perturbed or linearized distribution function.

The original gyro-kinetic work of RUTHERFORD and FRIEMAN (1968) and TAYLOR and HASTIE (1968) considers general geometries but employs a WKB or eikonal assumption for the spatial variation of the electrostatic potential Φ. Later work (JAMIN, 1971; CONNOR and HASTIE, 1975; and NEWBERGER, 1976) also employs the eikonal ansatz, with CONNOR and HASTIE being the first to employ a form for Φ satisfying both poloidal and toroidal periodicity constraints for axisymmetric geometries with finite magnetic shear. Later work by CATTO and TSANG (1977) attempts to remove the WKB assumption by employing a concentric magnetic surface model (KADOMSTEV and POGUTSE, 1969 and 1967).

In all of the preceding work magnetic coordinates were introduced prior to making the transformation to the guiding center variables. It is the transformation from the particle variables to the guiding center variables which permits finite gyroradius effects to be retained in lowest order. The present treatment avoids the substantial mathematical complications inherent in these prior treatments by introducing the transformation to the guiding center variables and performing the guiding center gyrophase average *before* specifying the magnetic coordinates to be employed. In this way the unperturbed, gyro-averaged Vlasov operator which retains finite gyro-effects is obtained in the most convenient manner for arbitrary unperturbed magnetic fields.

After the transformation has been performed, the magnetic coordinates can be introduced with relative simplicity in order to obtain the appropriate gyro-averaged Φ. The magnetic variables are only necessary to evaluate the gyro-average of the inhomogeneous term in the linearized Vlasov equation. In order to evaluate the inhomogeneous term a mode structure for Φ is required. To illustrate the technique to completion a Tokamak geometry is considered. A WKB or eikonal ansatz for the poloidal mode structure is avoided by employing a poloidal angle variable so that the poloidal variation can be strictly Fourier decomposed (TANG *et al.*, 1977).

A gyro-kinetic description is obtained by employing the guiding center variables $\mathbf{R}$, $\mathbf{E}$, μ, and ϕ where the guiding center variables are related to the original particle variables $\mathbf{r}$ and $\mathbf{v}$ via

$$\begin{aligned}\mathbf{R} &= \mathbf{r} + \Omega^{-1}\mathbf{v}\times\hat{\mathbf{n}},\\ \mathbf{v} &= v_\parallel\hat{\mathbf{n}} + (\hat{\boldsymbol{\psi}}\cos\phi + \hat{\mathbf{e}}\sin\phi) = v_\parallel\hat{\mathbf{n}} + v_\perp\hat{\mathbf{v}}_\perp,\end{aligned} \tag{1}$$

with $E = v^2/2$, $\mu = {v_\perp}^2/2B$, ${v_\parallel}^2 = 2(E-\mu B)$, ${v_\perp}^2 = [(\mathbf{I}-\hat{\mathbf{n}}\hat{\mathbf{n}})\cdot\mathbf{v}]^2$, $B = |\mathbf{B}|$, $\hat{\mathbf{n}} = \mathbf{B}/B$, $\Omega = ZeB/Mc$, $v = |\mathbf{v}|$, and the unit vectors $\hat{\boldsymbol{\psi}}$, $\hat{\mathbf{e}}$, and $\hat{\mathbf{n}}$ forming an orthogonal system in which $\hat{\mathbf{e}} = \hat{\mathbf{n}}\times\hat{\boldsymbol{\psi}}$. The quantities Z and M are the species charge number and mass; e and c are the magnitude of the charge on an electron and the speed of light. Unlike previous treatments, magnetic coordinates have not been explicitly introduced, thereby drastically simplifying the analysis that follows.

Changing to the guiding center variables, $\partial/\partial t \rightarrow \partial/\partial t$ while

$$\begin{aligned}\nabla &\rightarrow \nabla_R - [\nabla(\Omega^{-1}\hat{\mathbf{n}})\times\mathbf{v}]\cdot\nabla_R + \nabla\phi\frac{\partial}{\partial\phi} + \nabla\mu\frac{\partial}{\partial\mu},\\ \nabla_v &\rightarrow \nabla_V + \Omega^{-1}\mathbf{I}\times\hat{\mathbf{n}}\cdot\nabla_R,\end{aligned} \tag{2}$$

with $\nabla_V = \mathbf{v}\partial/\partial E + \mathbf{v}_\perp B^{-1}\partial/\partial\mu + \hat{\boldsymbol{\phi}} v_\perp^{-1}\partial/\partial\phi$, $\hat{\boldsymbol{\phi}} = \hat{\mathbf{n}}\times\hat{\mathbf{v}}_\perp$, $\mathbf{I}$ the unit dyadic, and

$$\nabla\phi = (v_\parallel/v_\perp)[-(\nabla\hat{\mathbf{n}})\cdot\hat{\mathbf{e}}\cos\phi + (\nabla\hat{\mathbf{n}})\cdot\hat{\boldsymbol{\psi}}\sin\phi] + (\nabla\hat{\mathbf{e}})\cdot\hat{\boldsymbol{\psi}} \tag{3}$$

$$\nabla\mu = -(\mu/B)\nabla B - (v_\perp v_\parallel/B)[(\nabla\hat{\mathbf{n}})\cdot\hat{\boldsymbol{\psi}}\cos\phi + (\nabla\hat{\mathbf{n}})\cdot\hat{\mathbf{e}}\sin\phi]. \tag{4}$$

Equations (3) and (4) are obtained by operating on $\hat{\boldsymbol{\psi}}\cdot\mathbf{v} = v_\perp\cos\phi$ and $\hat{\mathbf{e}}\cdot\mathbf{v} = v_\perp\sin\phi$ with ∇ and ∇_v and forming the appropriate combinations.

The unperturbed Vlasov operator in the guiding center variables is obtained from the change of variables (1) by noting that $(Ze/Mc)\mathbf{v}\times\mathbf{B}\cdot\nabla_v = -\Omega\partial/\partial\phi$ and $(\Omega\mathbf{v}\times\hat{\mathbf{n}})\cdot(\Omega^{-1}\mathbf{I}\times n\cdot\nabla_R) = -\mathbf{v}_\perp\cdot\nabla_R$ so that

$$\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + (Ze/Mc)\mathbf{v}\times\mathbf{B}\cdot\nabla_v \to -\Omega\frac{\partial}{\partial\phi} + \frac{\partial}{\partial t} + v_\parallel\hat{\mathbf{n}}\cdot\nabla_R + \mathbf{v}\cdot\left[\nabla\phi\frac{\partial}{\partial\phi} + \nabla\mu\frac{\partial}{\partial\mu} - \nabla(\Omega^{-1}\hat{\mathbf{n}})\times\mathbf{v}\cdot\nabla_R\right]. \tag{5}$$

For time variations slow compared to the gyromotion the dominant term in (5) is $-\Omega\partial/\partial\phi$ so that to lowest order the variable operated on by (5), usually a portion of distribution function, must be independent of ϕ. When the quantity being operated on is independent of the guiding center gyrophase, the unperturbed Vlasov operator may be gyro-averaged by employing $\mathbf{v}\cdot\nabla\phi\partial/\partial\phi \to 0$,

$$(2\pi)^{-1}\oint d\phi\,\mathbf{v}\mathbf{v} = (v_\perp{}^2/2)(\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}) + v_\parallel{}^2\hat{\mathbf{n}}\hat{\mathbf{n}},$$

$$\mathbf{v}_d = -(2\pi)^{-1}\oint d\phi\{\mathbf{v}\cdot[\nabla(\Omega^{-1}\hat{\mathbf{n}})\times\mathbf{v}]\cdot(\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}})\} = \mathbf{n}\times[(\mathbf{v}_\perp{}^2/2\Omega)\nabla\ln B + (v_\parallel{}^2/\Omega)\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}}] \tag{6}$$

$$u_\parallel = -(2\pi)^{-1}\oint d\phi\{\mathbf{v}\cdot[\nabla(\Omega^{-1}\hat{\mathbf{n}})\times\mathbf{v}]\cdot\mathbf{n}\} = -(v_\perp{}^2/2\Omega)\hat{\mathbf{n}}\cdot\nabla\times\hat{\mathbf{n}}, \tag{7}$$

and $(2\pi)^{-1}\oint d\phi\,\mathbf{v}\cdot\nabla\mu \approx 0$ to obtain

$$(2\pi)^{-1}\oint d\phi\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \left(\frac{Ze}{Mc}\right)\mathbf{v}\times\mathbf{B}\cdot\nabla_v\right] \to \frac{\partial}{\partial t} + (v_\parallel\hat{\mathbf{n}} + \mathbf{v}_d)\cdot\nabla_R. \tag{8}$$

