Chapter 1

Introduction to Dynamics

1.1 Introduction and Review

Dynamics is the science of how things *move*. A complete solution to the motion of a system means that we know the coordinates of all its constituent particles as functions of time. For a single point particle moving in three-dimensional space, this means we want to know its position vector $\boldsymbol{r}(t)$ as a function of time. If there are many particles, the motion is described by a set of functions $\boldsymbol{r}_i(t)$, where i labels which particle we are talking about. So generally speaking, solving for the motion means being able to predict where a particle will be at any given instant of time. Of course, knowing the function $\boldsymbol{r}_i(t)$ means we can take its derivative and obtain the velocity $\boldsymbol{v}_i(t) = d\boldsymbol{r}_i/dt$ at any time as well.

The complete motion for a system is not given to us outright, but rather is encoded in a set of differential equations, called the *equations of motion*. An example of an equation of motion is

$$m\frac{d^2x}{dt^2} = -mg \tag{1.1}$$

with the solution

$$x(t) = x_0 + v_0 t - \tfrac{1}{2}gt^2 \tag{1.2}$$

where x_0 and v_0 are constants. This describes the vertical motion of a particle of mass m moving near the earth's surface.

In this class, we shall discuss a general framework by which the equations of motion may be obtained, and methods for solving them. That "general framework" is Lagrangian Dynamics, which itself is really nothing more than an elegant restatement of Isaac Newton's Laws of Motion.

1.1.1 Newton's Laws

Aristotle held that objects move because they are somehow impelled to seek out their natural state. Thus, a rock falls because rocks belong on the earth, and flames rise because fire belongs in the heavens. To paraphrase Wolfgang Pauli, such notions are so vague as to be "not even wrong." It was only with the publication of Newton's *Principia* in 1687 that a theory of motion which had detailed predictive power was developed.

Newton's three Laws of Motion may be stated as follows:

I. A body remains in uniform motion unless acted on by a force.
II. Force equals rate of change of momentum: $\boldsymbol{F} = d\boldsymbol{p}/dt$.
III. Any two bodies exert equal and opposite forces on each other.

Newton's First Law states that a particle will move in a straight line at constant (possibly zero) velocity if it is subjected to no forces. Now this cannot be true in general, for suppose we encounter such a "free" particle and that indeed it is in uniform motion, so that $\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0 t$. Now $\boldsymbol{r}(t)$ is measured in some coordinate system, and if instead we choose to measure $\boldsymbol{r}(t)$ in a different coordinate system whose origin $\boldsymbol{R}$ moves according to the function $\boldsymbol{R}(t)$, then in this new "frame of reference" the position of our particle will be

$$\begin{aligned}\boldsymbol{r}'(t) &= \boldsymbol{r}(t) - \boldsymbol{R}(t) \\ &= \boldsymbol{r}_0 + \boldsymbol{v}_0 t - \boldsymbol{R}(t)\ .\end{aligned} \tag{1.3}$$

If the acceleration $d^2\boldsymbol{R}/dt^2$ is nonzero, then merely by shifting our frame of reference we have apparently falsified Newton's First Law – a free particle does *not* move in uniform rectilinear motion when viewed from an accelerating frame of reference. Thus, together with Newton's Laws comes an assumption about the existence of frames of reference – called *inertial frames* – in which Newton's Laws hold. A transformation from one frame $\mathcal{K}$ to another frame $\mathcal{K}'$ which moves at constant velocity $\boldsymbol{V}$ relative to $\mathcal{F}$ is called a *Galilean transformation*. The equations of motion of classical mechanics are *invariant* (do not change) under Galilean transformations.

At first, the issue of inertial and noninertial frames is confusing. Rather than grapple with this, we will try to build some intuition by solving mechanics problems assuming we *are* in an inertial frame. The earth's surface, where most physics experiments are done, is *not* an inertial frame, due to the centripetal accelerations associated with the earth's rotation about its own axis and its orbit around the sun. In this case, not only is our coordinate system's origin – somewhere in a laboratory on the surface of the earth – accelerating, but the coordinate axes themselves are rotating with respect to an inertial frame. The rotation of the earth leads to fictitious "forces" such as the Coriolis force, which have large-scale consequences. For example, hurricanes, when viewed from above, rotate counterclockwise in the northern hemisphere and clockwise in the southern hemisphere. Later on in the course we will devote ourselves to a detailed study of motion in accelerated coordinate systems.

