Chapter 10

Small Oscillations

10.1 Coupled Coordinates

We assume, for a set of n generalized coordinates $\{q_1, \ldots, q_n\}$, that the kinetic energy is a quadratic function of the velocities,

$$T = \tfrac{1}{2}\, T_{\sigma\sigma'}(q_1, \ldots, q_n)\, \dot{q}_\sigma\, \dot{q}_{\sigma'} \ , \tag{10.1}$$

where the sum on σ and σ' from 1 to n is implied. For example, expressed in terms of polar coordinates (r, θ, ϕ), the matrix T_{ij} is

$$T_{\sigma\sigma'} = m \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \qquad \Longrightarrow \qquad T = \tfrac{1}{2}\, m\big(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2\big) \ . \tag{10.2}$$

The potential $U(q_1, \ldots, q_n)$ is assumed to be a function of the generalized coordinates alone: $U = U(q)$. A more general formulation of the problem of small oscillations is given in the appendix, section 10.8.

The generalized momenta are

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} = T_{\sigma\sigma'}\, \dot{q}_{\sigma'} \ , \tag{10.3}$$

and the generalized forces are

$$F_\sigma = \frac{\partial L}{\partial q_\sigma} = \frac{1}{2}\, \frac{\partial T_{\sigma'\sigma''}}{\partial q_\sigma}\, \dot{q}_{\sigma'}\, \dot{q}_{\sigma''} - \frac{\partial U}{\partial q_\sigma} \ . \tag{10.4}$$

The Euler-Lagrange equations are then $\dot{p}_\sigma = F_\sigma$, or

$$T_{\sigma\sigma'}\, \ddot{q}_{\sigma'} + \left(\frac{\partial T_{\sigma\sigma'}}{\partial q_{\sigma''}} - \frac{1}{2}\, \frac{\partial T_{\sigma'\sigma''}}{\partial q_\sigma} \right) \dot{q}_{\sigma'}\, \dot{q}_{\sigma''} = -\frac{\partial U}{\partial q_\sigma} \tag{10.5}$$

which is a set of coupled nonlinear second order ODEs.

10.2 Expansion about Static Equilibrium

Small oscillation theory begins with the identification of a static equilibrium $\{\bar{q}_1, \ldots, \bar{q}_n\}$, which satisfies the n nonlinear equations

$$\frac{\partial U}{\partial q_\sigma}\bigg|_{q=\bar{q}} = 0 \ . \tag{10.6}$$

Once an equilibrium is found (note that there may be more than one static equilibrium), we expand about this equilibrium, writing

$$q_\sigma \equiv \bar{q}_\sigma + \eta_\sigma \ . \tag{10.7}$$

The coordinates $\{\eta_1, \ldots, \eta_n\}$ represent the *displacements relative to equilibrium.*

We next expand the Lagrangian to quadratic order in the generalized displacements, yielding

$$L = \tfrac{1}{2}\mathrm{T}_{\sigma\sigma'}\,\dot{\eta}_\sigma\,\dot{\eta}_{\sigma'} - \tfrac{1}{2}\mathrm{V}_{\sigma\sigma'}\,\eta_\sigma\eta_{\sigma'} \ , \tag{10.8}$$

where

$$\mathrm{T}_{\sigma\sigma'} = \frac{\partial^2 T}{\partial\dot{q}_\sigma\,\partial\dot{q}_{\sigma'}}\bigg|_{q=\bar{q}} \qquad , \qquad \mathrm{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial q_\sigma\,\partial q_{\sigma'}}\bigg|_{q=\bar{q}} \ . \tag{10.9}$$

Writing $\boldsymbol{\eta}^{\mathrm{t}}$ for the row-vector $(\eta_1, \ldots, \eta_n)$, we may suppress indices and write

$$L = \tfrac{1}{2}\,\dot{\boldsymbol{\eta}}^{\mathrm{t}}\,\mathrm{T}\,\dot{\boldsymbol{\eta}} - \tfrac{1}{2}\,\boldsymbol{\eta}^{\mathrm{t}}\,\mathrm{V}\,\boldsymbol{\eta} \ , \tag{10.10}$$

where T and V are the constant matrices of eqn. 10.9.

10.3 Method of Small Oscillations

The idea behind the method of small oscillations is to effect a coordinate transformation from the generalized displacements $\boldsymbol{\eta}$ to a new set of coordinates $\boldsymbol{\xi}$, which render the Lagrangian particularly simple. All that is required is a linear transformation,

$$\eta_\sigma = \mathrm{A}_{\sigma i}\,\xi_i \ , \tag{10.11}$$

where both σ and i run from 1 to n. The $n \times n$ matrix $\mathrm{A}_{\sigma i}$ is known as the *modal matrix.* With the substitution $\boldsymbol{\eta} = \mathrm{A}\,\boldsymbol{\xi}$ (hence $\boldsymbol{\eta}^{\mathrm{t}} = \boldsymbol{\xi}^{\mathrm{t}}\,\mathrm{A}^{\mathrm{t}}$, where $\mathrm{A}^{\mathrm{t}}_{i\sigma} = \mathrm{A}_{\sigma}$ is the matrix transpose), we have

$$L = \tfrac{1}{2}\,\dot{\boldsymbol{\xi}}^{\mathrm{t}}\,\mathrm{A}^{\mathrm{t}}\,\mathrm{T}\,\mathrm{A}\,\dot{\boldsymbol{\xi}} - \tfrac{1}{2}\,\boldsymbol{\xi}^{\mathrm{t}}\,\mathrm{A}^{\mathrm{t}}\,\mathrm{V}\,\mathrm{A}\,\boldsymbol{\xi} \ . \tag{10.12}$$

We now choose the matrix A such that

$$\mathrm{A}^{\mathrm{t}}\,\mathrm{T}\,\mathrm{A} = \mathbf{1} \tag{10.13}$$

$$\mathrm{A}^{\mathrm{t}}\,\mathrm{V}\,\mathrm{A} = \mathrm{diag}\left(\omega_1^2\,,\,\ldots\,,\,\omega_n^2\right)\ . \tag{10.14}$$

With this choice of A, the Lagrangian decouples:

$$L = \tfrac{1}{2}\sum_{i=1}^{n}\left(\dot{\xi}_i^2 - \omega_i^2\,\xi_i^2\right)\ , \tag{10.15}$$

with the solution

$$\xi_i(t) = C_i\,\cos(\omega_i\,t) + D_i\sin(\omega_i\,t)\ , \tag{10.16}$$

where $\{C_1,\ldots,C_n\}$ and $\{D_1,\ldots,D_n\}$ are $2n$ constants of integration, and where no sum is implied on i. Note that

$$\boldsymbol{\xi} = \mathrm{A}^{-1}\boldsymbol{\eta} = \mathrm{A}^{\mathrm{t}}\mathrm{T}\,\boldsymbol{\eta}\ . \tag{10.17}$$

In terms of the original generalized displacements, the solution is

$$\eta_\sigma(t) = \sum_{i=1}^{n}\mathrm{A}_{\sigma i}\left\{C_i\,\cos(\omega_i t) + D_i\sin(\omega_i t)\right\}\ , \tag{10.18}$$

and the constants of integration are linearly related to the initial generalized displacements and generalized velocities:

$$C_i = \mathrm{A}^{\mathrm{t}}_{i\sigma}\,\mathrm{T}_{\sigma\sigma'}\,\eta_{\sigma'}(0) \tag{10.19}$$

$$D_i = \omega_i^{-1}\mathrm{A}^{\mathrm{t}}_{i\sigma}\,\mathrm{T}_{\sigma\sigma'}\,\dot{\eta}_{\sigma'}(0)\ , \tag{10.20}$$

again with no implied sum on i on the RHS of the second equation, and where we have used $\mathrm{A}^{-1} = \mathrm{A}^{\mathrm{t}}\,\mathrm{T}$, from eqn. 10.13. (The implied sums in eqn. 10.20 are over σ and σ'.)

Note that the normal coordinates have unusual dimensions: $[\boldsymbol{\xi}] = \sqrt{M}\cdot L$, where L is length and M is mass.

10.3.1 Can you really just choose an A so that both these wonderful things happen in 10.13 and 10.14?

Yes.

10.3.2 Er...care to elaborate?

Both T and V are symmetric matrices. Aside from that, there is no special relation between them. In particular, they need not commute, hence they do not necessarily share any eigenvectors. Nevertheless, they may be simultaneously diagonalized as per 10.13 and 10.14. Here's why:

- Since T is symmetric, it can be diagonalized by an orthogonal transformation. That is, there exists a matrix $\mathcal{O}_1 \in \mathrm{O}(n)$ such that

$$\mathcal{O}_1^{\mathrm{t}}\,\mathrm{T}\,\mathcal{O}_1 = \mathrm{T}_{\mathrm{d}}\ , \tag{10.21}$$

  where T_{d} is diagonal.

- We may safely assume that T is positive definite. Otherwise the kinetic energy can become arbitrarily negative, which is unphysical. Therefore, one may form the matrix $\mathrm{T}_{\mathrm{d}}^{-1/2}$ which is the diagonal matrix whose entries are the inverse square roots of the corresponding entries of T_{d}. Consider the linear transformation $\mathcal{O}_1\,\mathrm{T}_{\mathrm{d}}^{-1/2}$. Its effect on T is

$$\mathrm{T}_{\mathrm{d}}^{-1/2}\,\mathcal{O}_1^{\mathrm{t}}\,\mathrm{T}\,\mathcal{O}_1\,\mathrm{T}_{\mathrm{d}}^{-1/2} = 1\ . \tag{10.22}$$

- Since $\mathcal{O}_1$ and T_{d} are wholly derived from T, the only thing we know about

$$\widetilde{\mathrm{V}} \equiv \mathrm{T}_{\mathrm{d}}^{-1/2}\,\mathcal{O}_1^{\mathrm{t}}\,\mathrm{V}\,\mathcal{O}_1\,\mathrm{T}_{\mathrm{d}}^{-1/2} \tag{10.23}$$

  is that it is explicitly a symmetric matrix. Therefore, it may be diagonalized by some orthogonal matrix $\mathcal{O}_2 \in \mathrm{O}(n)$. As T has already been transformed to the identity, the additional orthogonal transformation has no effect there. Thus, we have shown that there exist orthogonal matrices $\mathcal{O}_1$ and $\mathcal{O}_2$ such that

$$\mathcal{O}_2^{\mathrm{t}}\,\mathrm{T}_{\mathrm{d}}^{-1/2}\,\mathcal{O}_1^{\mathrm{t}}\,\mathrm{T}\,\mathcal{O}_1\,\mathrm{T}_{\mathrm{d}}^{-1/2}\,\mathcal{O}_2 = 1 \tag{10.24}$$

$$\mathcal{O}_2^{\mathrm{t}}\,\mathrm{T}_{\mathrm{d}}^{-1/2}\,\mathcal{O}_1^{\mathrm{t}}\,\mathrm{V}\,\mathcal{O}_1\,\mathrm{T}_{\mathrm{d}}^{-1/2}\,\mathcal{O}_2 = \mathrm{diag}\,(\omega_1^2, \ldots, \omega_n^2)\ . \tag{10.25}$$

  All that remains is to identify the modal matrix $\mathrm{A} = \mathcal{O}_1\,\mathrm{T}_{\mathrm{d}}^{-1/2}\,\mathcal{O}_2$.