In writing (8), the parallel velocity correction (HAZELTINE, 1973) $u_\parallel$ must be neglected because it is the same order as the gyroradius over scale length corrections to $\oint d\phi\,\mathbf{v}\cdot\nabla\mu$. The curvature and ∇B drifts, $\mathbf{v}_d$, are retained because perpendicular wavelengths are assumed small compared to parallel wavelengths which are of the order of unperturbed scale lengths. It should be noted that the variable operated on by (8) is a function of $\mathbf{R}$, E, and μ but not ϕ. Once this variable has been solved for in terms of the guiding center location $\mathbf{R}$ one must revert to the original particle variable $\mathbf{r} = \mathbf{R} - \Omega^{-1}\mathbf{v}\times\hat{\mathbf{n}}$, thereby introducing particle gyrophase dependence via the $\mathbf{v} = v_\parallel\hat{\mathbf{n}} + v_\perp(\hat{\boldsymbol{\psi}}\cos\phi + \hat{\mathbf{e}}\sin\phi)$. Because only the lowest order ϕ dependence is needed, distinctions between particle and guiding center gyrophase, energy, and magnetic moment are not required.

It should be noted that the gyro-kinetic technique outlined in the preceding paragraphs has not required that the magnetic coordinates be identified before carrying out the change of variables and gyro-average. The magnetic coordinates need only be introduced when it becomes necessary to evaluate the gyro-average of the inhomogeneous term in the linearized Vlasov equation. The technique is illustrated in the following paragraphs for a Tokamak geometry.

The unperturbed distribution function F_0 is taken to be a function of v^2 and the canonical angular momentum $(-Ze/c)\psi_0$, with $\psi_0 = \psi - (cMR/Ze)\hat{\boldsymbol{\zeta}}\cdot\mathbf{v}$, such that

$$F_0 = F_0(\psi_0, v^2) = N(\psi_0)[M/2\pi T(\psi_0)]^{3/2}\exp[-Mv^2/2T(\psi_0)]$$
$$\approx F_M\left\{1 - \hat{\boldsymbol{\zeta}}\cdot\mathbf{v}\frac{McR}{ZeN}\frac{\partial N}{\partial\psi}\left[1 + \eta\left(\frac{Mv^2}{2T} - \frac{3}{2}\right)\right]\right\}, \tag{9}$$

with $\eta = d\ln T/d\ln N$, $F_M \equiv F_0(\psi, v^2)$, and $N(\psi)$ and $T(\psi)$ the density and temperature as a function of poloidal flux ψ. The second form of (9) follows from a Taylor expansion of F_0 about $\psi_0 = \psi$, and ζ and R are the toroidal angle variable and the distance from the axis of symmetry to the point of interest ($|\nabla\zeta| = 1/R$).

Defining the non-adiabatic portion of the perturbed distribution function g in terms of the perturbed distribution function f via $g = f + (ZeF_0/T)\Phi$, the linearized Vlasov equation may be written as

$$\frac{\partial g}{\partial t} + \mathbf{v}\cdot\nabla g + \frac{Ze}{Mc}\mathbf{v}\times\mathbf{B}\cdot\nabla_v g = \frac{Ze}{T}F_0\frac{\partial\Phi}{\partial t} - c\frac{\partial F_0}{\partial\psi_0}\frac{\partial\Phi}{\partial\zeta}. \tag{10}$$

Employing the gyro-kinetic variables of (1) and considering time variations slow compared to a gyro-period, (10) yields $\partial g/\partial\phi = 0$ to lowest order. As a result, applying the results of the gyro-kinetic section to the next order equation for g results in the gyro-averaged equation for g.

$$\frac{\partial g}{\partial t}+(v_\parallel\hat{\mathbf{n}}+\mathbf{v}_d)\cdot\nabla_R g=\Omega\oint\frac{\mathrm{d}\phi}{2\pi\Omega}\left[\frac{ZeF_0}{T}\frac{\partial\Phi}{\partial t}(\mathbf{r},t)_{-c}\frac{\partial F_0}{\partial\psi_0}\frac{\partial\Phi}{\partial\zeta}(\mathbf{r},t)\right]. \tag{11}$$

In order to simplify the right-hand side of (11), the magnetic coordinates ψ, Θ, ζ are introduced, where $-\pi<\Theta\leq\pi$ is a poloidal angle variable. The potential $\Phi(\mathbf{r}, t)$ is assumed to be of the form

$$\Phi(\mathbf{r},t)=(2\pi)^{-2}\sum_{l,m}\int_{-\infty}^{\infty}\mathrm{d}\kappa\int_L\mathrm{d}\omega\Phi_{lm}(\kappa,\omega)\exp(-i\omega t+i\kappa\psi+im\Theta-il\zeta), \tag{12}$$

where L is the Landau contour (above all singularities in the complex plane). Away from the magnetic axis a change of variables from $\mathbf{R}$ to ψ', Θ', ζ' where $\psi'=\psi+\Omega^{-1}\mathbf{v}\times\hat{\mathbf{n}}\cdot\nabla\psi$, $\Theta'=\Theta+\Omega^{-1}\mathbf{v}\times\hat{\mathbf{n}}\cdot\nabla\Theta$, and $\zeta'=\zeta+\Omega^{-1}\mathbf{v}\times\hat{\mathbf{n}}\cdot\nabla\zeta$ (JAMIN, 1971 and NEWBERGER, 1976) gives $\exp i(\kappa\psi+m\Theta-l\zeta)=\exp[i(\kappa\psi'+m\Theta'-l\zeta')-i\Omega^{-1}\mathbf{v}\times\hat{\mathbf{n}}\cdot\mathbf{k}]$ with $\mathbf{k}=\kappa\nabla\psi+m\nabla\Theta-l\nabla\zeta$ and $\psi_0=\psi'+(cMR/Ze)\hat{\zeta}\cdot\mathbf{v}-\Omega^{-1}\mathbf{v}\times\hat{\mathbf{n}}\cdot\nabla\psi$. Expanding $F_0(\psi_0, v^2)$ and $\partial F_0(\psi_0, v^2)/\partial\psi_0$ about $\psi_0=\psi'$, neglecting gyroradius over unperturbed scale length corrections and employing $(2\pi)^{-1}\oint\mathrm{d}\phi\exp(-i\Omega^{-1}\mathbf{k}\cdot\mathbf{v}\times\mathbf{n})\approx J_0(k_\perp v_\perp/\Omega)$ so that

$$\begin{aligned}\bar{\Phi}&\equiv(2\pi)^{-1}\oint\mathrm{d}\phi\Phi(\mathbf{r},t)\\&=(2\pi)^{-2}\sum_{l,m}\int_{-\infty}^{\infty}\mathrm{d}\kappa\int_L\mathrm{d}\omega\Phi_{lm}J_0(k_\perp v_\perp/\Omega)\exp(-i\omega t+i\kappa\psi'+im\Theta'-il\zeta')\end{aligned} \tag{13}$$

with $k_\perp=|\hat{\mathbf{n}}\times(\mathbf{k}\times\hat{\mathbf{n}})|$, then (11) becomes

$$\frac{\partial g}{\partial t}+(v_\parallel\hat{\mathbf{n}}+\mathbf{v}_d)\cdot\nabla_R g=\frac{ZeF_M}{T}\left\{\frac{\partial\bar{\Phi}}{\partial t}-\frac{cT}{ZeN}\frac{\partial N}{\partial\psi'}\left[1+\eta\left(\frac{Mv^2}{2T}-\frac{3}{2}\right)\right]\frac{\partial\bar{\Phi}}{\partial\zeta'}\right\}, \tag{14}$$

where all unperturbed quantities are functions of the primed variables.