Newton's "quantity of motion" is the momentum $\boldsymbol{p}$, defined as the product $\boldsymbol{p} = m\boldsymbol{v}$ of a particle's mass m (how much stuff there is) and its velocity (how fast it is moving). In order to convert the Second Law into a meaningful equation, we must know how the force $\boldsymbol{F}$ depends on the coordinates (or possibly velocities) themselves. This is known as a *force law.* Examples of force laws include:

$$
\begin{aligned}
\text{Constant force:} &\qquad \boldsymbol{F} = -m\boldsymbol{g} \\[1em]
\text{Hooke's Law:} &\qquad F = -kx \\[1em]
\text{Gravitation:} &\qquad \boldsymbol{F} = -GMm\,\hat{\boldsymbol{r}}/r^2 \\[1em]
\text{Lorentz force:} &\qquad \boldsymbol{F} = q\,\boldsymbol{E} + q\,\frac{\boldsymbol{v}}{c} \times \boldsymbol{B} \\[1em]
\text{Fluid friction } (v \text{ small}): &\qquad \boldsymbol{F} = -b\,\boldsymbol{v}\ .
\end{aligned}
$$

Note that for an object whose mass does not change we can write the Second Law in the familiar form $\boldsymbol{F} = m\boldsymbol{a}$, where $\boldsymbol{a} = d\boldsymbol{v}/dt = d^2\boldsymbol{r}/dt^2$ is the acceleration. Most of our initial efforts will lie in using Newton's Second Law to solve for the motion of a variety of systems.

The Third Law is valid only for the extremely important case of *central forces* which we will discuss in great detail later on. Newtonian gravity – the force which makes the planets orbit the sun – is a central force. One consequence of the Third Law is that in free space two isolated particles will accelerate in such a way that $\boldsymbol{F}_1 = -\boldsymbol{F}_2$ and hence the accelerations are parallel to each other, with

$$
\frac{a_1}{a_2} = -\frac{m_2}{m_1}\ , \tag{1.4}
$$

where the minus sign is used here to emphasize that the accelerations are in opposite directions. We can also conclude that the *total momentum* $\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2$ is a constant, a result known as the *conservation of momentum.*

1.1.2 Aside : Inertial *vs.* Gravitational Mass

In addition to postulating the Laws of Motion, Newton also deduced the gravitational force law, which says that the force $\boldsymbol{F}_{ij}$ exerted by a particle i by another particle j

is

$$\boldsymbol{F}_{ij} = -Gm_i m_j \frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3} \ , \tag{1.5}$$

where G, the *Cavendish constant* (first measured by Henry Cavendish in 1798), takes the value

$$G = (6.6726 \pm 0.0008) \times 10^{-11} \mathrm{N} \cdot \mathrm{m}^2/\mathrm{kg}^2 \ . \tag{1.6}$$

Notice Newton's Third Law in action: $\boldsymbol{F}_{ij} + \boldsymbol{F}_{ji} = 0$. Now a very important and special feature of this "inverse square law" force is that a spherically symmetric mass distribution has the same force on an external body as it would if all its mass were concentrated at its center. Thus, for a particle of mass m near the surface of the earth, we can take $m_i = m$ and $m_j = M_\mathrm{e}$, with $\boldsymbol{r}_i - \boldsymbol{r}_j \simeq R_\mathrm{e}\hat{\boldsymbol{r}}$ and obtain

$$\boldsymbol{F} = -mg\hat{\boldsymbol{r}} \equiv -m\boldsymbol{g} \tag{1.7}$$

where $\hat{\boldsymbol{r}}$ is a radial unit vector pointing from the earth's center and $g = GM_\mathrm{e}/R_\mathrm{e}^2 \simeq 9.8\,\mathrm{m/s}^2$ is the acceleration due to gravity at the earth's surface. Newton's Second Law now says that $\boldsymbol{a} = -\boldsymbol{g}$, *i.e.* objects accelerate as they fall to earth. However, it is not *a priori* clear why the *inertial mass* which enters into the definition of momentum should be the same as the *gravitational mass* which enters into the force law. Suppose, for instance, that the gravitational mass took a different value, m'. In this case, Newton's Second Law would predict

$$\boldsymbol{a} = -\frac{m'}{m}\,\boldsymbol{g} \tag{1.8}$$

and unless the ratio m'/m were *the same number* for *all* objects, then bodies would fall with *different accelerations*. The experimental fact that bodies in a vacuum fall to earth at the same rate demonstrates the equivalence of inertial and gravitational mass, *i.e.* $m' = m$.