Note that it is *not possible* to simultaneously diagonalize *three* symmetric matrices in general.

10.3.3 Finding the Modal Matrix

While the above proof allows one to construct A by finding the two orthogonal matrices $\mathcal{O}_1$ and $\mathcal{O}_2$, such a procedure is extremely cumbersome. It would be much more convenient if A could be determined in one fell swoop. Fortunately, this is possible.

We start with the equations of motion, $\mathrm{T}\,\ddot{\boldsymbol{\eta}} + \mathrm{V}\,\boldsymbol{\eta} = 0$. In component notation, we have

$$\mathrm{T}_{\sigma\sigma'}\,\ddot{\eta}_{\sigma'} + \mathrm{V}_{\sigma\sigma'}\,\eta_{\sigma'} = 0 \ . \tag{10.26}$$

We now assume that $\boldsymbol{\eta}(t)$ oscillates with a single frequency ω, *i.e.* $\eta_\sigma(t) = \psi_\sigma\, e^{-i\omega t}$. This results in a set of linear algebraic equations for the components ψ_σ:

$$\big(\omega^2\,\mathrm{T}_{\sigma\sigma'} - \mathrm{V}_{\sigma\sigma'}\big)\,\psi_{\sigma'} = 0 \ . \tag{10.27}$$

These are n equations in n unknowns: one for each value of $\sigma = 1, \ldots, n$. Because the equations are homogeneous and linear, there is always a trivial solution $\boldsymbol{\psi} = 0$. In fact one might think this is the only solution, since

$$\big(\omega^2\,\mathrm{T} - \mathrm{V}\big)\,\boldsymbol{\psi} = 0 \qquad \overset{?}{\Longrightarrow} \qquad \boldsymbol{\psi} = \big(\omega^2\,\mathrm{T} - \mathrm{V}\big)^{-1}\,0 = 0 \ . \tag{10.28}$$

However, this fails when the matrix $\omega^2\,\mathrm{T} - \mathrm{V}$ is defective[1], *i.e.* when

$$\det\big(\omega^2\,\mathrm{T} - \mathrm{V}\big) = 0 \ . \tag{10.29}$$

Since T and V are of rank n, the above determinant yields an n^{th} order polynomial in ω^2, whose n roots are the desired squared eigenfrequencies $\{\omega_1^2, \ldots, \omega_n^2\}$.

Once the n eigenfrequencies are obtained, the modal matrix is constructed as follows. Solve the equations

$$\sum_{\sigma'=1}^{n} \big(\omega_i^2\,\mathrm{T}_{\sigma\sigma'} - \mathrm{V}_{\sigma\sigma'}\big)\,\psi^{(i)}_{\sigma'} = 0 \tag{10.30}$$

which are a set of $(n-1)$ linearly independent equations among the n components of the eigenvector $\boldsymbol{\psi}^{(i)}$. That is, there are n equations ($\sigma = 1, \ldots, n$), but one linear dependency since $\det\big(\omega_i^2\,\mathrm{T} - \mathrm{V}\big) = 0$. The eigenvectors may be chosen to satisfy a generalized orthogonality relationship,

$$\psi^{(i)}_{\sigma}\,\mathrm{T}_{\sigma\sigma'}\,\psi^{(j)}_{\sigma'} = \delta_{\sigma\sigma'} \ . \tag{10.31}$$

To see this, let us duplicate eqn. 10.30, replacing i with j, and multiply both equations as follows:

$$\psi^{(j)}_{\sigma} \times \big(\omega_i^2\,\mathrm{T}_{\sigma\sigma'} - \mathrm{V}_{\sigma\sigma'}\big)\,\psi^{(i)}_{\sigma'} = 0 \tag{10.32}$$

$$\psi^{(i)}_{\sigma} \times \big(\omega_j^2\,\mathrm{T}_{\sigma\sigma'} - \mathrm{V}_{\sigma\sigma'}\big)\,\psi^{(j)}_{\sigma'} = 0 \ . \tag{10.33}$$

[1]The label *defective* has a distastefully negative connotation. In modern parlance, we should instead refer to such a matrix as *determinantally challenged*.

Using the symmetry of T and V, upon subtracting these equations we obtain

$$(\omega_i^2 - \omega_j^2) \sum_{\sigma,\sigma'=1}^{n} \psi_\sigma^{(i)} \, \mathrm{T}_{\sigma\sigma'} \, \psi_{\sigma'}^{(j)} = 0 \ , \tag{10.34}$$

where the sums on i and j have been made explicit. This establishes that eigenvectors $\boldsymbol{\psi}^{(i)}$ and $\boldsymbol{\psi}^{(j)}$ corresponding to distinct eigenvalues $\omega_i^2 \neq \omega_j^2$ are orthogonal: $(\boldsymbol{\psi}^{(i)})^{\mathrm{t}} \, \mathrm{T} \, \boldsymbol{\psi}^{(j)} = 0$. For degenerate eigenvalues, the eigenvectors are not *a priori* orthogonal, but they may be orthogonalized via application of the Gram-Schmidt procedure. The remaining degrees of freedom - one for each eigenvector – are fixed by imposing the condition of normalization:

$$\psi_\sigma^{(i)} \to \psi_\sigma^{(i)} \Big/ \sqrt{\psi_\mu^{(i)} \, \mathrm{T}_{\mu\mu'} \, \psi_{\mu'}^{(i)}} \qquad \Longrightarrow \qquad \psi_\sigma^{(i)} \, \mathrm{T}_{\sigma\sigma'} \, \psi_{\sigma'}^{(j)} = \delta_{ij} \ . \tag{10.35}$$

The modal matrix is just the matrix of eigenvectors: $\mathrm{A}_{\sigma i} = \psi_\sigma^{(i)}$.

With the eigenvectors $\psi_\sigma^{(i)}$ thusly normalized, we have

$$\begin{aligned} 0 &= \psi_\sigma^{(i)} \left(\omega_j^2 \, \mathrm{T}_{\sigma\sigma'} - \mathrm{V}_{\sigma\sigma'} \right) \psi_{\sigma'}^{(j)} \\ &= \omega_j^2 \, \delta_{ij} - \psi_\sigma^{(i)} \, \mathrm{V}_{\sigma\sigma'} \, \psi_{\sigma'}^{(j)} \ , \end{aligned} \tag{10.36}$$

with no sum on j. This establishes the result

$$\mathrm{A}^{\mathrm{t}} \, \mathrm{V} \, \mathrm{A} = \mathrm{diag} \left(\omega_1^2 \, , \, \ldots \, , \, \omega_n^2 \right) \ . \tag{10.37}$$

10.4 Example: Masses and Springs

Two blocks and three springs are configured as in Fig. 10.1. All motion is horizontal. When the blocks are at rest, all springs are unstretched.

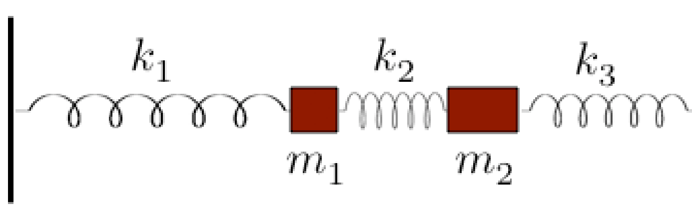

Figure 10.1: A system of masses and springs.

(a) Choose as generalized coordinates the displacement of each block from its equilibrium position, and write the Lagrangian.

(b) Find the T and V matrices.

(c) Suppose

$$m_1 = 2m \quad , \quad m_2 = m \quad , \quad k_1 = 4k \quad , \quad k_2 = k \quad , \quad k_3 = 2k \ ,$$

Find the frequencies of small oscillations.

(d) Find the normal modes of oscillation.

(e) At time $t = 0$, mass #1 is displaced by a distance b relative to its equilibrium position. *I.e.* $x_1(0) = b$. The other initial conditions are $x_2(0) = 0$, $\dot{x}_1(0) = 0$, and $\dot{x}_2(0) = 0$. Find t^*, the next time at which x_2 vanishes.

Solution

(a) The Lagrangian is

$$\boxed{L = \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 \dot{x}_2^2 - \tfrac{1}{2} k_1 x_1^2 - \tfrac{1}{2} k_2 (x_2 - x_1)^2 - \tfrac{1}{2} k_3 x_2^2}$$

(b) The T and V matrices are

$$\boxed{\mathrm{T}_{ij} = \frac{\partial^2 T}{\partial \dot{x}_i \, \partial \dot{x}_j} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}} \quad , \quad \boxed{\mathrm{V}_{ij} = \frac{\partial^2 U}{\partial x_i \, \partial x_j} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix}}$$

(c) We have $m_1 = 2m$, $m_2 = m$, $k_1 = 4k$, $k_2 = k$, and $k_3 = 2k$. Let us write $\omega^2 \equiv \lambda \, \omega_0^2$, where $\omega_0 \equiv \sqrt{k/m}$. Then

$$\omega^2 \mathrm{T} - \mathrm{V} = k \begin{pmatrix} 2\lambda - 5 & 1 \\ 1 & \lambda - 3 \end{pmatrix} \ .$$

The determinant is

$$\begin{aligned} \det\left(\omega^2 \mathrm{T} - \mathrm{V}\right) &= (2\lambda^2 - 11\lambda + 14) \, k^2 \\ &= (2\lambda - 7)\,(\lambda - 2)\, k^2 \ . \end{aligned}$$

There are two roots: $\lambda_- = 2$ and $\lambda_+ = \frac{7}{2}$, corresponding to the eigenfrequencies

$$\boxed{\omega_- = \sqrt{\frac{2k}{m}}} \quad , \quad \boxed{\omega_+ = \sqrt{\frac{7k}{2m}}}$$

(d) The normal modes are determined from $(\omega_a^2 \mathrm{T} - \mathrm{V})\,\vec{\psi}^{(a)} = 0$. Plugging in $\lambda = 2$ we have for the normal mode $\vec{\psi}^{(-)}$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \psi_1^{(-)} \\ \psi_2^{(-)} \end{pmatrix} = 0 \qquad \Rightarrow \qquad \boxed{\vec{\psi}^{(-)} = \mathcal{C}_- \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

Plugging in $\lambda = \frac{7}{2}$ we have for the normal mode $\vec{\psi}^{(+)}$

$$\begin{pmatrix} 2 & 1 \\ 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \psi_1^{(+)} \\ \psi_2^{(+)} \end{pmatrix} = 0 \qquad \Rightarrow \qquad \boxed{\vec{\psi}^{(+)} = \mathcal{C}_+ \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

The standard normalization $\psi_i^{(a)}\, \mathrm{T}_{ij}\, \psi_j^{(b)} = \delta_{ab}$ gives

$$\mathcal{C}_- = \frac{1}{\sqrt{3m}} \qquad , \qquad \mathcal{C}_2 = \frac{1}{\sqrt{6m}} \ . \tag{10.38}$$

