Errors = Uncertainties

$$x = x_{best} \pm \delta x$$

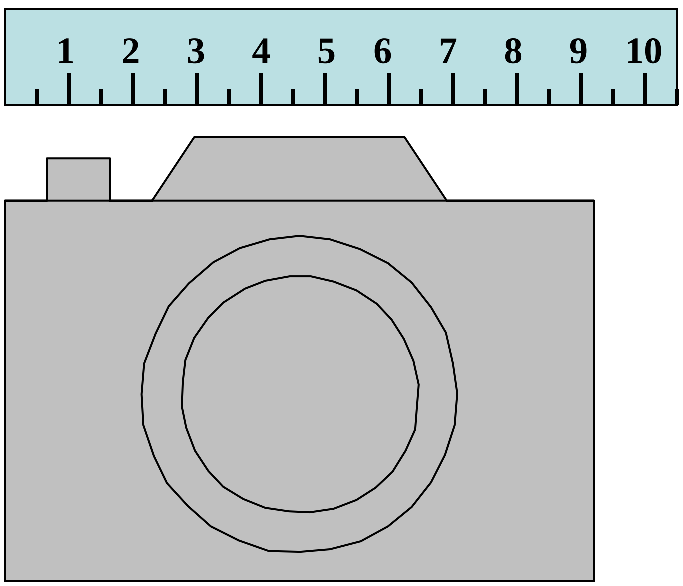
measured value of *x*

best estimate for x

uncertainty or error in the measurement

estimating uncertainties

when reading scales



$$l = 9.2 \pm 0.1 \text{ cm}$$

in repeatable measurements

measure period of a pendulum

exp.	1	2	3	4
T, s	2.3	2.4	2.5	2.4

$$T = 2.4 \pm 0.1 \text{ s}$$

Significant Figures

$$g = 9.82 \pm 0.02385 \text{ m/s}^2$$

$$g = 9.82 \pm 0.02 \text{ m/s}^2$$

Experimental uncertainties should be rounded to one significant figure (to two significant if the leading digit in the uncertainty is a 1)

$$g = 9.82 \pm 0.01437 \text{ m/s}^2$$

$$g = 9.82 \pm 0.014 \text{ m/s}^2$$

The last significant figure in any answer should be of the same order of magnitude (in the same decimal position) as the uncertainty

$$g = 9.82378 \pm 0.02 \text{ m/s}^2$$

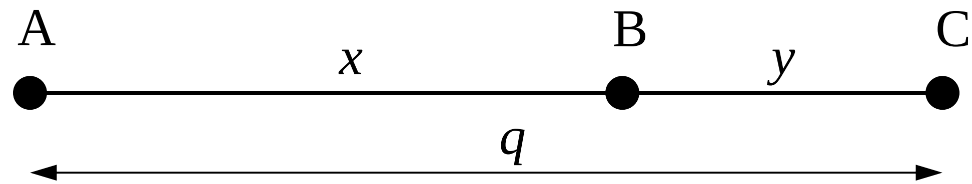
$$g = 9.82 \pm 0.02 \text{ m/s}^2$$

$$g = 9.82378 \pm 0.02385 \text{ m/s}^2 \;\rightarrow\; g = 9.82378 \pm 0.02 \text{ m/s}^2 \;\rightarrow\; g = 9.82 \pm 0.02 \text{ m/s}^2$$

$$v = 6051.78 \pm 32 \text{ m/s} \;\rightarrow\; v = 6051.78 \pm 30 \text{ m/s} \;\rightarrow\; v = 6050 \pm 30 \text{ m/s}$$

Propagation of Uncertainties

Uncertainty in a Sum



$$x = x_{best} \pm \delta x$$

$$y = y_{best} \pm \delta y$$

$$q = x + y = \left(x_{best} \pm \delta x\right) + \left(y_{best} \pm \delta y\right)$$

$$q_{\max} = \left(x_{best} + \delta x\right) + \left(y_{best} + \delta y\right) = \left(x_{best} + y_{best}\right) + \left(\delta x + \delta y\right)$$

$$q_{\min} = \left(x_{best} - \delta x\right) + \left(y_{best} - \delta y\right) = \left(x_{best} + y_{best}\right) - \left(\delta x + \delta y\right)$$

$$q = \left(x_{best} + y_{best}\right) \pm \left(\delta x + \delta y\right)$$

$$q = q_{best} \pm \delta q$$

$$q_{best} = x_{best} + y_{best}$$

$$\delta q = \delta x + \delta y$$

this equation overestimates δq because overestimate of x is accompanied by overestimate of y