In general (14) must be solved by integrating along the unperturbed guiding center trajectories. In order to further illustrate the treatment of the gyromotion effects, $v_\parallel\hat{\mathbf{n}}+\mathbf{v}_d$ will be assumed to be independent of ψ' and Θ'. This approximation is the one usually employed in evaluating the ion response for the trapped electron and collisionless drift instabilities. For $v_\parallel\hat{\mathbf{n}}+\mathbf{v}_d$ a constant and neglecting the ψ dependence of the right side of (14), g may be taken to be

$$g=(2\pi)^{-2}\sum_{l,m}\int_{-\infty}^{\infty}\mathrm{d}\kappa\int_L\mathrm{d}\omega g'_{lm}(\kappa,\omega)\exp(-i\omega t+i\kappa\psi'+im\Theta'-il\zeta'). \tag{15}$$

Employing (13) and (15), equation (14) may be transformed to obtain

$$g_{lm}'(\kappa,\omega)=\frac{(ZeF_M/T)(\omega-\omega_*{}^T)J_0(k_\perp v_\perp/\Omega)}{\omega-\mathbf{k}\cdot(v_\parallel\hat{\mathbf{n}}+\mathbf{v}_d)}\Phi_{lm}(\kappa,\omega) \tag{16}$$

where $\omega_*{}^T(\psi)=(lcT/ZeN)(\partial N/\partial\psi)\{1+\eta[(Mv^2/2T)-(3/2)]\}$. Reverting to the unprimed variables ψ, Θ, ζ by using (15),

$$g=(2\pi)^{-2}\sum_{l,m}\int_{-\infty}^{\infty}\mathrm{d}\kappa\int_L\mathrm{d}\omega g_{lm}(\kappa,\omega)\exp(-i\omega t+i\kappa\psi+im\Theta-il\zeta), \tag{17}$$

with $g_{lm}(\kappa,\omega)=g_{lm}'(\kappa,\omega)\exp(i\Omega^{-1}\mathbf{v}\times\hat{\mathbf{n}}\cdot\mathbf{k})$. The g from (17) is in terms of particle variables while that from (15) is in terms of the guiding center variables. The gyrophase dependence $\exp(i\Omega^{-1}\mathbf{k}\cdot\mathbf{v}\times\hat{\mathbf{n}})$ in $g_{lm}(\kappa,\omega)$ enters as it does when a trajectory integral over the full gyromotion is performed. Using $f_{lm}=-(ZeF_M/T)\Phi_{lm}+g_{lm}$ to form the perturbed density $n_{lm}=\int\mathrm{d}^3vf_{lm}$ and employing ϕ, $v_\perp$, $v_\parallel$ variables for the velocity integrations results in

$$n_{lm}=-\frac{ZeN}{T}\Phi_{lm}\left[1+\frac{\Gamma_0}{|k_\parallel|\,v_T}\left(\left\{\omega-\omega_*\left[1-\eta\left(\frac{1}{2}+b-\frac{b\Gamma_1}{\Gamma_0}-\xi^2\right)\right]\right\}Z(\xi)-\eta\omega_*\xi\right)\right], \tag{18}$$

with $\omega_*=(lcT/ZeN)(\partial N/\partial\psi)$, $v_T{}^2=2T/M$, $\Gamma_\nu=I_\nu(b)\exp(-b)$, $b=k_\perp{}^2T/M\Omega^2=k_\perp{}^2v^2/2\Omega^2$, $\xi=(\omega-\mathbf{k}\cdot\mathbf{v}_d)/|k_\parallel|\,v_T$, and $Z(\xi)$ the usual plasma dispersion function.

When the ψ variation of Φ is slow compared to a gyroradius then $k_\perp{}^2>\kappa^2\,|\nabla\psi|^2$ may be employed to expand Γ_ν about the poloidal wave vector squared, $k_p^2 = k_\perp^2 - \kappa^2 |\nabla\psi|^2$. If only terms to order κ^2 are retained then the inverse transform form κ space back to ψ will recover the second derivatives of $\Phi_{lm}(\kappa, \omega)$ with respect to ω because $\kappa^2 \rightarrow -\partial^2/\partial\psi^2$.

In summary, the linearized, unperturbed, gyro-averaged Vlasov operator containing finite gyroradius effects is obtained by a substantially simpler method than previous work, the technique transforms from the particle variables to the guiding center variables and gyro-averages before (rather than after) explicitly introducing the magnetic coordinates.

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REFERENCES

CATTO P. J. and TSANG K. T. (1977) *Physics Fluids* **20,** 396.
CONNOR J. W. and HASTIE R. J. (1975) *Plasma Phys.* **17,** 97.
HAZELTINE R. D. (1973) *Plasma Phys.* **15,** 77.
JAMIN E. (1971) Ph.D. Thesis, Princeton University.
KADOMSTEV B. B. and POGUTSE O. P. (1969) *Sov. Phys. Dokl.* **14,** 470.
KADOMSTEV B. B. and POGUTSE O. P. (1967) *Soviet Phys. JETP* **24,** 1172.
NEWBERGER B. S. (1976) Ph.D. Thesis, Princeton University.
RUTHERFORD P. H. and FRIEMAN E. A. (1968) *Physics Fluids* **11,** 569.
TANG W. M., ADAM J. C. COHEN B. I., FRIEMAN E. A., KROMMES J. A., REWOLDT G., ROSS D. W., ROSENBLUTH M. N., RUTHERFORD P. H., CATTO P. J., TSANG K. T. and CALLEN J. D. (1977) *Plasma Physics and Controlled Nuclear Fusion Research 1976* (Proc. 6th Int. Conf. Berchtesgaden) Vol. II, p. 323, I.A.E.A., Vienna.
TAYLOR J. B. and HASTIE R. J. (1968) *Plasma Phys.* **10,** 479.

II.) → Gyrokinetics : { Key ordering for magnetized plasma with $\rho/L_\perp \ll 1$.

- seek employ drifts, with gyro-radius finite, to reduce description of plasma dynamics from Vlasov → Gyrokinetic.

i.e. DKE: (GKE with $k_\perp \rho \to 0$)

$$\frac{\partial f}{\partial t} + V_z \frac{\partial f}{\partial z} - \frac{c}{B_0} \nabla\phi \times \vec{z} \cdot \nabla f + \frac{q}{m} E_z \frac{\partial f}{\partial v_z} = C(f)$$

1D motion + E×B drift replaces 6D phase space.