(e) The general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_- t) + B \begin{pmatrix} 1 \\ -2 \end{pmatrix} \cos(\omega_+ t) + C \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(\omega_- t) + D \begin{pmatrix} 1 \\ -2 \end{pmatrix} \sin(\omega_+ t) \ .$$

The initial conditions $x_1(0) = b$, $x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$ yield

$$A = \tfrac{2}{3} b \quad , \quad B = \tfrac{1}{3} b \quad , \quad C = 0 \quad , \quad D = 0 \ .$$

Thus,

$$\begin{aligned} x_1(t) &= \tfrac{1}{3} b \cdot \Big(2 \cos(\omega_- t) + \cos(\omega_+ t) \Big) \\ x_2(t) &= \tfrac{2}{3} b \cdot \Big(\cos(\omega_- t) - \cos(\omega_+ t) \Big) \ . \end{aligned}$$

Setting $x_2(t^*) = 0$, we find

$$\cos(\omega_- t^*) = \cos(\omega_+ t^*) \quad \Rightarrow \quad \pi - \omega_- t = \omega_+ t - \pi \quad \Rightarrow \quad \boxed{t^* = \frac{2\pi}{\omega_- + \omega_+}}$$

10.5 Example: Double Pendulum

As a second example, consider the double pendulum, with $m_1 = m_2 = m$ and $\ell_1 = \ell_2 = \ell$. The kinetic and potential energies are

$$T = m\ell^2 \dot{\theta}_1^2 + m\ell^2 \cos(\theta_1 - \theta_1)\, \dot{\theta}_1 \dot{\theta}_2 + \tfrac{1}{2} m\ell^2 \dot{\theta}_2^2 \tag{10.39}$$

$$V = -2mg\ell \cos\theta_1 - mg\ell \cos\theta_2 \ , \tag{10.40}$$

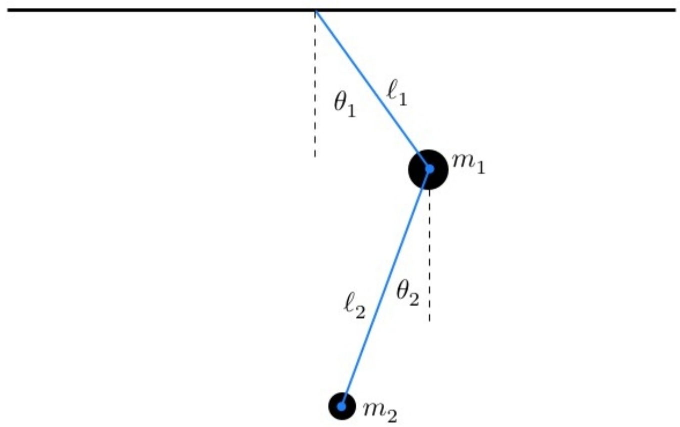

Figure 10.2: The double pendulum.

leading to

$$\mathrm{T} = \begin{pmatrix} 2m\ell^2 & m\ell^2 \\ m\ell^2 & m\ell^2 \end{pmatrix} \qquad , \qquad \mathrm{V} = \begin{pmatrix} 2mg\ell & 0 \\ 0 & mg\ell \end{pmatrix} \ . \tag{10.41}$$

Then

$$\omega^2 \mathrm{T} - \mathrm{V} = m\ell^2 \begin{pmatrix} 2\omega^2 - 2\omega_0^2 & \omega^2 \\ \omega^2 & \omega^2 - \omega_0^2 \end{pmatrix} \ , \tag{10.42}$$

with $\omega_0 = \sqrt{g/\ell}$. Setting the determinant to zero gives

$$2(\omega^2 - \omega_0^2)^2 - \omega^4 = 0 \quad \Rightarrow \quad \omega^2 = (2 \pm \sqrt{2})\, \omega_0^2 \ . \tag{10.43}$$

We find the unnormalized eigenvectors by setting $(\omega_i^2 \, \mathrm{T} - V)\, \psi^{(i)} = 0$. This gives

$$\psi^+ = C_+ \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \qquad , \qquad \psi^- = C_- \begin{pmatrix} 1 \\ +\sqrt{2} \end{pmatrix} \ , \tag{10.44}$$

where $C_\pm$ are constants. One can check $\mathrm{T}_{\sigma\sigma'} \, \psi_\sigma^{(i)} \, \psi_{\sigma'}^{(j)}$ vanishes for $i \neq j$. We then normalize by demanding $\mathrm{T}_{\sigma\sigma'} \, \psi_\sigma^{(i)} \, \psi_{\sigma'}^{(i)} = 1$ (no sum on i), which determines the coefficients $C_\pm = \frac{1}{2}\sqrt{(2 \pm \sqrt{2})/m\ell^2}$. Thus, the modal matrix is

$$\mathrm{A} = \begin{pmatrix} \psi_1^+ & \psi_1^- \\ \psi_2^+ & \psi_2^- \end{pmatrix} = \frac{1}{2\sqrt{m\ell^2}} \begin{pmatrix} \sqrt{2+\sqrt{2}} & \sqrt{2-\sqrt{2}} \\ -\sqrt{4+2\sqrt{2}} & +\sqrt{4-2\sqrt{2}} \end{pmatrix} \ . \tag{10.45}$$

10.6 Zero Modes

Recall Noether's theorem, which says that for every continuous one-parameter family of coordinate transformations,

$$q_\sigma \longrightarrow \tilde{q}_\sigma(q,\zeta) \qquad , \qquad \tilde{q}_\sigma(q,\zeta=0) = q_\sigma \ , \tag{10.46}$$

which leaves the Lagrangian invariant, *i.e.* $dL/d\zeta = 0$, there is an associated conserved quantity,

$$\varLambda = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} \qquad \text{satisfies} \qquad \frac{d\varLambda}{dt} = 0 \ . \tag{10.47}$$

For small oscillations, we write $q_\sigma = \bar{q}_\sigma + \eta_\sigma$, hence

$$\varLambda_k = \sum_\sigma C_{k\sigma}\, \dot{\eta}_\sigma \ , \tag{10.48}$$

where k labels the one-parameter families (in the event there is more than one continuous symmetry), and where

$$C_{k\sigma} = \sum_{\sigma'} \mathrm{T}_{\sigma\sigma'} \frac{\partial \tilde{q}_{\sigma'}}{\partial \zeta_k}\bigg|_{\zeta=0} \ . \tag{10.49}$$

Therefore, we can define the (unnormalized) normal mode

$$\xi_k = \sum_\sigma C_{k\sigma}\, \eta_\sigma \ , \tag{10.50}$$

which satisfies $\ddot{\xi}_k = 0$. Thus, in systems with continuous symmetries, to each such continuous symmetry there is an associated zero mode of the small oscillations problem, *i.e.* a mode with $\omega_k^2 = 0$.

10.6.1 Example of Zero Mode Oscillations

The simplest example of a zero mode would be a pair of masses m_1 and m_2 moving frictionlessly along a line and connected by a spring of force constant k. We know from our study of central forces that the Lagrangian may be written

$$\begin{aligned} L &= \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 \dot{x}_2^2 - \tfrac{1}{2} k (x_1 - x_2)^2 \\ &= \tfrac{1}{2} M \dot{X}^2 + \tfrac{1}{2} \mu \dot{x}^2 - \tfrac{1}{2} k x^2 \ , \end{aligned} \tag{10.51}$$

where $X = (m_1 x_1 + m_2 x_2)/(m_1 + m_2)$ is the center of mass position, $x = x_1 - x_2$ is the relative coordinate, $M = m_1 + m_2$ is the total mass, and $\mu = m_1 m_2/(m_1 + m_2)$

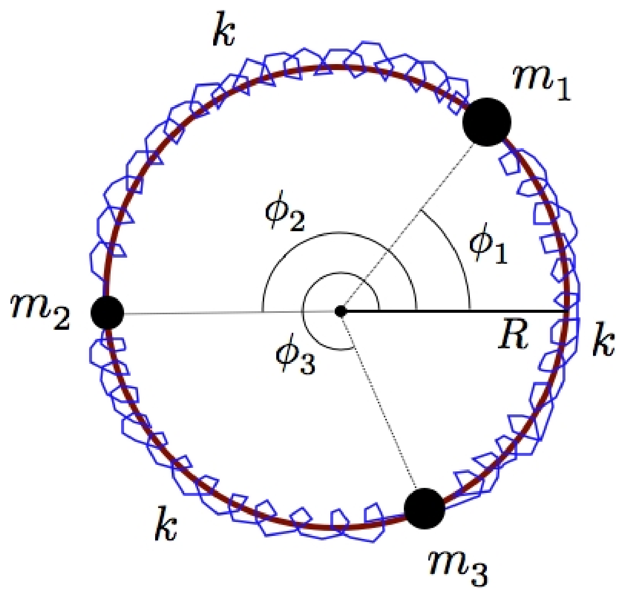

Figure 10.3: Coupled oscillations of three masses on a frictionless hoop of radius R. All three springs have the same force constant k, but the masses are all distinct.

is the reduced mass. The relative coordinate obeys $\ddot{x} = -\omega_0^2\, x$, where the oscillation frequency is $\omega_0 = \sqrt{k/\mu}$. The center of mass coordinate obeys $\ddot{X} = 0$, *i.e.* its oscillation frequency is zero. The center of mass motion is a zero mode.

Another example is furnished by the system depicted in fig. 10.3, where three distinct masses m_1, m_2, and m_3 move around a frictionless hoop of radius R. The masses are connected to their neighbors by identical springs of force constant k. We choose as generalized coordinates the angles ϕ_σ $(\sigma = 1, 2, 3)$, with the convention that

$$\phi_1 \le \phi_2 \le \phi_3 \le 2\pi + \phi_1 \ . \tag{10.52}$$

Let $R\chi$ be the equilibrium length for each of the springs. Then the potential energy is

$$\begin{aligned} U &= \tfrac{1}{2}kR^2\Big\{(\phi_2-\phi_1-\chi)^2+(\phi_3-\phi_2-\chi)^2+(2\pi+\phi_1-\phi_3-\chi)^2\Big\} \\ &= \tfrac{1}{2}kR^2\Big\{(\phi_2-\phi_1)^2+(\phi_3-\phi_2)^2+(2\pi+\phi_1-\phi_3)^2+3\chi^2-4\pi\chi\Big\} \ . \end{aligned} \tag{10.53}$$

Note that the equilibrium angle χ enters only in an additive constant to the potential energy. Thus, for the calculation of the equations of motion, it is irrelevant. It doesn't matter whether or not the equilibrium configuration is unstretched $(\chi = 2\pi/3)$ or not $(\chi \ne 2\pi/3)$.