$$\delta q = \sqrt{\left(\delta x\right)^2 + \left(\delta y\right)^2}$$

← quadratic sum

correct uncertainty in a sum for independent random errors δx and δy

Uncertainties in Sums and Differences

$$x = x_{best} \pm \delta x$$

$$y = y_{best} \pm \delta y$$

$$q = x + y$$

$$q_{best} = x_{best} + y_{best}$$

$$\delta q = \sqrt{(\delta x)^2 + (\delta y)^2}$$

$$q = x - y$$

$$q_{best} = x_{best} - y_{best}$$

$$\delta q = \sqrt{(\delta x)^2 + (\delta y)^2}$$

Fractional Uncertainties

$$x = x_{best} \pm \delta x$$

$$\text{fractional uncertainty} = \frac{\delta x}{|x_{best}|}$$

Example:

$$l = 30 \pm 0.3 \text{ cm}$$

$$\frac{\delta l}{|l_{best}|} = \frac{0.3 \text{ cm}}{30 \text{ cm}} = 0.01$$

Uncertainties in Products

$$x = x_{best} \pm \delta x = x_{best}\left(1 \pm \frac{\delta x}{|x_{best}|}\right) \qquad y = y_{best} \pm \delta y = y_{best}\left(1 \pm \frac{\delta y}{|y_{best}|}\right)$$

$$q = xy = x_{best}\left(1 \pm \frac{\delta x}{|x_{best}|}\right) y_{best}\left(1 \pm \frac{\delta y}{|y_{best}|}\right)$$

$$q_{\max} = x_{best} y_{best}\left(1 + \frac{\delta x}{|x_{best}|} + \frac{\delta y}{|y_{best}|} + \frac{\delta x}{|x_{best}|}\frac{\delta y}{|y_{best}|}\right) \approx x_{best} y_{best}\left(1 + \frac{\delta x}{|x_{best}|} + \frac{\delta y}{|y_{best}|}\right)$$

$$q_{\min} \approx x_{best} y_{best}\left(1 - \frac{\delta x}{|x_{best}|} - \frac{\delta y}{|y_{best}|}\right)$$

$$q = x_{best} y_{best}\left[1 \pm \left(\frac{\delta x}{|x_{best}|} + \frac{\delta y}{|y_{best}|}\right)\right] = q_{best}\left(1 \pm \frac{\delta q}{|q_{best}|}\right)$$

$$q_{best} = x_{best} y_{best}$$

$$\frac{\delta q}{|q|} = \frac{\delta x}{|x|} + \frac{\delta y}{|y|}$$

$$\frac{\delta q}{|q|} = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \left(\frac{\delta y}{y}\right)^2}$$

Example

Find momentum of a body with mass $m = 0.53 \pm 0.01$ kg

moving with velosity $v = 9.1 \pm 0.3$ m/s

$$p = mv = 0.53 \times 9.1 = 4.823 \text{ kg} \cdot \text{m/s}$$

$$\frac{\delta m}{m} = \frac{0.01}{0.53} = 0.02$$

$$\frac{\delta v}{v} = \frac{0.3}{9.1} = 0.03$$

$$\frac{\delta p}{p} = \sqrt{\left(\frac{\delta m}{m}\right)^2 + \left(\frac{\delta v}{v}\right)^2} = \sqrt{0.02^2 + 0.03^2} = 0.04$$

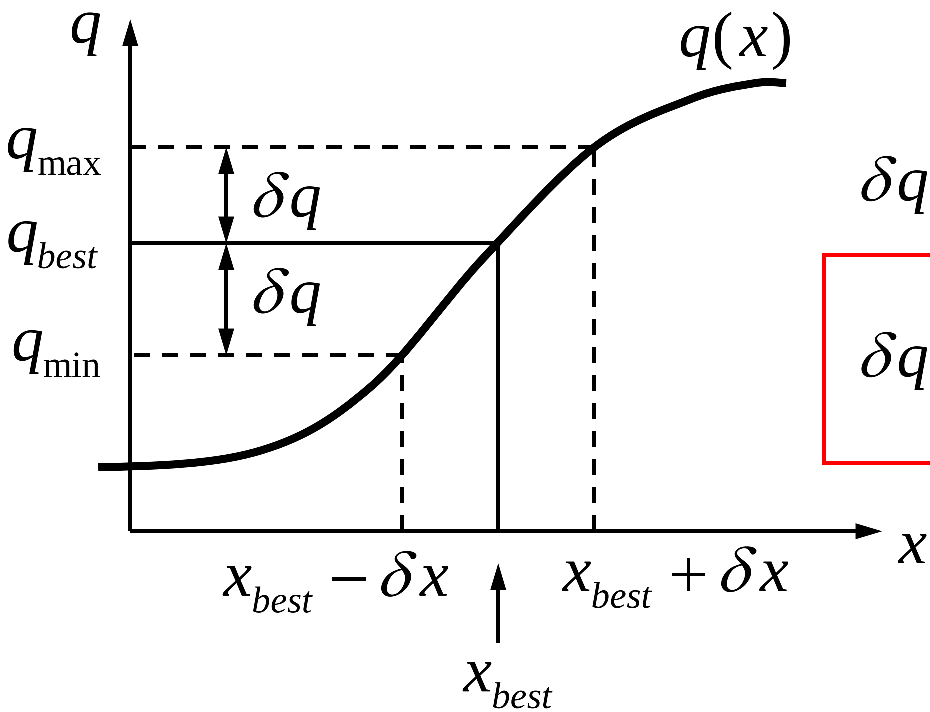
$$\delta p = 0.04\, p = 0.04 \times 4.823 = 0.193 \text{ kg} \cdot \text{m/s}$$