- central idea is guiding center transformation:
g.c. cyclotron

$$\underline{x} = \underline{X}_{gc} - \frac{\underline{v} \times \hat{n}}{\Omega}$$

$$V_\perp = \hat{\nu} \cos\phi + \hat{e} \sin\phi$$

$$\vec{e} = \hat{n} \times \hat{\nu}$$

$$\underline{v} = \underbrace{V_\parallel}_{\text{along field}} \hat{n} + V_\perp \underbrace{\hat{v}_\perp}_{\perp \text{ Velocity}}$$

$$\hat{n} = \underline{B}/|B|$$

where: $E = v^2/2$ → energy

$$\Omega = qB/mc$$

$$\mu = v_\perp^2/2B$$

$$v_\perp^2 = \left[(\underline{\underline{I}} - \hat{n}\hat{n}) \cdot \underline{v}\right]^2 = \left[\underline{v} - \hat{n}(\underline{v} \cdot \hat{n})\right]^2$$

$$\underbrace{V_\parallel^2 = 2(E - \mu B)}_{\text{Velocity variables}}$$

for linearized theory:

- simplifying LHS: $\frac{\partial}{\partial t} + \underline{v}\cdot\underline{\nabla} + \frac{q}{mc}\underline{v}\times\underline{B}\cdot\underline{\nabla}_v$

Now, changing to g.c. variables:

$$\underline{\nabla} \to \underline{\nabla}_R - [\nabla(\Omega^{-1}\hat{n})\times\underline{v}]\cdot\underline{\nabla}_R + \underline{\nabla\phi}\frac{\partial}{\partial\phi} + \underline{\nabla\mu}\cdot\frac{\partial}{\partial\mu}$$

$$\underline{\nabla}_v \to \underline{\nabla}_v + \frac{\underline{\underline{I}}\times\hat{n}\cdot\underline{\nabla}_R}{\Omega}$$

<u>so</u>

$$\underline{\nabla}_v = \underline{v}\frac{\partial}{\partial E} + \frac{\hat{\underline{v}}_\perp}{B}\frac{\partial}{\partial\mu} + \frac{\vec{\phi}}{\underline{v}_\perp}\frac{\partial}{\partial\phi} \qquad \vec{\phi} \equiv \hat{n}\times\hat{\underline{v}}_\perp$$

$$\underline{\nabla\mu} = -(\mu/B)\underline{\nabla}B - \left(\frac{v_\perp v_\parallel}{B}\right)\left[(\nabla\hat{n})\cdot\vec{\psi}\cos\phi + (\nabla\hat{n})\cdot\underline{e}\sin\phi\right]$$

$$\underline{\nabla}\phi = \frac{v_\parallel}{v_\perp}\left[-(\nabla\hat{n})\cdot\hat{e}\cos\phi + (\nabla\hat{n})\cdot\vec{\psi}\sin\phi\right] + \underline{\nabla}\vec{e}\cdot\vec{\psi}$$

<u>so</u>

$$\frac{q}{mc}\underline{v}\times\underline{B}\cdot\underline{\nabla}_v = -\Omega\frac{\partial}{\partial\phi}$$ → reduces to cyclotron motion

<u>and</u> $$(\Omega\underline{v}\times\hat{n})\cdot\left(\frac{\underline{\underline{I}}\times\hat{n}\cdot\underline{\nabla}_R}{\Omega}\right) = -\underline{v}_\perp\cdot\underline{\nabla}_R$$

↳ cancels ⊥ motion (fast term)

⇒

$$\frac{\partial}{\partial t} + \underline{v}\cdot\underline{\nabla} + \frac{q}{mc}\underline{v}\times\underline{B}\cdot\underline{\nabla}_v = -\Omega\frac{\partial}{\partial\phi} + \frac{\partial}{\partial t} + v_\parallel\hat{n}\cdot\underline{\nabla}_R$$

↳ single fast term

$$+ \underline{v}\cdot\left[\underline{\nabla}\phi\frac{\partial}{\partial\phi} + \underline{\nabla}\mu\frac{\partial}{\partial\mu} - \nabla(\Omega^{-1}\hat{n})\times\underline{v}\cdot\underline{\nabla}_R\right]$$

Now, can simplify for low frequency dynamics via

$$L = \frac{\partial}{\partial t} + \underline{v}\cdot\underline{\nabla} + \frac{q}{mc}\underline{v}\times\underline{B}\cdot\underline{\nabla}_v = -\Omega\frac{\partial}{\partial\phi} + \underset{\substack{\uparrow \\ \text{slow piece}}}{L_s}$$

$Lf = RHS$, in Vlasov Eqn.

$\Rightarrow$

$$-\Omega\frac{\partial}{\partial\phi}f + L_s f = RHS$$

i.e. $-\Omega\frac{\partial f^{(0)}}{\partial\phi} = 0 \Rightarrow f$ independent ϕ

1st. ord.

$$-\Omega\frac{\partial f^{(1)}}{\partial\phi} + L_s f_0 = RHS$$

f_0 indep. ϕ

$\oint\frac{d\phi}{2\pi} \Rightarrow \quad \langle L_s\rangle f_0 = \langle RHS\rangle$

where: $\langle L_s\rangle = \int\frac{d\phi}{2\pi}L_s$

$$\langle L_s\rangle f_0 = \langle RHS\rangle$$

'is' gyrokinetic equation.

Thus, remains to compute $\langle L_s\rangle f_0$, $\langle RHS\rangle$!

$$L_s = -\Omega\frac{\partial}{\partial\phi} + \frac{\partial}{\partial t} + v_\parallel\,\hat{n}\cdot\underline{\nabla}_R$$
$$+\ \underline{v}\cdot\left[\underline{\nabla}\phi\frac{\partial}{\partial\phi} + \underline{\nabla}\mu\frac{\partial}{\partial\mu} - \underline{\nabla}(\Omega^{-1}\hat{n})\times\underline{v}\cdot\nabla_R\right]$$

So

$$\langle L_s\rangle = \frac{\partial}{\partial t}\langle\ \rangle + \langle v_\parallel\,\hat{n}\cdot\nabla_R\rangle$$
$$+\left\langle\underline{v}\cdot\left[\underline{\nabla}\phi\frac{\partial}{\partial\phi} + \underline{\nabla}\mu\frac{\partial}{\partial\mu} - \underbrace{\underline{\nabla}(\Omega^{-1}\hat{n})}_{\text{drift piece i.e. } \nabla(1/\Omega)}\times\underline{v}\cdot\underline{\nabla}_R\right]\right\rangle$$

Now, can note:

$$\frac{1}{2\pi}\oint d\phi = \langle\ \rangle$$

and

$$\langle\underline{v}\,\underline{v}\rangle = \frac{v_\perp^2}{2}(\underline{\underline{I}} - \hat{n}\hat{n}) + v_\parallel^2\,\hat{n}\hat{n}$$

So

$$-\langle\underline{v}\cdot[\nabla(\Omega^{-1}\hat{n})\times\underline{v}]\cdot(\underline{\underline{I}} - \hat{n}\hat{n})\rangle$$
$$= \hat{n}\times\Big[\underbrace{(v_\perp^2/2\Omega)\nabla\ln B}_{v_{\nabla B}} + \underbrace{(v_\parallel^2/\Omega)\hat{n}\cdot\nabla\hat{n}}_{\underline{v}_{\text{curv}}}\Big]$$
$$= \langle\underline{v}_d\cdot\nabla_\perp\rangle$$

N.B. neglected $\langle v_d\cdot\nabla_\parallel\rangle$, as $k_\perp \gg k_\parallel$ assumed.

similarly;

$$\left.\begin{aligned}\left\langle \underline{v}\cdot\underline{\nabla\phi}\,\frac{\partial}{\partial\phi}\right\rangle &= 0\\ \left\langle \underline{v}\cdot\underline{\nabla}\mathcal{U}\right\rangle &= 0\end{aligned}\right\}\rightarrow\underline{\text{symmetry}}$$

Thus

$$\langle L_0\rangle = \frac{\partial}{\partial t} + v_\parallel\,\hat{n}\cdot\underline{\nabla}_R + \underline{v}_d\cdot\underline{\nabla}_R$$

Now, what of RHS? ⇒

∴ 2 issues $<$ f vs. g
density gradient

① f vs. g ?

Can always write:

$$f = \underbrace{-\frac{q\phi}{T}\langle f\rangle}_{\substack{\text{Boltzmann}\\ \text{(adiabatic)}}} + \underset{\hookrightarrow\text{(non-adiabatic)}}{g}$$

meaning: "adiabatic" → $k_\parallel v_\parallel \to \infty$
$k_\parallel v_\parallel > \omega$, etc.

piece.

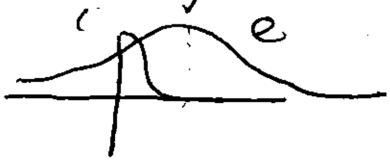
i.e. — corresponds to particles zipping along field lines

$f = \frac{e\phi}{T} f_0 + g$

- corresponds to all = adiabatic + everything else

- adiabatic particles should not drift, etc. ⇒ zip along field lines, dominantly

- utility : Current driven ion-acoustic



$v_{Ti} < \frac{\omega}{k_{\parallel}} < v_{Te}$ ⎨ prototype of one species adiabatic, one hydro.