1.2 Examples of Motion in One Dimension

To gain some experience with solving equations of motion in a physical setting, we consider some physically relevant examples of one-dimensional motion.

1.2.1 Uniform Force

With $F = -mg$, appropriate for a particle falling under the influence of a uniform gravitational field, we have $m\,d^2x/dt^2 = -mg$, or $\ddot{x} = -g$. Notation:

$$\dot{x} \equiv \frac{dx}{dt} \ , \qquad \ddot{x} \equiv \frac{d^2x}{dt^2} \ , \qquad \overset{\vdots\vdots}{x} = \frac{d^7x}{dt^7} \ , \qquad \textit{etc.} \tag{1.9}$$

With $v = \dot{x}$, we solve $dv/dt = -g$:

$$\int\limits_{v(0)}^{v(t)} dv = \int\limits_0^t ds\,(-g) \tag{1.10}$$

$$v(t) - v(0) = -gt\ . \tag{1.11}$$

Note that there is a constant of integration, $v(0)$, which enters our solution.

We are now in position to solve $dx/dt = v$:

$$\int\limits_{x(0)}^{x(t)} dx = \int\limits_0^t ds\, v(s) \tag{1.12}$$

$$x(t) = x(0) + \int\limits_0^t ds\,\big[v(0) - gs\big] \tag{1.13}$$

$$= x(0) + v(0)t - \tfrac{1}{2}gt^2\ . \tag{1.14}$$

Note that a second constant of integration, $x(0)$, has appeared.

1.2.2 Uniform force with linear frictional damping

In this case,

$$m\frac{dv}{dt} = -mg - \gamma v \tag{1.15}$$

which may be rewritten

$$\frac{dv}{v + mg/\gamma} = -\frac{\gamma}{m}dt \tag{1.16}$$

$$d\ln(v + mg/\gamma) = -(\gamma/m)dt\ . \tag{1.17}$$

Integrating then gives

$$\ln\left(\frac{v(t) + mg/\gamma}{v(0) + mg/\gamma}\right) = -\gamma t/m \tag{1.18}$$

$$v(t) = -\frac{mg}{\gamma} + \left(v(0) + \frac{mg}{\gamma}\right)e^{-\gamma t/m}\ . \tag{1.19}$$

Note that the solution to the first order ODE $m\dot{v} = -mg - \gamma v$ entails one constant of integration, $v(0)$.

One can further integrate to obtain the motion

$$x(t) = x(0) + \frac{m}{\gamma}\left(v(0) + \frac{mg}{\gamma}\right)(1 - e^{-\gamma t/m}) - \frac{mg}{\gamma}\,t \ . \tag{1.20}$$

The solution to the *second* order ODE $m\ddot{x} = -mg - \gamma\dot{x}$ thus entails *two* constants of integration: $v(0)$ and $x(0)$. Notice that as t goes to infinity the velocity tends towards the asymptotic value $v = -v_\infty$, where $v_\infty = mg/\gamma$. This is known as the *terminal velocity*. Indeed, solving the equation $\dot{v} = 0$ gives $v = -v_\infty$. The initial velocity is effectively "forgotten" on a time scale $\tau \equiv m/\gamma$.

Electrons moving in solids under the influence of an electric field also achieve a terminal velocity. In this case the force is not $F = -mg$ but rather $F = -eE$, where $-e$ is the electron charge ($e > 0$) and E is the electric field. The terminal velocity is then obtained from

$$v_\infty = eE/\gamma = e\tau E/m \ . \tag{1.21}$$

The *current density* is a product:

$$\text{current density} = (\text{number density}) \times (\text{charge}) \times (\text{velocity})$$

$$\begin{aligned} j &= n \cdot (-e) \cdot (-v_\infty) \\ &= \frac{ne^2\tau}{m}\,E \ . \end{aligned} \tag{1.22}$$

The ratio j/E is called the *conductivity* of the metal, σ. According to our theory, $\sigma = ne^2\tau/m$. This is one of the most famous equations of solid state physics! The dissipation is caused by electrons scattering off impurities and lattice vibrations ("phonons"). In high purity copper at low temperatures ($T \lesssim 4\,\mathrm{K}$), the *scattering time* τ is about a nanosecond ($\tau \approx 10^{-9}\,\mathrm{s}$).