The kinetic energy is simple:

$$T = \tfrac{1}{2}R^2\Big(m_1\,\dot\phi_1^2 + m_2\,\dot\phi_2^2 + m_3\,\dot\phi_3^2\Big) \ . \tag{10.54}$$

The T and V matrices are then

$$\mathrm{T} = \begin{pmatrix} m_1 R^2 & 0 & 0 \\ 0 & m_2 R^2 & 0 \\ 0 & 0 & m_3 R^2 \end{pmatrix} \quad , \qquad \mathrm{V} = \begin{pmatrix} 2kR^2 & -kR^2 & -kR^2 \\ -kR^2 & 2kR^2 & -kR^2 \\ -kR^2 & -kR^2 & 2kR^2 \end{pmatrix} \ . \tag{10.55}$$

We then have

$$\omega^2\,\mathrm{T} - \mathrm{V} = kR^2 \begin{pmatrix} \frac{\omega^2}{\Omega_1^2} - 2 & 1 & 1 \\ 1 & \frac{\omega^2}{\Omega_2^2} - 2 & 1 \\ 1 & 1 & \frac{\omega^2}{\Omega_3^2} - 2 \end{pmatrix} \ . \tag{10.56}$$

We compute the determinant to find the characteristic polynomial:

$$\begin{aligned} P(\omega) &= \det(\omega^2\,\mathrm{T} - \mathrm{V}) \qquad\qquad (10.57) \\ &= \frac{\omega^6}{\Omega_1^2\,\Omega_2^2\,\Omega_3^2} - 2\left(\frac{1}{\Omega_1^2\,\Omega_2^2} + \frac{1}{\Omega_2^2\,\Omega_3^2} + \frac{1}{\Omega_1^2\,\Omega_3^2}\right)\omega^4 + 3\left(\frac{1}{\Omega_1^2} + \frac{1}{\Omega_2^2} + \frac{1}{\Omega_3^2}\right)\omega^2 \ , \end{aligned}$$

where $\Omega_i^2 \equiv k/m_i$. The equation $P(\omega) = 0$ yields a cubic equation in ω^2, but clearly ω^2 is a factor, and when we divide this out we obtain a quadratic equation. One root obviously is $\omega_1^2 = 0$. The other two roots are solutions to the quadratic equation:

$$\omega_{2,3}^2 = \Omega_1^2 + \Omega_2^2 + \Omega_3^2 \pm \sqrt{\tfrac{1}{2}\big(\Omega_1^2 - \Omega_2^2\big)^2 + \tfrac{1}{2}\big(\Omega_2^2 - \Omega_3^2\big)^2 + \tfrac{1}{2}\big(\Omega_1^2 - \Omega_3^2\big)^2} \ . \tag{10.58}$$

To find the eigenvectors and the modal matrix, we set

$$\begin{pmatrix} \frac{\omega_j^2}{\Omega_1^2} - 2 & 1 & 1 \\ 1 & \frac{\omega_j^2}{\Omega_2^2} - 2 & 1 \\ 1 & 1 & \frac{\omega_j^2}{\Omega_3^2} - 2 \end{pmatrix} \begin{pmatrix} \psi_1^{(j)} \\ \psi_2^{(j)} \\ \psi_3^{(j)} \end{pmatrix} = 0 \ , \tag{10.59}$$

Writing down the three coupled equations for the components of $\boldsymbol{\psi}^{(j)}$, we find

$$\left(\frac{\omega_j^2}{\Omega_1^2} - 3\right)\psi_1^{(j)} = \left(\frac{\omega_j^2}{\Omega_2^2} - 3\right)\psi_2^{(j)} = \left(\frac{\omega_j^2}{\Omega_3^2} - 3\right)\psi_3^{(j)} \ . \tag{10.60}$$

We therefore conclude

$$\boldsymbol{\psi}^{(j)} = \mathcal{C}_j \begin{pmatrix} \left(\frac{\omega_j^2}{\Omega_1^2} - 3\right)^{-1} \\ \left(\frac{\omega_j^2}{\Omega_2^2} - 3\right)^{-1} \\ \left(\frac{\omega_j^2}{\Omega_3^2} - 3\right)^{-1} \end{pmatrix} \ . \tag{10.61}$$

The normalization condition $\psi_\sigma^{(i)}\,\mathrm{T}_{\sigma\sigma'}\,\psi_{\sigma'}^{(j)} = \delta_{ij}$ then fixes the constants $\mathcal{C}_j$:

$$\left[m_1\left(\frac{\omega_j^2}{\Omega_1^2} - 3\right)^{-2} + \, m_2\left(\frac{\omega_j^2}{\Omega_2^2} - 3\right)^{-2} + \, m_3\left(\frac{\omega_j^2}{\Omega_3^2} - 3\right)^{-2}\right]\big|\mathcal{C}_j\big|^2 = 1 \ . \tag{10.62}$$

The Lagrangian is invariant under the one-parameter family of transformations

$$\phi_\sigma \longrightarrow \phi_\sigma + \zeta \tag{10.63}$$

for all $\sigma = 1, 2, 3$. The associated conserved quantity is

$$\begin{aligned} \Lambda &= \sum_\sigma \frac{\partial L}{\partial \dot{\phi}_\sigma} \frac{\partial \tilde{\phi}_\sigma}{\partial \zeta} \\ &= R^2 \left(m_1 \dot{\phi}_1 + m_2 \dot{\phi}_2 + m_3 \dot{\phi}_3 \right) , \end{aligned} \tag{10.64}$$

which is, of course, the total angular momentum relative to the center of the ring. Thus, from $\dot{\Lambda} = 0$ we identify the zero mode as ξ_1, where

$$\xi_1 = \mathcal{C} \left(m_1 \phi_1 + m_2 \phi_2 + m_3 \phi_3 \right) , \tag{10.65}$$

where $\mathcal{C}$ is a constant. Recall the relation $\eta_\sigma = \mathrm{A}_{\sigma i} \, \xi_i$ between the generalized displacements η_σ and the normal coordinates ξ_i. We can invert this relation to obtain

$$\xi_i = \mathrm{A}^{-1}_{i\sigma} \, \eta_\sigma = \mathrm{A}^{\mathrm{t}}_{i\sigma} \, \mathrm{T}_{\sigma\sigma'} \, \eta_{\sigma'} \ . \tag{10.66}$$

Here we have used the result $\mathrm{A}^{\mathrm{t}} \, \mathrm{T} \, \mathrm{A} = 1$ to write

$$\mathrm{A}^{-1} = \mathrm{A}^{\mathrm{t}} \, \mathrm{T} \ . \tag{10.67}$$

This is a convenient result, because it means that if we ever need to express the normal coordinates in terms of the generalized displacements, we don't have to invert any matrices – we just need to do one matrix multiplication. In our case here, the T matrix is diagonal, so the multiplication is trivial. From eqns. 10.65 and 10.66, we conclude that the matrix $\mathrm{A}^{\mathrm{t}}\,\mathrm{T}$ must have a first *row* which is proportional to (m_1, m_2, m_3). Since these are the very diagonal entries of T, we conclude that A^{t} itself must have a first row which is proportional to $(1, 1, 1)$, which means that the first *column* of A is proportional to $(1, 1, 1)$. But this is confirmed by eqn. 10.60 when we take $j = 1$, since $\omega^2_{j=1} = 0$: $\psi^{(1)}_1 = \psi^{(1)}_2 = \psi^{(1)}_3$.

10.7 Chain of Mass Points

Next consider an infinite chain of identical masses, connected by identical springs of spring constant k and equilibrium length a. The Lagrangian is

$$\begin{aligned} L &= \tfrac{1}{2} m \sum_n \dot{x}_n^2 - \tfrac{1}{2} k \sum_n (x_{n+1} - x_n - a)^2 \\ &= \tfrac{1}{2} m \sum_n \dot{u}_n^2 - \tfrac{1}{2} k \sum_n (u_{n+1} - u_n)^2 \ , \end{aligned} \tag{10.68}$$

where $u_n \equiv x_n - na - b$ is the displacement from equilibrium of the n^{th} mass. The constant b is arbitrary. The Euler-Lagrange equations are

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_n}\right) = m\ddot{u}_n &= \frac{\partial L}{\partial u_n} \\ &= k(u_{n+1} - u_n) - k(u_n - u_{n-1}) \\ &= k(u_{n+1} + u_{n-1} - 2u_n) \ . \end{aligned} \tag{10.69}$$

Now let us assume that the system is placed on a large ring of circumference Na, where $N \gg 1$. Then $u_{n+N} = u_n$ and we may shift to Fourier coefficients,

$$u_n = \frac{1}{\sqrt{N}} \sum_q e^{iqan}\, \hat{u}_q \tag{10.70}$$

$$\hat{u}_q = \frac{1}{\sqrt{N}} \sum_n e^{-iqan}\, u_n \ , \tag{10.71}$$

where $q_j = 2\pi j/Na$, and both sums are over the set $j, n \in \{1, \ldots, N\}$. Expressed in terms of the $\{\hat{u}_q\}$, the equations of motion become

$$\begin{aligned} \ddot{\hat{u}}_q &= \frac{1}{\sqrt{N}} \sum_n e^{-iqna}\, \ddot{u}_n \\ &= \frac{k}{m}\frac{1}{\sqrt{N}} \sum_n e^{-iqan}\, (u_{n+1} + u_{n-1} - 2u_n) \\ &= \frac{k}{m}\frac{1}{\sqrt{N}} \sum_n e^{-iqan}\, (e^{-iqa} + e^{+iqa} - 2)\, u_n \\ &= -\frac{2k}{m}\, \sin^2\left(\tfrac{1}{2}qa\right) \hat{u}_q \end{aligned} \tag{10.72}$$

Thus, the $\{\hat{u}_q\}$ *are* the normal modes of the system (up to a normalization constant), and the eigenfrequencies are

$$\omega_q = \frac{2k}{m} \left| \sin\left(\tfrac{1}{2}qa\right) \right| \ . \tag{10.73}$$

This means that the modal matrix is

$$\mathrm{A}_{nq} = \frac{1}{\sqrt{Nm}}\, e^{iqan} \ , \tag{10.74}$$

where we've included the $\frac{1}{\sqrt{m}}$ factor for a proper normalization. (The normal modes themselves are then $\xi_q = \mathrm{A}^\dagger_{qn} \mathrm{T}_{nn'} u_{n'} = \sqrt{m}\, \hat{u}_q$. For complex A, the normalizations are $\mathrm{A}^\dagger \mathrm{T} \mathrm{A} = \mathbf{1}$ and $\mathrm{A}^\dagger \mathrm{V} \mathrm{A} = \mathrm{diag}(\omega_1^2, \ldots, \omega_N^2)$.