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{|e|}{m_e} E \frac{\partial f}{\partial v} = 0$$

$$f = \langle f \rangle + \hat{f}$$

$$\frac{\partial \hat{f}}{\partial t} + v\frac{\partial \hat{f}}{\partial x} = \frac{|e|}{m_e} \hat{E} \frac{\partial \langle f \rangle}{\partial v}$$

$$\langle f \rangle = \frac{n}{\sqrt{2\pi}} \exp\left[-(v-u_0)^2/2v_{th}^2\right]$$

$$\Rightarrow \quad \frac{\partial \hat{f}}{\partial t} + v\frac{\partial \hat{f}}{\partial x} = \frac{|e|}{\cancel{m_e}} \frac{(v-u_0)}{T_e/\cancel{m_e}} \left(-\frac{\partial \phi}{\partial x}\right) \langle f \rangle$$

Now $\quad \hat{f} = \frac{|e|\phi}{T} \langle f \rangle + \hat{g}$

$$\frac{\partial \hat{g}}{\partial t} + v\frac{\partial \hat{g}}{\partial x} = -v \cancel{\frac{|e|}{T}} \frac{\partial \hat{\phi}}{\partial x} \langle f \rangle - \frac{|e|}{T}\frac{\partial \phi}{\partial t} \langle f \rangle$$

$$+ \frac{|e|}{T} (\cancel{v} - u_0) \left(-\frac{\partial \phi}{\partial x}\right) \langle f \rangle$$

$$= -\frac{|e|}{T} \frac{\partial \hat{\phi}}{\partial t} \langle f \rangle + \frac{|e|}{T} u_0 \frac{\partial \phi}{\partial x} \langle f \rangle$$

$$\sim (\omega - k u_0)$$

∴ of clear utility to isolate non-adiabatic piece, as

$$RHS \sim \frac{\omega}{k} - u_0 \rightarrow \text{flips sign !}$$

② why not drift ?

Consider drift-kinetic-equation:

$$\frac{\partial f}{\partial t} + v_z \frac{\partial f}{\partial z} - \frac{c}{B} \underline{\nabla} \phi \times \hat{z} \cdot \underline{\nabla} f - \frac{|e|}{m} E_z \frac{\partial f}{\partial v_z} = 0$$

$k_\perp >> k_\parallel \Rightarrow$ ($E_\parallel$ nonlinearity small)

$$\frac{\partial \tilde{f}}{\partial t} + v_z \frac{\partial \tilde{f}}{\partial z} - \frac{c}{B} \underline{\nabla} \phi \times \hat{z} \cdot \underline{\nabla} \tilde{f}$$

$$= \frac{c}{B} \underline{\nabla} \tilde{\phi} \times \hat{z} \cdot \underline{\nabla} \langle f \rangle + \frac{|e|}{m} \tilde{E}_z \frac{\partial \langle f \rangle}{\partial v_z}$$

$$E_z = -\frac{\partial \phi}{\partial z} \qquad f = \frac{|e| \tilde{\phi}}{T} \langle f_0 \rangle + g$$

$\Rightarrow$

$$\frac{\partial}{\partial t} \left(\frac{|e| \tilde{\phi}}{T} \langle f_0 \rangle + \tilde{g} \right) + v_z \frac{\partial}{\partial z} \left(\cancel{\frac{|e| \tilde{\phi}}{T} \langle f_0 \rangle} + \tilde{g} \right)$$

$$- \frac{c}{B} \underline{\nabla} \phi \times \hat{z} \cdot \underline{\nabla} \left(\cancel{\frac{|e| \tilde{\phi}}{T} \langle f_0 \rangle} + \tilde{g} \right) = \frac{c}{B} \underline{\nabla} \tilde{\phi} \times \hat{z} \cdot \underline{\nabla} \langle f \rangle + \cancel{\frac{|e|}{m} \tilde{E}_z \frac{\partial \langle f \rangle}{\partial v_z}}$$

NL (E×B drift) annihilates adiabatic part only non-adiabatic part advanced.

$$\Rightarrow \frac{\partial \hat{g}}{\partial t} + v_z \frac{\partial}{\partial z}\hat{g} - \frac{c}{B_0}\underline{\nabla}\phi\times\hat{z}\cdot\underline{\nabla}\hat{g}$$

$$= -\frac{|e|}{T}\langle f_0\rangle\frac{\partial\vec{\phi}}{\partial t} + \frac{c}{B_0}\underline{\nabla}\phi\times\hat{z}\cdot\underline{\nabla}\langle f\rangle$$

∴ bottom line, in electrostatics, is:

+ → ions
− → electrons

$$RHS = \pm\frac{|e|}{T_0} f_0 \frac{\partial\langle\phi\rangle}{\partial t} + \frac{c}{B}\underline{\nabla}\langle\phi\rangle\times\hat{z}\cdot\underline{\nabla}\langle f\rangle$$

⇒ meaning $\langle\ \rangle$?

$$\phi = \sum_{\underline{k}} \phi_{\underline{k}}\, e^{i\underline{k}\cdot x - i\omega t}$$

$$\underline{x} = \underline{R} - \frac{\underline{v}\times\hat{n}}{\Omega}$$

$$\Rightarrow \underline{k}\cdot\underline{x} = \underline{k}\cdot\underline{R} - k\cdot\frac{\underline{v}\times\vec{n}}{\Omega}$$

$$= k\cdot R + k\rho\sin(\phi-\psi)$$

$\psi = \measuredangle k$

$$\langle\phi_k\rangle = \frac{1}{2\pi}\oint d\phi\, \phi_k\, e^{i\underline{k}\cdot\underline{R}}\, e^{ik\rho\sin(\phi-\psi)}$$

but $\oint \frac{d\phi}{2\pi} e^{ik\rho\sin(\phi-\psi)} \equiv J_0(k_\perp\rho)$

so, in fourier components, can write gyrokinetic equation as:

$$-i(\omega - \underline{k}\cdot\underline{V}_d - k_{\parallel}V_{\parallel})\hat{g}_{\underline{k}} \qquad \overset{\text{gyro-}}{\downarrow}\ \text{avg. factor}$$

$$= \mp \frac{i|e|}{T_{i,e}}(\omega - \omega_{*i,e}) f_0 \, J_0(k_\perp\rho)\,\phi_{\underline{k}}$$

$$\omega_* = \frac{1}{\langle f\rangle}\frac{Tc}{B_0 q}k_\theta \frac{\partial\langle f\rangle}{\partial r}$$

$$= + k_\theta\left(\frac{(T/m)mc}{qB_0}\right) = \qquad k_\theta\rho V\frac{1}{\langle f_0\rangle}\frac{\partial\langle f_0\rangle}{\partial r}$$

$$\sim \frac{k_\theta\rho V}{L_\perp} \equiv \text{diamagnetic frequency}$$

$V_{dia} \sim \frac{\rho V}{L_\perp}$ → diamagnetic velocity

→ Toward Macroscopics

- To compute anything of relevance (i.e. dispersion relation, wave frequency, etc.), its necessary to compute moments

i.e.

$$\left.\begin{aligned}\hat{n} &= \int d^3v\, f \\ \hat{J}_{\parallel_{i,e}} &= \int d^3v\, qV_\parallel f\end{aligned}\right\} \text{etc.}$$

so as solving GKE obtains g_k in g.c. variables ↔ must

o transform to real/particles variables

$$\Rightarrow \quad g = \sum_{\underline{k}} g_{\underline{k}} e^{i\underline{k}\cdot\underline{x}}$$

$$= \sum_{\underline{k}} \underbrace{g_{k} e^{i\underline{k}\cdot\underline{R}}}_{g_{g.c.}} e^{i k_\perp \rho \sin(\phi - \chi)}$$

and, as will seek ϕ integrated quantity

i.e. $n = \int d^3v \; f$ etc.