$$p = 4.82 \pm 0.19 \text{ kg} \cdot \text{m/s}$$

↑

always indicate units

Arbitrary Functions of One Variable



$$\delta q = q(x_{best} + \delta x) - q(x_{best}) = \frac{dq}{dx}\delta x$$

$$\delta q = \left|\frac{dq}{dx}\right|\delta x$$

Example

Find side *a* of a square with area *S* = 25 ± 2 cm^2.

$$a = \sqrt{S} = \sqrt{25} = 5 \text{ cm}$$

$$\delta a = \left|\frac{da}{dS}\right|\delta S = \frac{1}{2\sqrt{S}}\delta S = \frac{1}{2\cdot\sqrt{25}}2 = 0.2 \text{ cm}$$

$$a = 5.0 \pm 0.2 \text{ cm}$$

General Formula for Error Propagation

$$q = q(x, y)$$

$$q_{best} = q(x_{best}, y_{best})$$

partial derivative of q with respect to x

$$q(x_{best} + \delta x, y_{best} + \delta y) = q(x_{best}, y_{best}) + \frac{\partial q}{\partial x}\delta x + \frac{\partial q}{\partial y}\delta y$$

extreme values of q: $$q(x_{best}, y_{best}) \pm \left(\left|\frac{\partial q}{\partial x}\right|\delta x + \left|\frac{\partial q}{\partial y}\right|\delta y \right)$$

$$\delta q = \left|\frac{\partial q}{\partial x}\right|\delta x + \left|\frac{\partial q}{\partial y}\right|\delta y$$

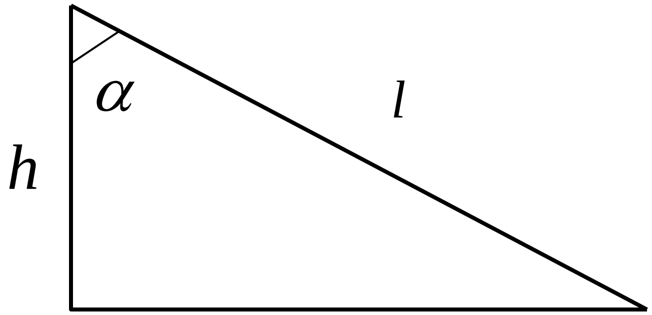
overestimated for independent δx and δy

$$\delta q = \sqrt{\left(\frac{\partial q}{\partial x}\delta x\right)^2 + \left(\frac{\partial q}{\partial y}\delta y\right)^2}$$

main formula for error propagation
always use this formula

for independent random errors δx and δy

Example



$l = 10 \pm 0.1$ m

$\alpha = 20 \pm 3^0$

Find *h*.

$$h = l \cdot \cos\alpha = 10 \cdot \cos 20^0 = 10 \cdot 0.94 = 9.4 \text{ m}$$

$$\delta h = \sqrt{\left(\frac{\partial h}{\partial l}\delta l\right)^2 + \left(\frac{\partial h}{\partial \alpha}\delta \alpha\right)^2}$$

$$\frac{\partial h}{\partial l} = \cos\alpha$$

$$\frac{\partial h}{\partial \alpha} = l \cdot (-\sin\alpha)$$

always use radians when calculating the errors on trig functions

$$\delta\alpha = 3^0 = \frac{2\pi \text{ rad}}{360^0} \cdot 3^0 = 0.05 \text{ rad}$$

$$\delta h = \sqrt{(\cos\alpha \cdot \delta l)^2 + (l \cdot (-\sin\alpha) \cdot \delta\alpha)^2} = \sqrt{(0.94 \cdot 0.1)^2 + (10 \cdot [-0.34] \cdot 0.05)^2} = 0.2 \text{ m}$$

$$h = 9.4 \pm 0.2 \text{ m}$$

Notations

$\sqrt{a^2+b^2} = a \oplus b$ shorthand notation for quadratic sum

quadratic sum = addition in quadrature

for independent random errors $\delta x \leftrightarrow \sigma_x$

↑

σ = (sigma) = Standard Deviation