Note that

$$\mathrm{T}_{nn'} = m\,\delta_{n,n'} \tag{10.75}$$

$$\mathrm{V}_{nn'} = 2k\,\delta_{n,n'} - k\,\delta_{n,n'+1} - k\,\delta_{n,n'-1} \tag{10.76}$$

and that

$$\begin{aligned}(\mathrm{A}^\dagger \mathrm{T} \mathrm{A})_{qq'} &= \sum_{n=1}^N \sum_{n'=1}^N \mathrm{A}^*_{nq} \mathrm{T}_{nn'} \mathrm{A}_{n'q'} \\ &= \frac{1}{Nm} \sum_{n=1}^N \sum_{n'=1}^N e^{-iqan}\, m\,\delta_{nn'}\, e^{iq'an'} \\ &= \frac{1}{N} \sum_{n=1}^N e^{i(q'-q)an} = \delta_{qq'} \ ,\end{aligned} \tag{10.77}$$

and

$$\begin{aligned}(\mathrm{A}^\dagger \mathrm{V} \mathrm{A})_{qq'} &= \sum_{n=1}^N \sum_{n'=1}^N \mathrm{A}^*_{nq} \mathrm{T}_{nn'} \mathrm{A}_{n'q'} \\ &= \frac{1}{Nm} \sum_{n=1}^N \sum_{n'=1}^N e^{-iqan} \Big(2k\,\delta_{n,n'} - k\,\delta_{n,n'+1} - k\,\delta_{n,n'-1} \Big)\, e^{iq'an'} \\ &= \frac{k}{m} \frac{1}{N} \sum_{n=1}^N e^{i(q'-q)an} \left(2 - e^{-iq'a} - e^{iq'a} \right) \\ &= \frac{4k}{m} \sin^2\!\left(\tfrac{1}{2}qa\right) \delta_{qq'} = \omega_q^2\, \delta_{qq'}\end{aligned} \tag{10.78}$$

Since $\hat{x}_{q+G} = \hat{x}_q$, where $G = \frac{2\pi}{a}$, we may choose any set of q values such that no two are separated by an integer multiple of G. The set of points $\{jG\}$ with $j \in \mathbf{Z}$ is called the *reciprocal lattice*. For a linear chain, the reciprocal lattice is itself a linear chain[2]. One natural set to choose is $q \in \left[-\frac{\pi}{a}, \frac{\pi}{a}\right]$. This is known as the *first Brillouin zone* of the reciprocal lattice.

Finally, we can write the Lagrangian itself in terms of the $\{u_q\}$. One easily finds

$$L = \tfrac{1}{2}\, m \sum_q \dot{\hat{u}}^*_q\, \dot{\hat{u}}_q - \ k \sum_q (1 - \cos qa)\, \hat{u}^*_q\, \hat{u}_q \ , \tag{10.79}$$

where the sum is over q in the first Brillouin zone. Note that

$$\hat{u}_{-q} = \hat{u}_{-q+G} = \hat{u}^*_q \ . \tag{10.80}$$

[2]For higher dimensional Bravais lattices, the reciprocal lattice is often different than the real space ("direct") lattice. For example, the reciprocal lattice of a face-centered cubic structure is a body-centered cubic lattice.

This means that we can restrict the sum to half the Brillouin zone:

$$L = \tfrac{1}{2} m \sum_{q \in [0, \frac{\pi}{a}]} \left\{ \dot{\hat{u}}_q^* \, \dot{\hat{u}}_q - \frac{4k}{m} \, \sin^2\!\big(\tfrac{1}{2} qa\big) \, \hat{u}_q^* \, \hat{u}_q \right\} . \tag{10.81}$$

Now $\hat{u}_q$ and $\hat{u}_q^*$ may be regarded as linearly independent, as one regards complex variables z and z^*. The Euler-Lagrange equation for $\hat{u}_q^*$ gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\hat{u}}_q^*}\right) = \frac{\partial L}{\partial \hat{u}_q^*} \quad \Rightarrow \quad \ddot{\hat{u}}_q = -\omega_q^2 \, \hat{u}_q \ . \tag{10.82}$$

Extremizing with respect to $\hat{u}_q$ gives the complex conjugate equation.

10.7.1 Continuum Limit

Let us take $N \to \infty$, $a \to 0$, with $L_0 = Na$ fixed. We'll write

$$u_n(t) \longrightarrow u(x = na, t) \tag{10.83}$$

in which case

$$T = \tfrac{1}{2} m \sum_n \dot{u}_n^2 \qquad \longrightarrow \qquad \tfrac{1}{2} m \int \frac{dx}{a} \left(\frac{\partial u}{\partial t}\right)^2 \tag{10.84}$$

$$V = \tfrac{1}{2} k \sum_n (u_{n+1} - u_n)^2 \qquad \longrightarrow \qquad \tfrac{1}{2} k \int \frac{dx}{a} \left(\frac{u(x+a) - u(x)}{a}\right)^2 a^2 \tag{10.85}$$

Recognizing the spatial derivative above, we finally obtain

$$\begin{aligned} L &= \int dx \, \mathcal{L}(u, \partial_t u, \partial_x u) \\ \mathcal{L} &= \tfrac{1}{2} \, \mu \left(\frac{\partial u}{\partial t}\right)^2 - \tfrac{1}{2} \, \tau \left(\frac{\partial u}{\partial x}\right)^2 , \end{aligned} \tag{10.86}$$

where $\mu = m/a$ is the linear mass density and $\tau = ka$ is the tension[3]. The quantity $\mathcal{L}$ is the *Lagrangian density*; it depends on the field $u(x,t)$ as well as its partial derivatives $\partial_t u$ and $\partial_x u$[4]. The action is

$$S\big[u(x,t)\big] = \int_{t_a}^{t_b} \!\! dt \int_{x_a}^{x_b} \!\! dx \, \mathcal{L}(u, \partial_t u, \partial_x u) \ , \tag{10.87}$$

[3]For a proper limit, we demand μ and τ be neither infinite nor infinitesimal.

[4]$\mathcal{L}$ may also depend explicitly on x and t.

where $\{x_a, x_b\}$ are the limits on the x coordinate. Setting $\delta S = 0$ gives the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_t u\right)}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_x u\right)}\right) = 0 \ . \tag{10.88}$$

For our system, this yields the Helmholtz equation,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \ , \tag{10.89}$$

where $c = \sqrt{\tau/\mu}$ is the velocity of wave propagation. This is a linear equation, solutions of which are of the form

$$u(x,t) = C\, e^{iqx}\, e^{-i\omega t} \ , \tag{10.90}$$

where

$$\omega = cq \ . \tag{10.91}$$

Note that in the continuum limit $a \to 0$, the dispersion relation derived for the chain becomes

$$\omega_q^2 = \frac{4k}{m}\sin^2\!\left(\tfrac{1}{2}qa\right) \longrightarrow \frac{ka^2}{m}\, q^2 = c^2\, q^2 \ , \tag{10.92}$$

and so the results agree.

10.8 Appendix I : General Formulation

In the development in section 10.1, we assumed that the kinetic energy T is a homogeneous function of degree 2, and the potential energy U a homogeneous function of degree 0, in the generalized velocities $\dot{q}_\sigma$. However, we've encountered situations where this is not so: problems with time-dependent holonomic constraints, such as the mass point on a rotating hoop, and problems involving charged particles moving in magnetic fields. The general Lagrangian is of the form

$$L = \tfrac{1}{2}\, T_{2\,\sigma\sigma'}(q)\,\dot{q}_\sigma\,\dot{q}_{\sigma'} + T_{1\,\sigma}(q)\,\dot{q}_\sigma + T_0(q) - U_{1\,\sigma}(q)\,\dot{q}_\sigma - U_0(q) \ , \tag{10.93}$$

where the subscript 0, 1, or 2 labels the degree of homogeneity of each term in the generalized velocities. The generalized momenta are then

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} = T_{2\,\sigma\sigma'}\,\dot{q}_{\sigma'} + T_{1\,\sigma} - U_{1\,\sigma} \tag{10.94}$$

and the generalized forces are

$$F_\sigma = \frac{\partial L}{\partial q_\sigma} = \frac{\partial (T_0 - U_0)}{\partial q_\sigma} + \frac{\partial (T_{1\,\sigma'} - U_{1\,\sigma'})}{\partial q_\sigma}\,\dot{q}_{\sigma'} + \frac{1}{2}\,\frac{\partial T_{2\,\sigma'\sigma''}}{\partial q_\sigma}\,\dot{q}_{\sigma'}\,\dot{q}_{\sigma''} \ , \tag{10.95}$$

and the equations of motion are again $\dot{p}_\sigma = F_\sigma$. Once we solve

In equilibrium, we seek a time-independent solution of the form $q_\sigma(t) = \bar{q}_\sigma$. This entails

$$\frac{\partial}{\partial q_\sigma}\bigg|_{q=\bar{q}} \Big(U_0(q) - T_0(q)\Big) = 0 \ , \tag{10.96}$$

which give us n equations in the n unknowns $(q_1, \ldots, q_n)$. We then write $q_\sigma = \bar{q}_\sigma + \eta_\sigma$ and expand in the notionally small quantities η_σ. It is important to understand that we assume η and all of its time derivatives as well are small. Thus, we can expand L to quadratic order in $(\eta, \dot{\eta})$ to obtain

$$L = \tfrac{1}{2}\, \mathrm{T}_{\sigma\sigma'}\, \dot{\eta}_\sigma\, \dot{\eta}_{\sigma'} - \tfrac{1}{2}\, \mathrm{B}_{\sigma\sigma'}\, \eta_\sigma\, \dot{\eta}_{\sigma'} - \tfrac{1}{2}\, \mathrm{V}_{\sigma\sigma'}\, \eta_\sigma\, \eta_{\sigma'} \ , \tag{10.97}$$

where

$$\mathrm{T}_{\sigma\sigma'} = T_{2\,\sigma\sigma'}(\bar{q}) \quad , \quad \mathrm{V}_{\sigma\sigma'} = \frac{\partial^2\big(U_0 - T_0\big)}{\partial q_\sigma\, \partial q_{\sigma'}}\bigg|_{q=\bar{q}} \quad , \quad \mathrm{B}_{\sigma\sigma'} = 2\, \frac{\partial\big(U_{1\,\sigma'} - T_{1\,\sigma'}\big)}{\partial q_\sigma}\bigg|_{q=\bar{q}} \ . \tag{10.98}$$

Note that the T and V matrices are symmetric. The $\mathrm{B}_{\sigma\sigma'}$ term is new.