$J_\parallel = \int d^3v \; v_\parallel f$

$$\Rightarrow \quad g_{\underline{k}} = J_0(k_\perp \rho) \, g_{\underline{k}}^{g.c.}$$

For electromagnetic perturbations, note:

- magnetic field lines 'wiggle' ↔ Alfvenic oscillations

$$v_\parallel(\hat{n}\cdot\nabla) = v_\parallel\left(\partial_z + \frac{\tilde{B}_\perp}{B_0}\cdot\nabla_\perp\right)$$

but, for $\tilde{B}_\perp >> \tilde{B}_\parallel$ (low β)

$\tilde{B} = \nabla\times(A_\parallel \hat{z})$

↳ component of vector potential along $\underline{B_0}$

so

$$v_\parallel \hat{n}\cdot\underline{\nabla} - \frac{c}{B}\underline{\nabla}\phi\times\hat{z}\cdot\underline{\nabla} = v_\parallel \partial_z - \frac{c}{B_0}\underset{\downarrow\ \text{total}\ \perp\ \text{scattering}}{\overset{\mathcal{L}}{\underline{\nabla}\left(\hat{\phi} - \frac{v_\parallel}{c}\hat{A}_\parallel\right)}}\times\hat{z}\cdot\underline{\nabla}$$

so, in ω_* term $\hat{\phi} \rightarrow \hat{\phi} - \frac{v_\parallel}{c}A_\parallel$

- $\hat{E}_\parallel$ contains inductive piece

$$\hat{E}_\parallel = \underset{\uparrow}{-\frac{1}{c}\frac{\partial A_\parallel}{\partial t}} - \nabla_\parallel\hat{\phi}$$

$$\frac{}{m}\hat{E}_\parallel\frac{\partial f_0}{\partial v_\parallel} = \frac{q}{\cancel{m}}\Big(\underset{\substack{\uparrow\\ \underline{\underline{\text{New}}}}}{+\frac{1}{c}\,i\omega A_{\parallel\,\substack{\underline{k}\\ \omega}}} - \underset{\substack{\uparrow\\ \text{cancels, leaving on RHS}\\ \omega\hat{\phi}\ \text{term}}}{i k_z\hat{\phi}_{\substack{\underline{k}\\ \omega}}}\Big)\frac{-v_\parallel}{T/\cancel{m}}\overline{f_0}$$

$$= \text{old} - \frac{i\omega}{c}\frac{q v_\parallel}{T}A_\parallel\, q\overline{f_0}$$

$$= \omega\left(\phi - \frac{v_\parallel A_\parallel}{c}\right)\frac{q\overline{f_0}}{T}$$

so $\hat{\phi} \rightarrow \left(\hat{\phi} - \frac{v_\parallel A_\parallel}{c}\right)$ in all RHS

$$(\omega - \omega_{d_i} - k_\parallel v_\parallel)\hat{g}_{\underline{k}} = -\frac{i|q|}{T_i}\overline{f_0}\,(\omega - \omega_{*i})\left(\hat{\phi} - \frac{v_\parallel}{c}\hat{A}_\parallel\right)_{\underline{k}} J_0(k_\perp\rho_i)$$

is linear, e.m. gyrokinetic equation.

Using Gyrokinetics - Kinetic Shear Alfven Wave

Now, $$-i(\omega-\omega_{D_i}-k_\parallel v_{\parallel i})\hat{g}_{i\underline{k}, \omega} = -\frac{i|e|}{T_i}(\omega-\omega_{*i})\left(\hat{\phi}-\frac{v_{\parallel i}}{c}\hat{A}_\parallel\right)_{\underline{k}, \omega}\overline{f}_{0i}\overline{J}_0$$

ions $\overline{J}_0 = J_0(k_\perp\rho_i)$

$$-i(\omega-\omega_{De}-k_\parallel v_{\parallel e})\hat{g}_{e\underline{k}, \omega} = \frac{i|e|}{T_e}(\omega-\omega_{*e})\left(\hat{\phi}-\frac{v_{\parallel e}}{c}A_\parallel\right)_{\underline{k}, \omega}\overline{f}_{0e}$$

electrons

$$\left\{\begin{array}{l}\hat{n}_e = \hat{n}_i \rightarrow \text{quasineutrality} \qquad (k^2\lambda_D^2 \ll 1) \\ \nabla_\perp^2\hat{A}_\parallel = -\frac{4\pi}{c}(\tilde{J}_{\parallel e}+\tilde{J}_{\parallel i}) \rightarrow \text{Ampere's Law}\end{array}\right.$$

$\rightarrow$ macroscopic relations $\Rightarrow$ 'shear Alfven' / 'reduced MHD' fluctuations!

$$\frac{\hat{n}_e}{n_0} = \frac{|e|}{T_e}\hat{\phi} + \int d^3v\, \hat{g}_e$$

$$\frac{\hat{n}_i}{n_0} = -\frac{|e|\hat{\phi}}{T_i} + \int d^3v\, J_0(k_\perp\rho)\hat{g}_i$$

and basic ordering: $v_{Ti} < \frac{\omega}{k_\parallel} < v_{Te}$
QN

For electrons:

$$i\underset{\uparrow}{k_\parallel} v_{\parallel e}\hat{g}_{e\underline{k}, \omega} - i(\omega-\omega_{De})\hat{g}_{e\underline{k}, \omega} = \frac{i|e|}{T_e}(\omega-\omega_{*e})\left(\hat{\phi}-\underset{\uparrow}{\frac{v_\parallel}{c}}\hat{A}_\parallel\right)_{\underline{k}, \omega}\overline{f}_{0e}$$

$v_\parallel \gg \omega/k_\parallel \Rightarrow \uparrow$'s balance;
(adiabatic limit expansion) - like electrons in ion acoustic

$$\hat{g}_{e\underline{k}, \omega} = \frac{i|e|}{T_e}\frac{(\omega-\omega_{*e})}{ik_\parallel \cancel{v_{\parallel e}}}\left(-\frac{\cancel{v_\parallel}}{c}\hat{A}_\parallel\right)_{\underline{k}, \omega}\overline{f}_{0e}$$

$$\hat{g}_{e\,\underset{\omega}{k}} = \frac{-\cancel{i}|e|}{\cancel{i}T_e}\,\frac{\omega-\omega_{*e}}{ck_\parallel}\,\hat{A}_{\parallel\,\underset{\omega}{k}}\,\bar{f}_{0e}$$

$$\Rightarrow \frac{\hat{n}_e}{n_0}{}_{\underset{\omega}{k}} = \frac{|e|}{T_e}\cdot\left[\hat{\phi} - \left(1-\frac{\omega_{*e}}{\omega}\right)\frac{\omega\hat{A}_\parallel}{ck_\parallel}\right]_{k,\omega}.$$

For ions; $\omega > k_\parallel v_{\parallel i}$ → hydrodynamic expansion ~~a~~ a' ions in ion acoustic

$$-i\left(\underset{\uparrow}{\omega}-\omega_{Di}-k_\parallel v_{\parallel i}\right)\hat{g}_{i\,\underset{\omega}{k}} = \frac{-i|e|}{T_i}\left(\omega-\omega_{*i}\right)\left(\underset{\uparrow}{\hat{\phi}}-\frac{v_{\parallel i}}{c}\hat{A}_\parallel\right)_{\underset{\omega}{k}}\bar{f}_{0i}\,J_0(k_\perp\rho)$$

$$\Rightarrow \hat{g}_{i\,\underset{\omega}{k}} = \frac{|e|}{T_i}\left[\frac{(\omega-\omega_{*i})\,\bar{f}_0\,J_0(k_\perp\rho)}{(\omega-\omega_{Di}-k_\parallel v_{\parallel i})}\right]\left(\hat{\phi}-\frac{v_\parallel}{c}\hat{A}_\parallel\right)_{k,\omega}$$