1.2.3 Uniform force with quadratic frictional damping

At higher velocities, the frictional damping is proportional to the *square* of the velocity. The frictional force is then $F_\mathrm{f} = -cv^2\,\mathrm{sgn}\,(v)$, where $\mathrm{sgn}\,(v)$ is the *sign* of v: $\mathrm{sgn}\,(v) = +1$ if $v > 0$ and $\mathrm{sgn}\,(v) = -1$ if $v < 0$. (Note one can also write $\mathrm{sgn}\,(v) = v/|v|$ where $|v|$ is the *absolute value*.) Why all this trouble with $\mathrm{sgn}\,(v)$? Because it is important that the frictional force *dissipate* energy, and therefore that F_f be *oppositely directed* with respect to the velocity v. We will assume that $v < 0$ always, hence $F_\mathrm{f} = +cv^2$.

Notice that there is a terminal velocity, since setting $\dot{v} = -g + (c/m)v^2 = 0$ gives

$v = \pm v_\infty$, where $v_\infty = \sqrt{mg/c}$. One can write the equation of motion as

$$\frac{dv}{dt} = \frac{g}{v_\infty^2}(v^2 - v_\infty^2) \tag{1.23}$$

and using

$$\frac{1}{v^2 - v_\infty^2} = \frac{1}{2v_\infty}\left[\frac{1}{v - v_\infty} - \frac{1}{v + v_\infty}\right] \tag{1.24}$$

we obtain

$$\begin{aligned} \frac{dv}{v^2 - v_\infty^2} &= \frac{1}{2v_\infty}\frac{dv}{v - v_\infty} - \frac{1}{2v_\infty}\frac{dv}{v + v_\infty} \\ &= \frac{1}{2v_\infty} d\ln\left(\frac{v_\infty - v}{v_\infty + v}\right) \\ &= \frac{g}{v_\infty^2}\, dt \ . \end{aligned} \tag{1.25}$$

Assuming $v(0) = 0$, we integrate to obtain

$$\frac{1}{2v_\infty}\ln\left(\frac{v_\infty - v(t)}{v_\infty + v(t)}\right) = \frac{gt}{v_\infty^2} \tag{1.26}$$

which may be massaged to give the final result

$$v(t) = -v_\infty \tanh(gt/v_\infty) \ . \tag{1.27}$$

Recall that the *hyperbolic tangent* function $\tanh(x)$ is given by

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \ . \tag{1.28}$$

Again, as $t \to \infty$ one has $v(t) \to -v_\infty$, *i.e.* $v(\infty) = -v_\infty$.

Advanced Digression: To gain an understanding of the constant c, consider a flat surface of area S moving through a fluid at velocity v ($v > 0$). During a time Δt, all the fluid molecules inside the volume $\Delta V = S \cdot v\Delta t$ will have executed an elastic collision with the moving surface. Since the surface is assumed to be much more massive than each fluid molecule, the center of mass frame for the surface-molecule collision is essentially the frame of the surface itself. If a molecule moves with velocity u is the laboratory frame, it moves with velocity $u - v$ in the center of mass (CM) frame, and since the collision is elastic, its final CM frame velocity is reversed, to $v - u$. Thus, in the laboratory frame the molecule's velocity has become $2v - u$ and it has suffered a change in velocity of $\Delta u = 2(v - u)$. The total momentum change is obtained by multiplying Δu by the total mass $M = \varrho\Delta V$, where ϱ is the mass density of the fluid. But then the total momentum imparted to the fluid is

$$\Delta P = 2(v - u) \cdot \varrho\, S\, v\Delta t \tag{1.29}$$

and the force on the fluid is

$$F = \frac{\Delta P}{\Delta t} = 2S\,\varrho\,v(v-u)\ . \tag{1.30}$$

Now it is appropriate to average this expression over the microscopic distribution of molecular velocities u, and since on average $\langle u\rangle = 0$, we obtain the result $\langle F\rangle = 2S\varrho v^2$, where $\langle\cdots\rangle$ denotes a microscopic average over the molecular velocities in the fluid. (There is a subtlety here concerning the effect of fluid molecules striking the surface from either side – you should satisfy yourself that this derivation is sensible!) Newton's Third Law then states that the frictional force imparted to the moving surface by the fluid is $F_{\rm f} = -\langle F\rangle = -cv^2$, where $c = 2S\varrho$. In fact, our derivation is too crude to properly obtain the numerical prefactors, and it is better to write $c = \mu\varrho S$, where μ is a dimensionless constant which depends on the *shape* of the moving object.