Now we can always write $\mathrm{B} = \frac{1}{2}(\mathrm{B}^{\mathrm{s}} + \mathrm{B}^{\mathrm{a}})$ as a sum over symmetric and antisymmetric parts, with $\mathrm{B}^{\mathrm{s}} = \mathrm{B} + \mathrm{B}^{\mathrm{t}}$ and $\mathrm{B}^{\mathrm{a}} = \mathrm{B} - \mathrm{B}^{\mathrm{t}}$. Since,

$$\mathrm{B}^{\mathrm{s}}_{\sigma\sigma'}\, \eta_\sigma\, \dot{\eta}_{\sigma'} = \frac{d}{dt}\Big(\tfrac{1}{2}\, \mathrm{B}^{\mathrm{s}}_{\sigma\sigma'}\, \eta_\sigma\, \eta_{\sigma'}\Big) \ , \tag{10.99}$$

any symmetric part to B contributes a total time derivative to L, and thus has no effect on the equations of motion. Therefore, we can project B onto its antisymmetric part, writing

$$\mathrm{B}_{\sigma\sigma'} = \left(\frac{\partial\big(U_{1\,\sigma'} - T_{1\,\sigma'}\big)}{\partial q_\sigma} - \frac{\partial\big(U_{1\,\sigma} - T_{1\,\sigma}\big)}{\partial q_{\sigma'}}\right)_{q=\bar{q}} \ . \tag{10.100}$$

We now have

$$p_\sigma = \frac{\partial L}{\partial \dot{\eta}_\sigma} = \mathrm{T}_{\sigma\sigma'}\, \dot{\eta}_{\sigma'} + \tfrac{1}{2}\, \mathrm{B}_{\sigma\sigma'}\, \eta_{\sigma'} \ , \tag{10.101}$$

and

$$F_\sigma = \frac{\partial L}{\partial \eta_\sigma} = -\tfrac{1}{2}\, \mathrm{B}_{\sigma\sigma'}\, \dot{\eta}_{\sigma'} - \mathrm{V}_{\sigma\sigma'}\, \eta_{\sigma'} \ . \tag{10.102}$$

The equations of motion, $\dot{p}_\sigma = F_\sigma$, then yield

$$\mathrm{T}_{\sigma\sigma'}\, \ddot{\eta}_{\sigma'} + \mathrm{B}_{\sigma\sigma'}\, \dot{\eta}_{\sigma'} + \mathrm{V}_{\sigma\sigma'}\, \eta_{\sigma'} = 0 \ . \tag{10.103}$$

Let us write $\boldsymbol{\eta}(t) = \boldsymbol{\eta}\, e^{-i\omega t}$. We then have

$$\big(\omega^2\,\mathrm{T} + i\omega\,\mathrm{B} - \mathrm{V}\big)\,\boldsymbol{\eta} = 0\ . \tag{10.104}$$

To solve eqn. 10.104, we set $P(\omega) = 0$, where $P(\omega) = \det\big[\mathrm{Q}(\omega)\big]$, with

$$\mathrm{Q}(\omega) \equiv \omega^2\,\mathrm{T} + i\omega\,\mathrm{B} - \mathrm{V}\ . \tag{10.105}$$

Since T, B, and V are real-valued matrices, and since $\det(M) = \det(M^{\mathrm{t}})$ for any matrix M, we can use $\mathrm{B}^{\mathrm{t}} = -\mathrm{B}$ to obtain $P(-\omega) = P(\omega)$ and $P(\omega^*) = \big[P(\omega)\big]^*$. This establishes that if $P(\omega) = 0$, *i.e.* if ω is an eigenfrequency, then $P(-\omega) = 0$ and $P(\omega^*) = 0$, *i.e.* $-\omega$ and ω^* are also eigenfrequencies (and hence $-\omega^*$ as well).

10.9 Appendix II : Additional Examples

10.9.1 Right Triatomic Molecule

A molecule consists of three identical atoms located at the vertices of a 45° right triangle. Each pair of atoms interacts by an effective spring potential, with all spring constants equal to k. Consider only planar motion of this molecule.

(a) Find three 'zero modes' for this system (*i.e.* normal modes whose associated eigenfrequencies vanish).

(b) Find the remaining three normal modes.

Solution

It is useful to choose the following coordinates:

$$(X_1, Y_1) = (x_1\ ,\ y_1) \tag{10.106}$$

$$(X_2, Y_2) = (a + x_2\ ,\ y_2) \tag{10.107}$$

$$(X_3, Y_3) = (x_3\ ,\ a + y_3)\ . \tag{10.108}$$

The three separations are then

$$
\begin{aligned}
d_{12} &= \sqrt{(a + x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= a + x_2 - x_1 + \ldots
\end{aligned}
\tag{10.109}
$$

$$
\begin{aligned}
d_{23} &= \sqrt{(-a + x_3 - x_2)^2 + (a + y_3 - y_2)^2} \\
&= \sqrt{2}\, a - \tfrac{1}{\sqrt{2}}\big(x_3 - x_2\big) + \tfrac{1}{\sqrt{2}}\big(y_3 - y_2\big) + \ldots
\end{aligned}
\tag{10.110}
$$

$$
\begin{aligned}
d_{13} &= \sqrt{(x_3 - x_1)^2 + (a + y_3 - y_1)^2} \\
&= a + y_3 - y_1 + \ldots\ .
\end{aligned}
\tag{10.111}
$$

The potential is then

$$
U = \tfrac{1}{2}k\,\big(d_{12} - a\big)^2 + \tfrac{1}{2}k\,\big(d_{23} - \sqrt{2}\,a\big)^2 + \tfrac{1}{2}k\,\big(d_{13} - a\big)^2 \tag{10.112}
$$

$$
\begin{aligned}
&= \tfrac{1}{2}k\big(x_2 - x_1\big)^2 + \tfrac{1}{4}k\big(x_3 - x_2\big)^2 + \tfrac{1}{4}k\big(y_3 - y_2\big)^2 \\
&\qquad - \tfrac{1}{2}k\big(x_3 - x_2\big)\big(y_3 - y_2\big) + \tfrac{1}{2}k\big(y_3 - y_1\big)^2
\end{aligned}
\tag{10.113}
$$

Defining the row vector

$$
\boldsymbol{\eta}^{\mathrm{t}} \equiv \big(x_1\ ,\ y_1\ ,\ x_2\ ,\ y_2\ ,\ x_3\ ,\ y_3\big)\ , \tag{10.114}
$$

we have that U is a quadratic form:

$$
U = \tfrac{1}{2}\eta_\sigma \mathrm{V}_{\sigma\sigma'}\eta_{\sigma'} = \tfrac{1}{2}\boldsymbol{\eta}^{\mathrm{t}}\,\mathrm{V}\,\boldsymbol{\eta}, \tag{10.115}
$$

with

$$
\mathrm{V} = \mathrm{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial q_\sigma\,\partial q_{\sigma'}}\bigg|_{\mathrm{eq.}} = \ k \begin{pmatrix}
1 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 \\
-1 & 0 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
0 & -1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2}
\end{pmatrix} \tag{10.116}
$$

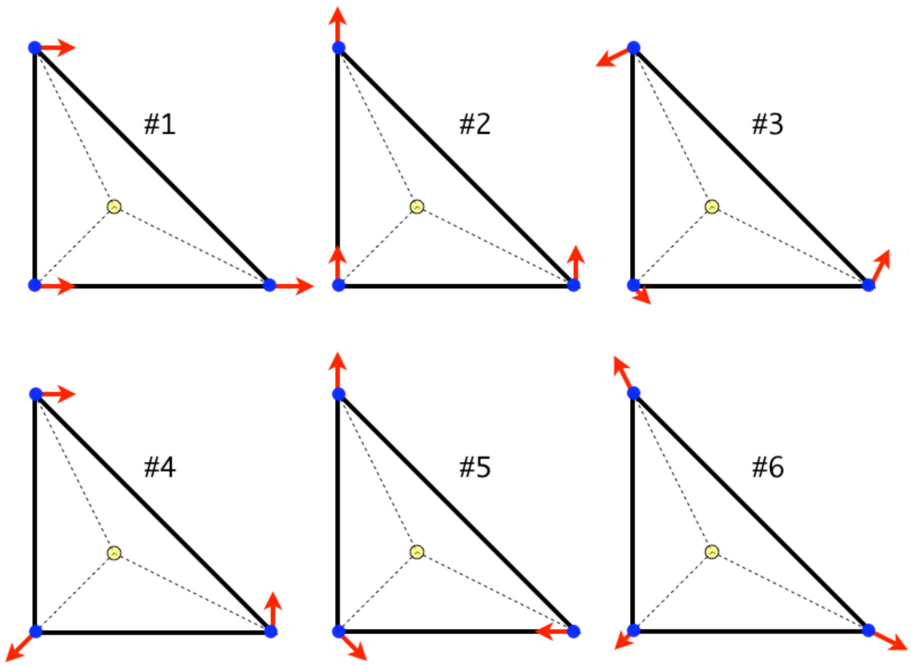

Figure 10.4: Normal modes of the 45° right triangle. The yellow circle is the location of the CM of the triangle.

The kinetic energy is simply

$$T = \tfrac{1}{2}m\big(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2 + \dot{x}_3^2 + \dot{y}_3^2\big) \ , \tag{10.117}$$

which entails

$$\mathrm{T}_{\sigma\sigma'} = m\,\delta_{\sigma\sigma'} \ . \tag{10.118}$$

(b) The three zero modes correspond to x-translation, y-translation, and rotation. Their eigenvectors, respectively, are

$$\boldsymbol{\psi}_1 = \frac{1}{\sqrt{3m}}\begin{pmatrix}1\\0\\1\\0\\1\\0\end{pmatrix} \quad , \quad \boldsymbol{\psi}_2 = \frac{1}{\sqrt{3m}}\begin{pmatrix}0\\1\\0\\1\\0\\1\end{pmatrix} \quad , \quad \boldsymbol{\psi}_3 = \frac{1}{2\sqrt{3m}}\begin{pmatrix}1\\-1\\1\\2\\-2\\-1\end{pmatrix} \ . \tag{10.119}$$

To find the unnormalized rotation vector, we find the CM of the triangle, located at $\big(\frac{a}{3}\,,\,\frac{a}{3}\big)$, and sketch orthogonal displacements $\hat{\boldsymbol{z}} \times (\boldsymbol{R}_i - \boldsymbol{R}_{\rm CM})$ at the position of mass point i.

(c) The remaining modes may be determined by symmetry, and are given by

$$\boldsymbol{\psi}_4 = \frac{1}{2\sqrt{m}} \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{\psi}_5 = \frac{1}{2\sqrt{m}} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad , \quad \boldsymbol{\psi}_6 = \frac{1}{2\sqrt{3m}} \begin{pmatrix} -1 \\ -1 \\ 2 \\ -1 \\ -1 \\ 2 \end{pmatrix} \quad , \tag{10.120}$$

with

$$\omega_1 = \sqrt{\frac{k}{m}} \quad , \quad \omega_2 = \sqrt{\frac{2k}{m}} \quad , \quad \omega_3 = \sqrt{\frac{3k}{m}} \ . \tag{10.121}$$

Since $\mathrm{T} = m \cdot 1$ is a multiple of the unit matrix, the orthogonormality relation $\psi^a_i \, \mathrm{T}_{ij} \, \psi^b_j = \delta^{ab}$ entails that the eigenvectors are mutually orthogonal in the usual dot product sense, with $\boldsymbol{\psi}_a \cdot \boldsymbol{\psi}_b = m^{-1} \, \delta_{ab}$. One can check that the eigenvectors listed here satisfy this condition.

The simplest of the set $\{\boldsymbol{\psi}_4, \boldsymbol{\psi}_5, \boldsymbol{\psi}_6\}$ to find is the uniform dilation $\boldsymbol{\psi}_6$, sometimes called the 'breathing' mode. This must keep the triangle in the same shape, which means that the deviations at each mass point are proportional to the distance to the CM. Next, it is simplest to find $\boldsymbol{\psi}_4$, in which the long and short sides of the triangle oscillate out of phase. Finally, the mode $\boldsymbol{\psi}_5$ must be orthogonal to all the remaining modes. No heavy lifting (*e.g.* `Mathematica`) is required!