Now: → $\omega > k_\parallel v_{\parallel i}$ $\omega > \omega_{Di}$

⇒ fluid ions

→ $\bar{f}_{0i}$ even → $\hat{A}_\parallel$ term cancels

→ $k_\perp\rho_i \ll 1$

$$\Rightarrow \hat{g}_{i\,\underset{\omega}{k}}\,J_0(k_\perp\rho_i) = \frac{|e|}{T_i}\,\frac{(\omega-\omega_{*i})}{\omega}\,\bar{f}_0\,J_0^2(k_\perp\rho_i)\,\hat{\phi}_{\underset{\omega}{k}}$$

$$= \frac{|e|\hat{\phi}_{\underset{\omega}{k}}}{T_i}\left(1-\frac{\omega_{*i}}{\omega}\right)\bar{f}_0\,J_0^2(k_\perp\rho_i)$$

$$\frac{\hat{n}_i}{n_0} = -\frac{|e|}{T_i}\,\hat{\phi}_{\underset{\omega}{k}}\left[1-\int d^3v\,\bar{f}_0\,J_0^2(k_\perp\rho_i)\left(1-\frac{\omega_{*i}}{\omega}\right)\right]$$

$$\int d^3v\, \bar{f}_0\, J_0^2(k_\perp \rho_i) = I_0(b_i)\, e^{-b_i}$$

$$b_i \equiv k_\perp^2 \rho_{iT}^2$$

$$\therefore \quad \frac{\hat{n}_i}{n_0} = \frac{-|e|}{T_i}\, \hat{\phi}_{\underline{k},\omega} \left[1 - \left(1 - \frac{\omega_{*i}}{\omega} \right) I_0(b_i)\, e^{-b_i} \right]$$

expanding in $b_i \ll 1$,

$$\frac{\hat{n}_i}{n_0} \cong \frac{-|e|}{T_i}\, \hat{\phi}_{\underline{k},\omega} \left[1 - \left(1 - \frac{\omega_{*i}}{\omega} \right) (1 - b_i) \right]$$

Now in MHD ordering: $\omega > \omega_{*i}, \omega_{*e}$

$\Rightarrow$

$$\frac{|e|}{T_e} \left[\hat{\phi} - \frac{\omega \hat{A}_\parallel}{c k_\parallel} \right]_{\underline{k},\omega} = -\frac{|e|}{T_i}\, \hat{\phi}_{\underline{k},\omega} \left[b_i \right] \qquad \rho_s^2 \equiv c_s^2/\Omega_i^2$$

$$= -k_\perp^2 \rho_s^2\, \frac{|e|}{T_e}\, \hat{\phi}_{\underline{k},\omega}$$

$$\therefore \quad \left[\hat{\phi} - \frac{\omega \hat{A}_\parallel}{c k_\parallel} \right]_{\underline{k},\omega} = -k_\perp^2 \rho_s^2\, \hat{\phi}_{\underline{k},\omega}$$

Note:

$$-\text{LHS} \sim \hat{E}_{\parallel\, \underline{k},\omega} / k_\parallel \qquad E_\parallel = -\nabla_\parallel \phi - \frac{1}{c}\frac{\partial A}{\partial t}$$

$$\therefore \quad QN \Rightarrow \hat{E}_{\parallel\, \underline{k},\omega} \sim +k_\parallel k_\perp^2 \rho_s^2\, \hat{\phi}_{\underline{k},\omega}$$

⇒ for $k_\perp^2 \rho_s^2 \lll 1$ ⇒ $\hat{E}_\parallel \cong 0$

in MHD (ideal)'s: $\underline{E} + \frac{\underline{v}\times\underline{B}}{c} \cong 0$

$$\hat{n}\cdot\left[\underline{E}+\frac{\underline{v}\times\underline{B}}{c}\right] = 0 \Rightarrow \hat{E}_\parallel = 0$$

∴ QN ⟺ effective 'Ohm's Law'.

- content

$$\underbrace{\left(\hat{\phi} - \frac{\omega \hat{A}_\parallel}{c k_\parallel}\right)_{\underline{k},\omega}}_{\sim \hat{E}_\parallel} = \underbrace{-k_\perp^2 \rho_s^2}_{\text{polarization drift (ions only)}} \hat{\phi}_{\underline{k},\omega}$$

i.e. $\frac{\tilde{n}_e}{n} \sim \hat{E}_\parallel$ (electron flow along field) balances $\frac{\tilde{n}_i}{n_0} \sim -k_\perp^2 \rho^2 \hat{\phi}$

as E×B drifts cancel.

Ampere's Law

- Now, for 2nd equation, can compute $\hat{J}_{\parallel e}$, $\hat{J}_{\parallel i}$ and plug into Ampere's Law but can notice, from basic equations:

$$\int d^3v\, J_0(k_\perp \rho_i)\, \hat{g}^i \text{ eqn} - \int d^3v\, \hat{g}^e \text{ eqn} \Rightarrow \text{forms current!}$$

$$\Rightarrow \overset{①}{} J_0(k_\perp\rho_i)\left(-i(\omega-\omega_{di}-k_\parallel v_{\parallel i})\hat{g}_i\right) = -\frac{i|e|}{T_i}(\omega-\omega_{*i})J_0^2(k_\perp\rho_i)\bar{f}_0\left(\hat{\phi} - \frac{v_\parallel}{c}\hat{A}_\parallel\right)_{\underline{k}}$$

②

$$-i(\omega-\omega_{de}-k_\parallel v_{\parallel e})\hat{g}_e = \frac{i|e|}{T_e}(\omega-\omega_{*e})\left(\hat{\phi}-\frac{v_\parallel}{c}\hat{A}_\parallel\right)_{\underline{k}}\bar{f}_0$$

$\int d^3v_i$ ① $-\int d^3v_e$ ② $\Rightarrow$

$$-i\omega\left[\int d^3v\, J_0(k_\perp\rho_i)\hat{g}_i - \int d^3v\, \hat{g}_e\right] + i k_\parallel \frac{\hat{J}_\parallel}{n_0 q}$$

$$+ i\left[\int d^3v\, \omega_{di}\hat{g}_i J_0(k_\perp\rho_i) - \int d^3v\, \omega_{de}\hat{g}_e\right]$$

$$= -\frac{i|e|}{T_i}(\omega-\omega_{*i})\left(\int d^3v\, \bar{f}_0 J_0^2(k_\perp\rho_i)\right)\hat{\phi}_{\underline{k}}$$

$$-\frac{i|e|}{T_e}(\omega-\omega_{*e})\hat{\phi}_{\underline{k},\omega}$$ by even $\bar{f}_0$ in $v_\parallel$,

Now, ① QN $\Rightarrow$ $-\frac{|e|\hat{\phi}}{T_i} + \int d^3v\, J_0(k_\perp\rho_i)\hat{g}_i$

$$= \frac{|e|\hat{\phi}}{T_e} + \int d^3v\, \hat{g}_e$$

② $\int d^3v\, \bar{f}_0 J_0^2 = I_0(b_i)\, e^{-b_i}$

$$\cong (1-k_\perp^2\rho_i^2) \qquad k_\perp^2\rho_i^2 \ll 1$$

$$\Rightarrow \quad i k_\parallel \frac{\hat{J}_\parallel}{n_0 q} + i\left[\int d^3v\, \omega_{di}\, \hat{g}_{i_{\mathbf{k}}}\, J_0(k_\perp \rho_i) - \int d^3v\, \omega_{de}\, \hat{g}_{e_{\mathbf{k}}}\right]$$

$$= +\frac{i|e|}{T_i}\, \omega\, k_\perp^2 \rho_i^2\, \hat{\phi}_{\mathbf{k}} + i|e|\left(\frac{\omega_{*i}}{T_i} I_0(b_i) e^{-b_i} + \frac{\omega_{*e}}{T_e}\right)\hat{\phi}_{\mathbf{k}}$$

but further :

$$= \frac{i|e|}{T_i}\, \omega\, k_\perp^2 \rho_i^2\, \hat{\phi}_{\mathbf{k}} + i|e|\left(\frac{\omega_*}{T_i}\left(\overset{A}{\cancel{1}} - b_i\right) + \overset{A}{\cancel{\frac{\omega_{*e}}{T_e}}}\right)\hat{\phi}_{\mathbf{k}}$$

$$\frac{\omega_{*i}}{T_i} = -\frac{\omega_{*e}}{T_e}$$

(species E×B drift together)

