PHYSICS 110A : CLASSICAL MECHANICS
MIDTERM EXAM #1

[1] A particle of mass m moves in the one-dimensional potential

$$U(x) = \frac{U_0}{a^4}\left(x^2 - a^2\right)^2 \, . \qquad (1)$$

(a) Sketch $U(x)$. Identify the location(s) of any local minima and/or maxima, and be sure that your sketch shows the proper behavior as $x \to \pm\infty$.
`[15 points]`

Solution : Clearly the minima lie at $x = \pm a$ and there is a local maximum at $x = 0$.

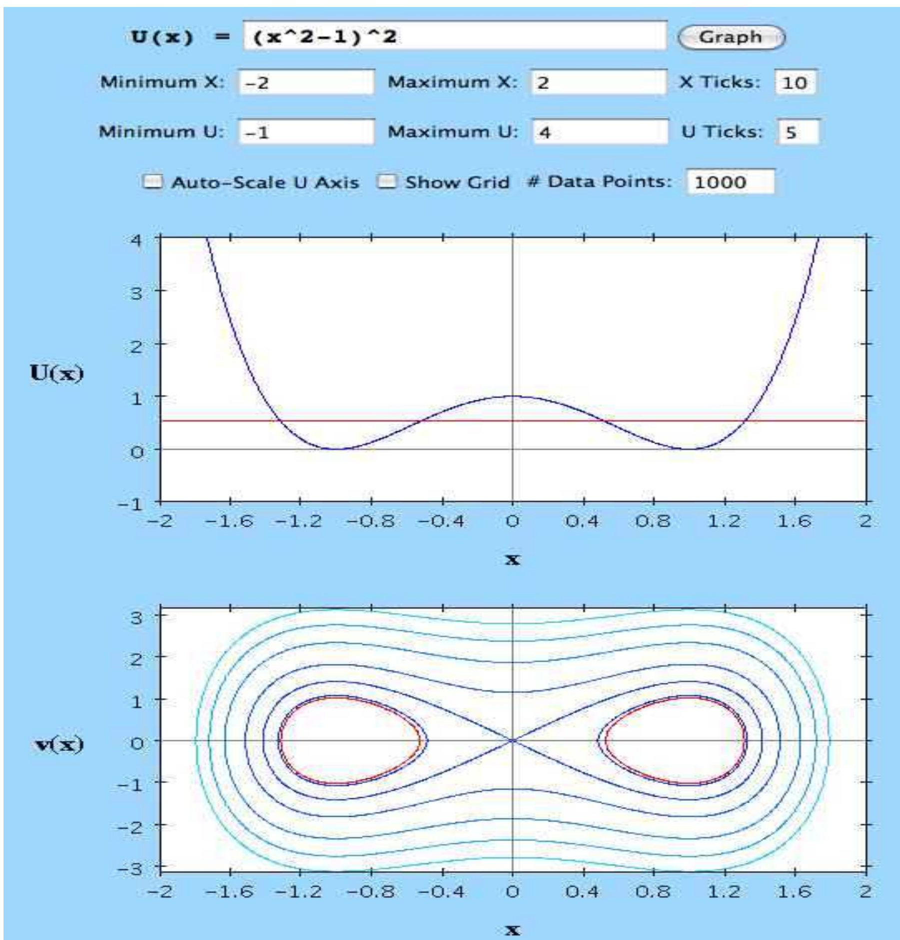

Figure 1: Sketch of the double well potential $U(x) = (U_0/a^4)(x^2 - a^2)^2$, here with distances in units of a, and associated phase curves. The separatrix is the phase curve which runs through the origin. Shown in red is the phase curve for $U = \frac{1}{2}\,U_0$, consisting of two deformed ellipses. For $U = 2\,U_0$, the phase curve is connected, lying outside the separatrix.