10.9.2 Triple Pendulum

Consider a triple pendulum consisting of three identical masses m and three identical rigid massless rods of length ℓ, as depicted in Fig. 10.5.

(a) Find the T and V matrices.

(b) Find the equation for the eigenfrequencies.

(c) Numerically solve the eigenvalue equation for ratios ω_a^2/ω_0^2, where $\omega_0 = \sqrt{g/\ell}$. Find the three normal modes.

Solution

The Cartesian coordinates for the three masses are

$$x_1 = \ell \sin\theta_1 \qquad y_1 = -\ell \cos\theta_1$$
$$x_2 = \ell \sin\theta_1 + \ell \sin\theta_2 \qquad y_2 = -\ell \cos\theta_1 - \ell \cos\theta_2$$
$$x_3 = \ell \sin\theta_1 + \ell \sin\theta_2 + \ell \sin\theta_3 \qquad y_3 = -\ell \cos\theta_1 - \ell \cos\theta_2 - \ell \cos\theta_3 \ .$$

By inspection, we can write down the kinetic energy:

$$\begin{aligned} T &= \tfrac{1}{2} m \big(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2 + \dot{x}_3^3 + \dot{y}_3^2\big) \\ &= \tfrac{1}{2} m\, \ell^2 \Big\{ 3\,\dot{\theta}_1^2 + 2\,\dot{\theta}_2^2 + \dot{\theta}_3^2 + 4\,\cos(\theta_1 - \theta_2)\,\dot{\theta}_1\,\dot{\theta}_2 \\ &\qquad + 2\,\cos(\theta_1 - \theta_3)\,\dot{\theta}_1\,\dot{\theta}_3 + 2\,\cos(\theta_2 - \theta_3)\,\dot{\theta}_2\,\dot{\theta}_3 \Big\} \end{aligned}$$

The potential energy is

$$U = -mg\ell \Big\{ 3\,\cos\theta_1 + 2\,\cos\theta_2 + \cos\theta_3 \Big\} \ ,$$

and the Lagrangian is $L = T - U$:

$$\begin{aligned} L = \tfrac{1}{2} m\, \ell^2 \Big\{ & 3\,\dot{\theta}_1^2 + 2\,\dot{\theta}_2^2 + \dot{\theta}_3^2 + 4\,\cos(\theta_1 - \theta_2)\,\dot{\theta}_1\,\dot{\theta}_2 + 2\,\cos(\theta_1 - \theta_3)\,\dot{\theta}_1\,\dot{\theta}_3 \\ & + 2\,\cos(\theta_2 - \theta_3)\,\dot{\theta}_2\,\dot{\theta}_3 \Big\} + mg\ell \Big\{ 3\,\cos\theta_1 + 2\,\cos\theta_2 + \cos\theta_3 \Big\} \ . \end{aligned}$$

The Cartesian coordinates for the three masses are

$$x_1 = \ell \sin\theta_1 \qquad y_1 = -\ell \cos\theta_1$$
$$x_2 = \ell \sin\theta_1 + \ell \sin\theta_2 \qquad y_2 = -\ell \cos\theta_1 - \ell \cos\theta_2$$
$$x_3 = \ell \sin\theta_1 + \ell \sin\theta_2 + \ell \sin\theta_3 \qquad y_3 = -\ell \cos\theta_1 - \ell \cos\theta_2 - \ell \cos\theta_3 \ .$$

By inspection, we can write down the kinetic energy:

$$\begin{aligned} T &= \tfrac{1}{2} m \big(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2 + \dot{x}_3^3 + \dot{y}_3^2\big) \\ &= \tfrac{1}{2} m\, \ell^2 \Big\{ 3\,\dot{\theta}_1^2 + 2\,\dot{\theta}_2^2 + \dot{\theta}_3^2 + 4\,\cos(\theta_1 - \theta_2)\,\dot{\theta}_1\,\dot{\theta}_2 \\ &\qquad + 2\,\cos(\theta_1 - \theta_3)\,\dot{\theta}_1\,\dot{\theta}_3 + 2\,\cos(\theta_2 - \theta_3)\,\dot{\theta}_2\,\dot{\theta}_3 \Big\} \end{aligned}$$

The potential energy is

$$U = -mg\ell \Big\{ 3\,\cos\theta_1 + 2\,\cos\theta_2 + \cos\theta_3 \Big\} \ ,$$

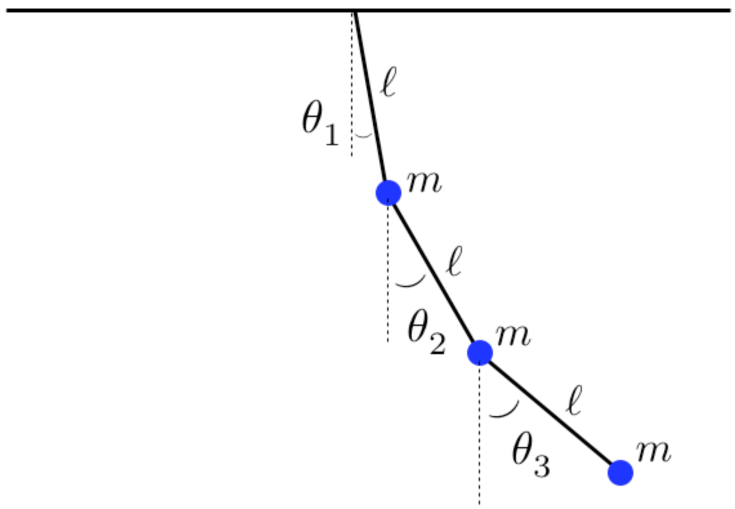

Figure 10.5: The triple pendulum.

and the Lagrangian is $L = T - U$:

$$L = \tfrac{1}{2} m\,\ell^2 \Big\{ 3\,\dot\theta_1^2 + 2\,\dot\theta_2^2 + \dot\theta_3^2 + 4\,\cos(\theta_1 - \theta_2)\,\dot\theta_1\,\dot\theta_2 + 2\,\cos(\theta_1 - \theta_3)\,\dot\theta_1\,\dot\theta_3 \\ + 2\,\cos(\theta_2 - \theta_3)\,\dot\theta_2\,\dot\theta_3 \Big\} + mg\ell \Big\{ 3\,\cos\theta_1 + 2\,\cos\theta_2 + \cos\theta_3 \Big\}\ .$$

Write down expressions for the conjugate momenta. The momenta are given by

$$\begin{aligned}
\pi_1 &= \frac{\partial L}{\partial \dot\theta_1} = m\,\ell^2 \Big\{ 3\,\dot\theta_1 + 2\,\dot\theta_2\,\cos(\theta_1 - \theta_2) + \dot\theta_3\,\cos(\theta_1 - \theta_3) \Big\} \\
\pi_2 &= \frac{\partial L}{\partial \dot\theta_2} = m\,\ell^2 \Big\{ 2\,\dot\theta_2 + 2\,\dot\theta_1\,\cos(\theta_1 - \theta_2) + \dot\theta_3\,\cos(\theta_2 - \theta_3) \Big\} \\
\pi_3 &= \frac{\partial L}{\partial \dot\theta_2} = m\,\ell^2 \Big\{ \dot\theta_3 + \dot\theta_1\,\cos(\theta_1 - \theta_3) + \dot\theta_2\,\cos(\theta_2 - \theta_3) \Big\}\ .
\end{aligned}$$

The only conserved quantity is the total energy, $E = T + U$.

(a) As for the T and V matrices, we have

$$\mathrm{T}_{\sigma\sigma'} = \frac{\partial^2 T}{\partial\theta_\sigma\,\partial\theta_{\sigma'}}\bigg|_{\boldsymbol{\theta}=0} = m\ell^2 \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and

$$\mathrm{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial\theta_\sigma\,\partial\theta_{\sigma'}}\bigg|_{\boldsymbol{\theta}=0} = mg\ell \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\ .$$

(b) The eigenfrequencies are roots of the equation $\det(\omega^2\,\mathrm{T} - \mathrm{V}) = 0$. Defining $\omega_0 \equiv \sqrt{g/\ell}$, we have

$$\omega^2\,\mathrm{T} - \mathrm{V} = m\ell^2 \begin{pmatrix} 3(\omega^2-\omega_0^2) & 2\,\omega^2 & \omega^2 \\ 2\,\omega^2 & 2(\omega^2-\omega_0^2) & \omega^2 \\ \omega^2 & \omega^2 & (\omega^2-\omega_0^2) \end{pmatrix}$$

and hence

$$\begin{aligned} \det(\omega^2\mathrm{T}-\mathrm{V}) &= 3(\omega^2-\omega_0^2)\cdot\Big[2(\omega^2-\omega_0^2)^2-\omega^4\Big] - 2\,\omega^2\cdot\Big[2\,\omega^2(\omega^2-\omega_0^2)-\omega^4\Big] \\ &\qquad + \omega^2\cdot\Big[2\,\omega^4 - 2\,\omega^2(\omega^2-\omega_0^2)\Big] \\ &= 6\,(\omega^2-\omega_0^2)^3 - 9\,\omega^4\,(\omega^2-\omega_0^2) + 4\omega^6 \\ &= \omega^6 - 9\,\omega_0^2\,\omega^4 + 18\,\omega_0^4\,\omega^2 - 6\,\omega_0^6\ . \end{aligned}$$

(c) The equation for the eigenfrequencies is

$$\lambda^3 - 9\,\lambda^2 + 18\,\lambda - 6 = 0\ , \tag{10.122}$$

where $\omega^2 = \lambda\,\omega_0^2$. This is a cubic equation in λ. Numerically solving for the roots, one finds

$$\omega_1^2 = 0.415774\,\omega_0^2 \quad,\quad \omega_2^2 = 2.29428\,\omega_0^2 \quad,\quad \omega_3^2 = 6.28995\,\omega_0^2\ . \tag{10.123}$$

I find the (unnormalized) eigenvectors to be

$$\boldsymbol{\psi}_1 = \begin{pmatrix} 1 \\ 1.2921 \\ 1.6312 \end{pmatrix} \quad,\quad \boldsymbol{\psi}_2 = \begin{pmatrix} 1 \\ 0.35286 \\ -2.3981 \end{pmatrix} \quad,\quad \boldsymbol{\psi}_3 = \begin{pmatrix} 1 \\ -1.6450 \\ 0.76690 \end{pmatrix}\ . \tag{10.124}$$

10.9.3 Equilateral Linear Triatomic Molecule

Consider the vibrations of an equilateral triangle of mass points, depicted in figure 10.6 . The system is confined to the (x, y) plane, and in equilibrium all the strings are unstretched and of length a.

(a) Choose as generalized coordinates the Cartesian displacements (x_i, y_i) with respect to equilibrium. Write down the exact potential energy.

(b) Find the T and V matrices.