1.2.4 Crossed Electric and Magnetic Fields

Consider now a three-dimensional example of a particle of charge q moving in mutually perpendicular $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. We'll throw in gravity for good measure. We take $\boldsymbol{E} = E\hat{\boldsymbol{x}}$, $\boldsymbol{B} = B\hat{\boldsymbol{z}}$, and $\boldsymbol{g} = -g\hat{\boldsymbol{z}}$. The equation of motion is Newton's 2nd Law again:

$$m\,\ddot{\boldsymbol{r}} = m\boldsymbol{g} + q\boldsymbol{E} + \tfrac{q}{c}\,\dot{\boldsymbol{r}}\times\boldsymbol{B}\ . \tag{1.31}$$

The RHS (right hand side) of this equation is a vector sum of the forces due to gravity plus the Lorentz force of a moving particle in an electromagnetic field. In component notation, we have

$$m\ddot{x} = qE + \frac{qB}{c}\,\dot{y} \tag{1.32}$$

$$m\ddot{y} = -\frac{qB}{c}\,\dot{x} \tag{1.33}$$

$$m\ddot{z} = -mg\ . \tag{1.34}$$

The equations for coordinates x and y are coupled, while that for z is independent and may be immediately solved to yield

$$z(t) = z(t) + \dot{z}(0)\,t - \tfrac{1}{2}gt^2\ . \tag{1.35}$$

The remaining equations may be written in terms of the velocities $v_x = \dot{x}$ and $v_y = \dot{y}$:

$$\dot{v}_x = \omega_{\rm c}(v_y + u_{\rm D}) \tag{1.36}$$

$$\dot{v}_y = -\omega_{\rm c}\,v_x\ , \tag{1.37}$$

where $\omega_c = qB/mc$ is the *cyclotron frequency* and $u_D = cE/B$ is the *drift speed* for the particle. As we shall see, these are the equations for a harmonic oscillator. The solution is

$$v_x(t) = v_x(0)\,\cos(\omega_c t) + \big(v_y(0) + u_D\big)\,\sin(\omega_c t) \tag{1.38}$$

$$v_y(t) = -u_D + \big(v_y(0) + u_D\big)\cos(\omega_c t) - v_x(0)\,\sin(\omega_c t)\ . \tag{1.39}$$

Integrating again, the full motion is given by:

$$x(t) = x(0) + A\,\sin\delta + A\,\sin(\omega_c t - \delta) \tag{1.40}$$

$$y(r) = y(0) - u_D\,t - A\cos\delta + A\,\cos(\omega_c t - \delta)\ , \tag{1.41}$$

where

$$A = \frac{1}{\omega_c}\sqrt{\dot{x}^2(0) + \big(\dot{y}(0) + u_D\big)^2} \quad , \quad \delta = \tan^{-1}\left(\frac{\dot{y}(0) + u_D}{\dot{x}(0)}\right) . \tag{1.42}$$

Thus, in the full solution of the motion there are *six* constants of integration:

$$x(0)\ ,\ y(0)\ ,\ z(0)\ ,\ A\ ,\ \delta\ ,\ \dot{z}(0)\ . \tag{1.43}$$

Of course instead of A and δ one may choose as constants of integration $\dot{x}(0)$ and $\dot{y}(0)$.

1.3 Pause for Reflection

In mechanical systems, for each coordinate, or "degree of freedom," there exists a corresponding second order ODE. The full solution of the motion of the system entails two constants of integration for each degree of freedom.

1.4 Phase Space Dynamics

Dynamics is the study of motion through phase space. For our purposes, we will take $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_N)$ to be an N-tuple, *i.e.* a point in R^N. The equation of motion is then

$$\frac{d}{dt}\,\boldsymbol{\varphi}(t) = \boldsymbol{V}\big(\boldsymbol{\varphi}, t\big)\ . \tag{1.44}$$

Note that any N^{th} order ODE, of the general form

$$\frac{d^N x}{dt^N} = H\left(x, \frac{dx}{dt}, \ldots, \frac{d^{N-1}x}{dt^{N-1}}\right)\ , \tag{1.45}$$

may be represented by the first order system $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$. To see this, define $\varphi_k = d^{k-1}x/dt^{k-1}$, with $k = 1, \ldots, N$. Thus, for $j < N$ we have $\dot{\varphi}_j = \varphi_{j+1}$, and $\dot{\varphi}_N = f$. In other words,

$$\overbrace{\frac{d}{dt}\begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_{N-1} \\ \varphi_N \end{pmatrix}}^{\dot{\boldsymbol{\varphi}}} = \overbrace{\begin{pmatrix} \varphi_2 \\ \vdots \\ \varphi_N \\ F\big(\varphi_1, \ldots, \varphi_N\big) \end{pmatrix}}^{\boldsymbol{V}(\boldsymbol{\varphi})} \quad . \tag{1.46}$$