Now;

$$\boxed{\overset{\textcircled{1}}{i k_\parallel \frac{J_\parallel}{n_0|e|}} + i\left[\overset{\textcircled{2}}{\int d^3v\, \omega_{di}\, \underset{\textcircled{3}}{\hat{g}_{i_{\mathbf{k}}}}\, J_0(k_\perp\rho_i)} - \int d^3v\, \omega_{de}\, \hat{g}_{e_{\mathbf{k}}}\right] = \frac{i|e|}{T_i}\hat{\phi}_{\mathbf{k}}(\omega - \omega_{*i})\, k_\perp^2 \rho_i^2}$$

- equivalent to $\nabla \cdot J = 0$

i.e. $\frac{\partial \hat{\rho}}{\partial t} + \underline{\nabla} \cdot \underline{J} = 0$ $\qquad \frac{\partial \hat{\rho}}{\partial t} = -i\omega\, \underbrace{\left(\hat{n}_i|e| - \hat{n}_e|e|\right)}_{=0}$

for $k^2 \lambda_D^2 < 1$

$\therefore \quad \nabla \cdot J = 0$

$$\Rightarrow \quad \nabla_\parallel J_\parallel = -\underline{\nabla}_\perp \cdot \underline{\hat{J}}_\perp$$

$$\hat{J}_\perp = \cancel{J_i^{E\times B}} - \cancel{J_e^{E\times B}} + J_i^{pol} + J^{drift}$$

E×B drift of species equal

$\therefore$ ① → $k_\parallel \hat{J}_\parallel \rightarrow \nabla_\parallel J_\parallel$

③ → $\nabla_\perp \cdot J_{\perp\, pol}$.

② → $\nabla_\perp \cdot J_\perp^{drift}$.

⇒ quite general expression for $k_\perp \rho_i < 1$. Can evaluate for various cases.

Thus have gyrokinetic moment equations for $k_\perp^2 \rho_i^2 < 1$ ⇒ $\omega > \omega_{*i}$ (MHD)

$$\hat{\phi}_{\underline{k}} - \frac{\omega \hat{A}_{\parallel \underline{k}}}{c k_\parallel} = -k_\perp^2 \rho_s^2 \hat{\phi}_{\underline{k}}$$

$$\frac{i k_\parallel}{n_0 |e|} \hat{J}_{\parallel_{\underline{k}}} + i \left[\int d^3v \, \omega_{di} \hat{g}_{i\underline{k}} J_0(k_\perp \rho_i) - \int d^3v \, \omega_{de} \hat{g}_{e\underline{k}} \right] = \frac{i|e|}{T_e} k_\perp^2 \rho_s^2 \, \omega \hat{\phi}_{\underline{k}}$$

Now, for basic excitation/wave: Kinetic Shear Alfven Wave!

$$\hat{J}_{\parallel_{\underline{k}}} = +\frac{c}{4\pi} k_\perp^2 \hat{A}_{\parallel_{\underline{k}}} \quad ; \quad (1 + k_\perp^2 \rho_s^2) \hat{\phi}_{\underline{k}} = \frac{\omega}{c k_\parallel} \hat{A}_{\parallel_{\underline{k}}}$$

$\Rightarrow$ and, furthurmore $\omega_d \to 0$ (straight field or/and $\omega \gg \omega_d$)

$$\frac{\cancel{i}k_\parallel}{n_0|e|}\frac{c}{4\pi}\cancel{k_\perp^2}\frac{ck_\parallel}{\omega}\frac{I(\ )}{|e|}\frac{|e|\hat\phi}{T_0} = \cancel{i}\frac{|e|}{T_0}\hat\phi\,\cancel{k_\perp^2}\rho_s^2\,\omega$$

$$\frac{c^2 m_i^2}{|e|^2 B_0^2}\frac{T_e}{m_i}\left(\frac{B_0^2}{4\pi n_0 m_i}\right)\frac{k_\parallel^2}{\omega}(1+k_\perp^2\rho_s^2) = \omega\rho_s^2$$

$$\cancel{\rho_s^2}\, V_A^2 k_\parallel^2(1+k_\perp^2\rho_s^2) = \omega^2\cancel{\rho_s^2}$$

$\Rightarrow$ KSAW dispersion relation:

$$\boxed{\begin{aligned} &\omega^2 = k_\parallel^2 V_A^2(1+\underbrace{k_\perp^2\rho_s^2}) \\ \text{or}\quad &k_\perp^2 = \left(\frac{\omega^2}{k_\parallel^2 V_A^2}-1\right)\Big/\rho_s^2. \end{aligned}}$$

$\to$ cross-field propagation from polarization drift.

- $V_A^2 = (B_0^2/4\pi)/(n_0 m_i)$ $\to$ analogue to waves on string

$\downarrow$ magnetic tension

$\hookrightarrow$ ion inertia

$\mu \to n_0 m_i$

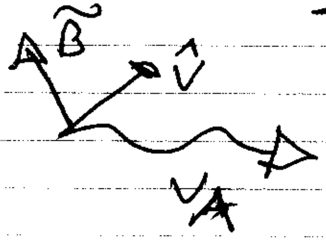
$T \to B^2/4\pi$.

- Note: $\hat\phi - \frac{\omega}{c}\frac{\hat A_\parallel}{k_\parallel} \cong 0$ $\qquad$ $\hat\phi = \frac{V_A}{c}\hat A_\parallel$

$\hat\phi - \frac{k_\parallel V_A}{c k_\parallel}\hat A_\parallel \cong 0$ $\qquad$ i.e. $\hat\phi = \hat A_\parallel$, within factor.

- wave propagates along $\underline{B_0}$!

$$\begin{cases} \text{fluid motion} \quad \underline{V} = -\frac{\nabla\phi \times \hat{z}}{B} \cdot \quad \perp \; B_0 \\ \text{magnetic perturbation} \quad \delta B_\perp \sim \underline{V}, \quad \perp \; B_0 \end{cases}$$



- note can re-write as:

$$\hat{E}_\parallel = \hat{\phi} - \frac{\omega \hat{A}_\parallel}{c k_\parallel} = 0$$

$$-i\nabla_\parallel \phi = \frac{\omega \hat{A}_\parallel}{c} \Rightarrow \frac{1}{c}\frac{\partial \hat{A}_\parallel}{\partial t} = -\nabla_\parallel \phi$$

(Ohms Law)

and
$$\frac{\nabla_\parallel \hat{J}_\parallel}{n_0 |e|} = +\rho_s^2 \frac{\partial}{\partial t} \nabla_\perp^2 \frac{|e| \hat{\phi}}{T_e}$$

$(\underline{\nabla} \cdot \underline{J} = 0)$

⇒ corresponds to linearized reduced MHD!

i.e. upon re-scale:

$$\frac{d}{dt} \nabla_\perp^2 \phi = \nabla_\parallel \hat{J}_\parallel \qquad J_\parallel = \nabla_\perp^2 A_\parallel$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{V}_E \cdot \underline{\nabla}$$

and $$\frac{\partial A}{\partial t} + \underline{V}_E \cdot \underline{\nabla} A = - \nabla_{\parallel} \hat{\phi} .$$

Bottom Line :

$$\boxed{\begin{array}{ll} \text{Reduced MHD} \Leftrightarrow & \text{Gyrokinetic} \\ \quad \Updownarrow & \text{Ordering.} \\ \text{incompr. MHD} & \end{array}}$$