(c) There are three normal modes of oscillation for which the corresponding eigenfrequencies all vanish: $\omega_a = 0$. Write down these modes explicitly, and provide a

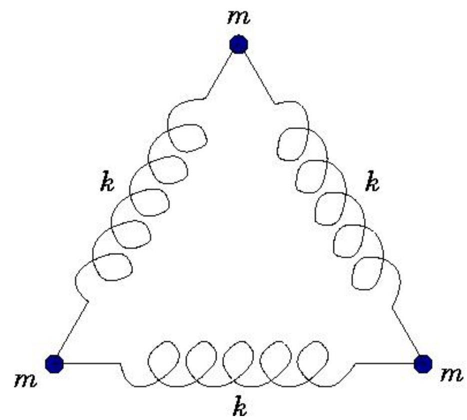

Figure 10.6: An equilateral triangle of identical mass points and springs.

physical interpretation for why $\omega_a = 0$. Since this triplet is degenerate, there is no unique answer – any linear combination will also serve as a valid 'zero mode'. However, if you think physically, a natural set should emerge.

`(d)` The three remaining modes all have finite oscillation frequencies. They correspond to distortions of the triangular shape. One such mode is the "breathing mode" in which the triangle uniformly expands and contracts. Write down the eigenvector associated with this normal mode and compute its associated oscillation frequency.

`(e)` The fifth and sixth modes are degenerate. They must be orthogonal (with respect to the inner product defined by T) to all the other modes. See if you can figure out what these modes are, and compute their oscillation frequencies. As in `(a)`, any linear combination of these modes will also be an eigenmode.

`(f)` Write down your full expression for the modal matrix A_{ai}, and check that it is correct by using `Mathematica`.

Solution

Choosing as generalized coordinates the Cartesian displacements relative to equilibrium, we have the following:

$$\begin{aligned}
&\#1 : \big(x_1, y_1\big) \\
&\#2 : \big(a + x_2, y_2\big) \\
&\#3 : \big(\tfrac{1}{2}a + x_3, \tfrac{\sqrt{3}}{2}a + y_3\big) \ .
\end{aligned}$$

Let d_ij be the separation of particles i and j. The potential energy of the spring connecting them is then $\frac{1}{2}\,k\,(d_{ij}-a)^2$.

$$
\begin{aligned}
d_{12}^2 &= \big(a + x_2 - x_1\big)^2 + \big(y_2 - y_1\big)^2 \\
d_{23}^2 &= \big(-\tfrac{1}{2}a + x_3 - x_2\big)^2 + \big(\tfrac{\sqrt{3}}{2}a + y_3 - y_2\big)^2 \\
d_{13}^2 &= \big(\tfrac{1}{2}a + x_3 - x_1\big)^2 + \big(\tfrac{\sqrt{3}}{2}a + y_3 - y_1\big)^2 \ .
\end{aligned}
$$

The full potential energy is

$$
U = \tfrac{1}{2}\,k\,\big(d_{12}-a\big)^2 + \tfrac{1}{2}\,k\,\big(d_{23}-a\big)^2 + \tfrac{1}{2}\,k\,\big(d_{13}-a\big)^2 \ . \qquad (10.125)
$$

This is a cumbersome expression, involving square roots.

To find T and V, we need to write T and V as quadratic forms, neglecting higher order terms. Therefore, we must expand $d_{ij}-a$ to linear order in the generalized coordinates. This results in the following:

$$
\begin{aligned}
d_{12} &= a + \big(x_2 - x_1\big) + \ldots \\
d_{23} &= a - \tfrac{1}{2}\,\big(x_3 - x_2\big) + \tfrac{\sqrt{3}}{2}\,\big(y_3 - y_2\big) + \ldots \\
d_{13} &= a + \tfrac{1}{2}\,\big(x_3 - x_1\big) + \tfrac{\sqrt{3}}{2}\,\big(y_3 - y_1\big) + \ldots \ .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
U = \tfrac{1}{2}\,k\,\big(x_2 - x_1\big)^2 &+ \tfrac{1}{8}\,k\,\big(x_2 - x_3 - \sqrt{3}\,y_2 + \sqrt{3}\,y_3\big)^2 \\
&+ \tfrac{1}{8}\,k\,\big(x_3 - x_1 + \sqrt{3}\,y_3 - \sqrt{3}\,y_1\big)^2 + \text{higher order terms} \ .
\end{aligned}
$$

Defining

$$
\big(q_1, q_2, q_3, q_4, q_5, q_6\big) = \big(x_1, y_1, x_2, y_2, x_3, y_3\big) \ ,
$$

we may now read off

$$
\mathrm{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial q_\sigma\,\partial q_{\sigma'}}\bigg|_{\bar{\mathrm{q}}} = k
\begin{pmatrix}
5/4 & \sqrt{3}/4 & -1 & 0 & -1/4 & -\sqrt{3}/4 \\
\sqrt{3}/4 & 3/4 & 0 & 0 & -\sqrt{3}/4 & -3/4 \\
-1 & 0 & 5/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\
0 & 0 & -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\
-1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 & 1/2 & 0 \\
-\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & -3/4 & 0 & 3/2
\end{pmatrix}
$$

The T matrix is trivial. From

$$
T = \tfrac{1}{2}m\big(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2 + \dot{x}_3^2 + \dot{y}_3^2\big) \ .
$$

we obtain

$$
\mathrm{T}_{ij} = \frac{\partial^2 T}{\partial \dot{q}_i\,\partial \dot{q}_j} = m\,\delta_{ij} \ ,
$$

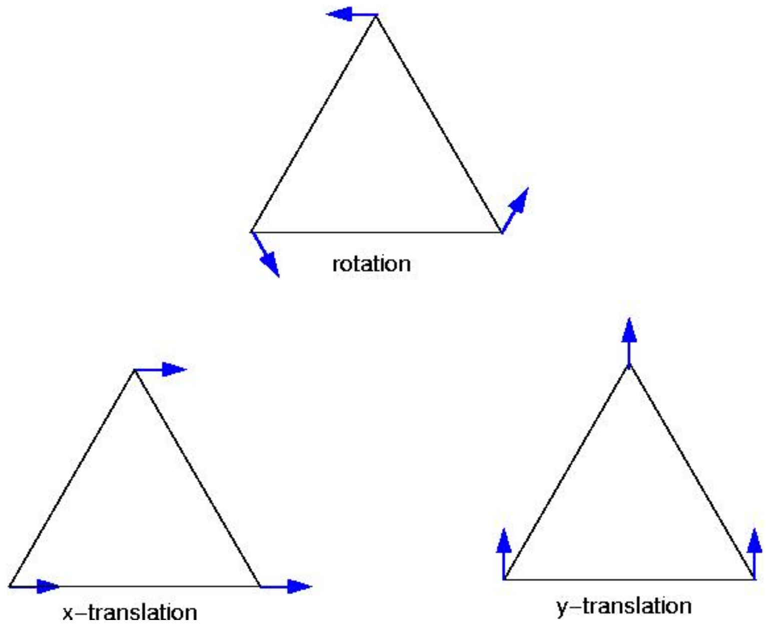

Figure 10.7: Zero modes of the mass-spring triangle.

and $\mathrm{T} = m \cdot \mathbf{1}$ is a multiple of the unit matrix.

The zero modes are depicted graphically in figure 10.7. Explicitly, we have

$$\boldsymbol{\xi}_x = \frac{1}{\sqrt{3m}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{\xi}_y = \frac{1}{\sqrt{3m}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad , \quad \boldsymbol{\xi}_{\mathrm{rot}} = \frac{1}{\sqrt{3m}} \begin{pmatrix} 1/2 \\ -\sqrt{3}/2 \\ 1/2 \\ \sqrt{3}/2 \\ -1 \\ 0 \end{pmatrix} .$$

That these are indeed zero modes may be verified by direct multiplication: $\mathrm{V}\,\boldsymbol{\xi}_{x.y} = \mathrm{V}\,\boldsymbol{\xi}_{\mathrm{rot}} = 0$.

The three modes with finite oscillation frequency are depicted graphically in figure 10.8. Explicitly, we have

$$\boldsymbol{\xi}_{\mathrm{A}} = \frac{1}{\sqrt{3m}} \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \\ -1/2 \\ \sqrt{3}/2 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{\xi}_{\mathrm{B}} = \frac{1}{\sqrt{3m}} \begin{pmatrix} -\sqrt{3}/2 \\ 1/2 \\ \sqrt{3}/2 \\ 1/2 \\ 0 \\ -1 \end{pmatrix} \quad , \quad \boldsymbol{\xi}_{\mathrm{dil}} = \frac{1}{\sqrt{3m}} \begin{pmatrix} -\sqrt{3}/2 \\ -1/2 \\ \sqrt{3}/2 \\ -1/2 \\ 0 \\ 1 \end{pmatrix} .$$

The oscillation frequencies of these modes are easily checked by multiplying the eigenvectors by the matrix V. Since $\mathrm{T} = m \cdot \mathbf{1}$ is diagonal, we have $\mathrm{V}\,\boldsymbol{\xi}_a = m\omega_a^2\,\boldsymbol{\xi}_a$.

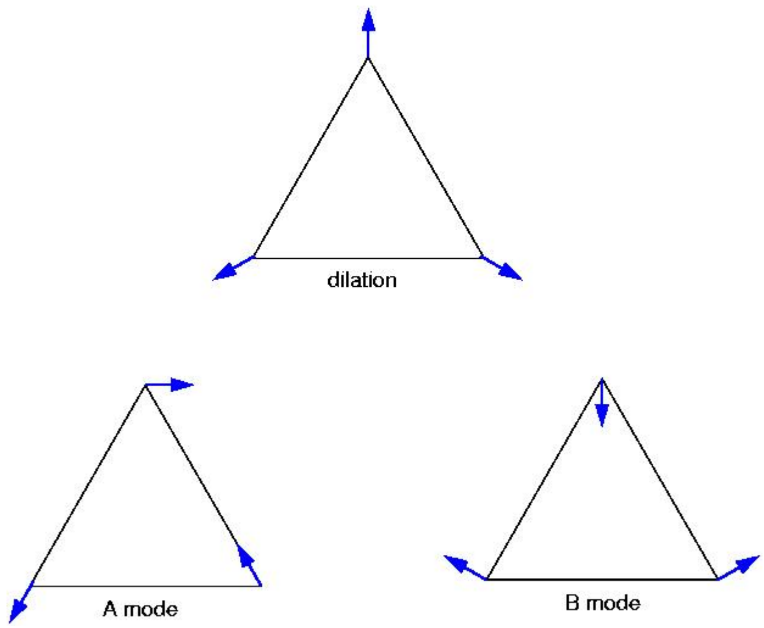

Figure 10.8: Finite oscillation frequency modes of the mass-spring triangle.

One finds

$$\omega_{\rm A} = \omega_{\rm B} = \sqrt{\frac{3k}{2m}} \quad , \quad \omega_{\rm dil} = \sqrt{\frac{3k}{m}} \ .$$

`Mathematica`? I don't need no stinking `Mathematica`.

Figure 10.9: *John Henry*, statue by Charles O. Cooper (1972). “Now the man that invented the steam drill, he thought he was mighty fine. But John Henry drove fifteen feet, and the steam drill only made nine.” - from *The Ballad of John Henry*.