Mechanical systems are dynamical systems. We have for each 'generalized coordinate' q_i an equation of motion of the form

$$\ddot{q}_\sigma = Q_\sigma(q_1, \ldots, q_K; \dot{q}_1, \ldots, \dot{q}_K) \ , \tag{1.47}$$

where K is the number of degrees of freedom the system possesses. If there are no constraints, $K = \mathcal{N} \cdot d$, where $\mathcal{N}$ is the number of particles and d is the dimension of space. If we then identify

$$\boldsymbol{\varphi}_\sigma = q_\sigma \quad , \quad \boldsymbol{\varphi}_{\sigma+K} = \dot{q}_\sigma \ , \tag{1.48}$$

and

$$V_\sigma = \dot{q}_\sigma \quad , \quad V_{\sigma+K} = Q_\sigma\big(\{q_\nu\}; \{\dot{q}_\nu\}\big) \ , \tag{1.49}$$

for $\sigma = 1, \ldots, K$, then we arrive at the general form of eqn. 1.44 for a dynamical system, with $N = 2K$.

In autonomous cases, where $\boldsymbol{V}(\boldsymbol{\varphi}, t) = \boldsymbol{V}(\boldsymbol{\varphi})$ alone, $\boldsymbol{V}(\boldsymbol{\varphi})$ is called a *vector field* over the phase space. A solution $\boldsymbol{\varphi}(t)$ to the dynamical system of eqn. 1.44 is called an *integral curve*. It entails N constants of integration, *i.e.* $\boldsymbol{\varphi}(0)$. The set of all integral curves is called the *phase flow* of the dynamical system.

1.4.1 Existence/Uniqueness/Extension Theorems

Theorem : Given $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$ and $\boldsymbol{\varphi}(0)$, if each $\boldsymbol{V}(\boldsymbol{\varphi})$ is a smooth vector field over some open set $\mathcal{D} \in \mathbf{R}^N$, then for $\boldsymbol{\varphi}(0) \in \mathcal{D}$ the initial value problem has a solution on some finite time interval $(-\tau, +\tau)$ and the solution is unique. Furthermore, the solution has a unique extension forward or backward in time, either indefinitely or until $\boldsymbol{\varphi}(t)$ reaches the boundary of $\mathcal{D}$.

Corollary : *Different trajectories never intersect!*

1.4.2 Linear Differential Equations

A homogeneous linear N^{th} order ODE,

$$\frac{d^N x}{dt^N} + c_{N-1}\,\frac{d^{N-1}x}{dt^{N-1}} + \ldots + c_1\,\frac{dx}{dt} + c_0\,x = 0 \tag{1.50}$$

may be written in matrix form, as

$$\frac{d}{dt}\begin{pmatrix}\varphi_1\\ \varphi_2\\ \vdots\\ \varphi_N\end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ -c_0 & -c_1 & -c_2 & \cdots & -c_{N-1}\end{pmatrix}}^{M}\begin{pmatrix}\varphi_1\\ \varphi_2\\ \vdots\\ \varphi_N\end{pmatrix} . \tag{1.51}$$

Thus,

$$\dot{\boldsymbol{\varphi}} = M\boldsymbol{\varphi} , \tag{1.52}$$

and if the coefficients c_k are time-independent, *i.e.* the ODE is *autonomous*, the solution is obtained by exponentiating the constant matrix Q:

$$\boldsymbol{\varphi}(t) = \exp(Mt)\,\boldsymbol{\varphi}(0) \; ; \tag{1.53}$$

the exponential of a matrix may be given meaning by its Taylor series expansion. If the ODE is not autonomous, then $M = M(t)$ is time-dependent, and the solution is given by the 'path-ordered exponential',

$$\boldsymbol{\varphi}(t) = \mathcal{P}\exp\left\{\int\limits_0^t dt'\,M(t')\right\}\boldsymbol{\varphi}(0) \; , \tag{1.54}$$

As defined, the equation $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$ is autonomous, since g_t depends only on t and on no other time variable. However, by extending the phase space from $\mathcal{M}$ to $\mathrm{R}\times\mathcal{M}$, which is of dimension $(N+1)$, one can describe arbitrary time-dependent ODEs.