(b) Sketch a representative set of phase curves. Be sure to sketch any separatrices which exist, and identify their energies. Also sketch all the phase curves for motions with total energy $E = \frac{1}{2}\,U_0$. Do the same for $E = 2\,U_0$.
[15 points]

Solution : See Fig. 1 for the phase curves. Clearly $U(\pm a) = 0$ is the minimum of the potential, and $U(0) = U_0$ is the local maximum and the energy of the separatrix. Thus, $E = \frac{1}{2}\,U_0$ cuts through the potential in both wells, and the phase curves at this energy form two disjoint sets. For $E < U_0$ there are four turning points, at

$$x_{1,<} = -a\sqrt{1+\sqrt{\frac{E}{U_0}}} \quad , \quad x_{1,>} = -a\sqrt{1-\sqrt{\frac{E}{U_0}}}$$

and

$$x_{2,<} = a\sqrt{1-\sqrt{\frac{E}{U_0}}} \quad , \quad x_{2,>} = a\sqrt{1+\sqrt{\frac{E}{U_0}}}$$

For $E = 2\,U_0$, the energy is above that of the separatrix, and there are only two turning points, $x_{1,<}$ and $x_{2,>}$. The phase curve is then connected.

(c) The phase space dynamics are written as $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$, where $\boldsymbol{\varphi} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$. Find the upper and lower components of the vector field $\boldsymbol{V}$.
[10 points]

Solution :

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ -\frac{1}{m}\,U'(x) \end{pmatrix} = \begin{pmatrix} \dot{x} \\ -\frac{4U_0}{a^2}\,x\,(x^2-a^2) \end{pmatrix} \ . \tag{2}$$

(d) Derive and expression for the period T of the motion when the system exhibits small oscillations about a potential minimum.
[10 points]

Solution : Set $x = \pm a + \eta$ and Taylor expand:

$$U(\pm a + \eta) = \frac{4U_0}{a^2}\,\eta^2 + \mathcal{O}(\eta^3) \ . \tag{3}$$

Equating this with $\frac{1}{2}k\,\eta^2$, we have the effective spring constant $k = 8U_0/a^2$, and the small oscillation frequency

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{8\,U_0}{ma^2}} \ . \tag{4}$$

The period is $2\pi/\omega_0$.

[**2**] An R-L-C circuit is shown in fig. 2. The resistive element is a light bulb. The inductance is $L = 400\,\mu$H; the capacitance is $C = 1\,\mu$F; the resistance is $R = 32\,\Omega$. The voltage $V(t)$ oscillates sinusoidally, with $V(t) = V_0 \cos(\omega t)$, where $V_0 = 4\,$V. In this problem, you may neglect all transients; we are interested in the late time, steady state operation of this circuit. Recall the relevant MKS units:

$$1\,\Omega = 1\,\text{V}\cdot\text{s}\,/\,\text{C} \quad , \quad 1\,\text{F} = 1\,\text{C}\,/\,\text{V} \quad , \quad 1\,\text{H} = 1\,\text{V}\cdot\text{s}^2/\,\text{C}\ .$$

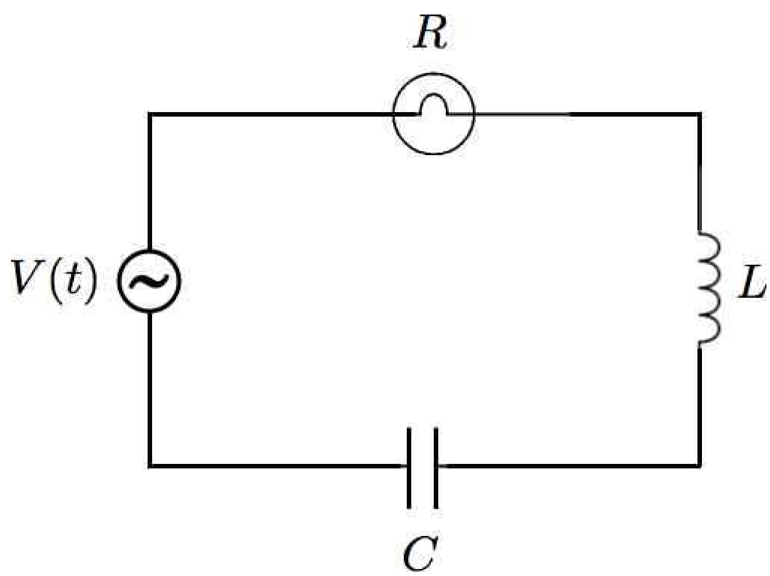

Figure 2: An R-L-C circuit in which the resistive element is a light bulb.

(a) Is this circuit underdamped or overdamped?
`[10 points]`

Solution : We have

$$\omega_0 = (LC)^{-1/2} = 5 \times 10^4\,\text{s}^{-1} \quad , \quad \beta = \frac{R}{2L} = 4 \times 10^4\,\text{s}^{-1}\ .$$

Thus, $\omega_0^2 > \beta^2$ and the circuit is *underdamped*.

(b) Suppose the bulb will only emit light when the average power dissipated by the bulb is greater than a threshold $P_{\text{th}} = \frac{2}{9}\,W$. For fixed $V_0 = 4\,$V, find the frequency range for ω over which the bulb emits light. Recall that the instantaneous power dissipated by a resistor is $P_R(t) = I^2(t)R$. (Average this over a cycle to get the average power dissipated.)
`[20 points]`

Solution : The charge on the capacitor plate obeys the ODE

$$L\,\ddot{Q} + R\,\dot{Q} + \frac{Q}{C} = V(t)\ .$$

The solution is

$$Q(t) = Q_{\text{hom}}(t) + A(\omega)\,\frac{V_0}{L}\,\cos\big(\omega t - \delta(\omega)\big)\ ,$$

with

$$A(\omega) = \Big[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2\Big]^{-1/2} \quad , \quad \delta(\omega) = \tan^{-1}\bigg(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\bigg) \ .$$

Thus, ignoring the transients, the power dissipated by the bulb is

$$\begin{aligned} P_R(t) &= \dot{Q}^2(t)\, R \\ &= \omega^2 A^2(\omega)\, \frac{V_0^2 R}{L^2} \sin^2\!\big(\omega t - \delta(\omega)\big) \ . \end{aligned}$$

Averaging over a period, we have $\langle\, \sin^2(\omega t - \delta)\,\rangle = \frac{1}{2}$, so

$$\langle\, P_R \,\rangle = \omega^2 A^2(\omega)\, \frac{V_0^2 R}{2L^2} = \frac{V_0^2}{2\,R} \cdot \frac{4\beta^2\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \ .$$

Now $V_0^2/2R = \frac{1}{4}\,\mathrm{W}$. So $P_{\mathrm{th}} = \alpha V_0^2/2R$, with $\alpha = \frac{8}{9}$. We then set $\langle P_R \rangle = P_{\mathrm{th}}$, whence

$$(1-\alpha)\cdot 4\beta^2\omega^2 = \alpha\,(\omega_0^2 - \omega^2)^2 \ .$$

The solutions are

$$\omega = \pm\sqrt{\frac{1-\alpha}{\alpha}}\,\beta + \sqrt{\bigg(\frac{1-\alpha}{\alpha}\bigg)\beta^2 + \omega_0^2} = \big(3\sqrt{3} \pm \sqrt{2}\big) \times 1000\,\mathrm{s}^{-1} \ .$$

(c) Compare the expressions for the instantaneous power dissipated by the voltage source, $P_V(t)$, and the power dissipated by the resistor $P_R(t) = I^2(t)R$. If $P_V(t) \neq P_R(t)$, where does the power extra power go or come from? What can you say about the averages of P_V and $P_R(t)$ over a cycle? Explain your answer.
`[20 points]`

Solution : The instantaneous power dissipated by the voltage source is

$$\begin{aligned} P_V(t) = V(t)\, I(t) &= -\omega\, A\, \frac{V_0}{L}\, \sin(\omega t - \delta)\, \cos(\omega t) \\ &= \omega\, A\, \frac{V_0}{2L}\, \Big(\sin\delta - \sin(2\omega t - \delta)\Big) \ . \end{aligned}$$

As we have seen, the power dissipated by the bulb is

$$P_R(t) = \omega^2 A^2\, \frac{V_0^2 R}{L^2} \sin^2(\omega t - \delta) \ .$$

These two quantities are not identical, but they do have identical time averages over one cycle:

$$\langle\, P_V(t)\,\rangle = \langle\, P_R(t)\,\rangle = \frac{V_0^2}{2R} \cdot 4\beta^2\,\omega^2\, A^2(\omega) \ .$$

Energy conservation means

$$P_V(t) = P_R(t) + \dot{E}(t) \ ,$$

where

$$E(t) = \frac{L\dot{Q}^2}{2} + \frac{Q^2}{2C}$$

is the energy in the inductor and capacitor. Since $Q(t)$ is periodic, the average of $\dot{E}$ over a cycle must vanish, which guarantees $\langle\, P_V(t)\,\rangle = \langle\, P_R(t)\,\rangle$.