

Chapter 2

Systems of Particles

2.1 Work-Energy Theorem

Consider a system of many particles, with positions \mathbf{r}_i and velocities $\dot{\mathbf{r}}_i$. The kinetic energy of this system is

$$T = \sum_i T_i = \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 . \quad (2.1)$$

Now let's consider how the kinetic energy of the system changes in time. Assuming each m_i is time-independent, we have

$$\frac{dT_i}{dt} = m_i \dot{\mathbf{r}}_i \cdot \ddot{\mathbf{r}}_i . \quad (2.2)$$

Here, we've used the relation

$$\frac{d}{dt} (\mathbf{A}^2) = 2 \mathbf{A} \cdot \frac{d\mathbf{A}}{dt} . \quad (2.3)$$

We now invoke Newton's 2nd Law, $m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i$, to write eqn. 2.2 as $\dot{T}_i = \mathbf{F}_i \cdot \dot{\mathbf{r}}_i$. We integrate this equation from time t_A to t_B :

$$\begin{aligned} T_i^{(B)} - T_i^{(A)} &= \int_{t_A}^{t_B} dt \frac{dT_i}{dt} \\ &= \int_{t_A}^{t_B} dt \mathbf{F}_i \cdot \dot{\mathbf{r}}_i \equiv \sum_i W_i^{(A \rightarrow B)} , \end{aligned} \quad (2.4)$$

where $W_i^{(A \rightarrow B)}$ is the total *work done* on particle i during its motion from state A to state B . Clearly the total kinetic energy is $T = \sum_i T_i$ and the total work done on all

particles is $W^{(A \rightarrow B)} = \sum_i W_i^{(A \rightarrow B)}$. Eqn. 2.4 is known as the *work-energy theorem*. It says that

In the evolution of a mechanical system, the change in total kinetic energy is equal to the total work done: $T^{(B)} - T^{(A)} = W^{(A \rightarrow B)}$.

2.2 Conservative and Nonconservative Forces

For the sake of simplicity, consider a single particle with kinetic energy $T = \frac{1}{2}m\dot{\mathbf{r}}^2$. The work done on the particle during its mechanical evolution is

$$W^{(A \rightarrow B)} = \int_{t_A}^{t_B} dt \mathbf{F} \cdot \mathbf{v} , \quad (2.5)$$

where $\mathbf{v} = \dot{\mathbf{r}}$. This is the most general expression for the work done. If the force \mathbf{F} depends only on the particle's position \mathbf{r} , we may write $d\mathbf{r} = \mathbf{v} dt$, and then

$$W^{(A \rightarrow B)} = \int_{\mathbf{r}_A}^{\mathbf{r}_B} d\mathbf{r} \cdot \mathbf{F}(\mathbf{r}) . \quad (2.6)$$

Consider now the force

$$\mathbf{F}(\mathbf{r}) = K_1 y \hat{\mathbf{x}} + K_2 x \hat{\mathbf{y}} , \quad (2.7)$$

where $K_{1,2}$ are constants. Let's evaluate the work done along each of the two paths in fig. 2.1:

$$W^{(I)} = K_1 \int_{x_A}^{x_B} dx y_A + K_2 \int_{y_A}^{y_B} dy x_B = K_1 y_A (x_B - x_A) + K_2 x_B (y_B - y_A) \quad (2.8)$$

$$W^{(II)} = K_1 \int_{y_A}^{y_B} dy x_A + K_2 \int_{x_A}^{x_B} dx y_B = K_1 x_A (y_B - y_A) + K_2 y_B (x_B - x_A) . \quad (2.9)$$

Note that in general $W^{(I)} \neq W^{(II)}$. Thus, if we start at point A, the kinetic energy at point B will depend on the path taken, since the work done is path-dependent.

The difference between the work done along the two paths is

$$W^{(I)} - W^{(II)} = (K_2 - K_1) (x_B - x_A) (y_B - y_A) . \quad (2.10)$$

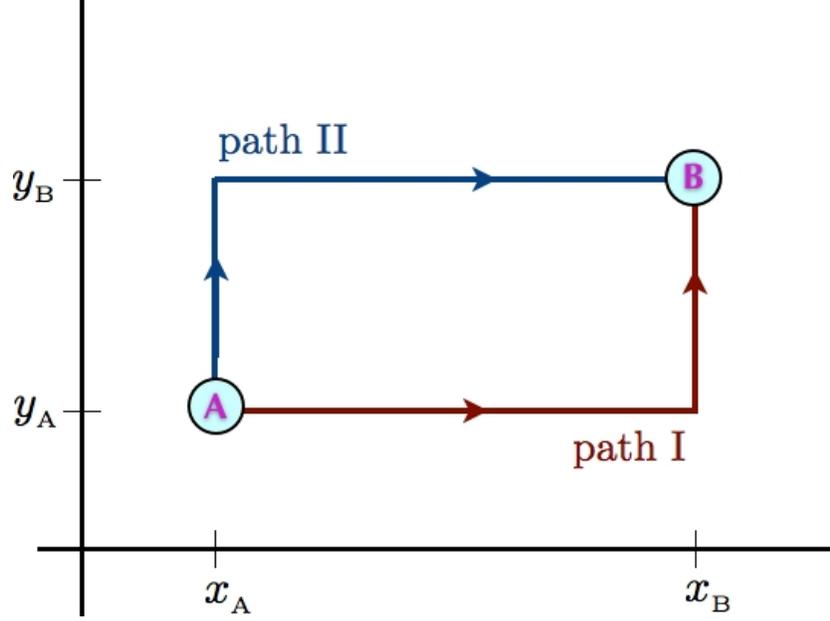


Figure 2.1: Two paths joining points A and B.

Thus, we see that if $K_1 = K_2$, the work is the same for the two paths. In fact, if $K_1 = K_2$, the work would be path-independent, and would depend only on the endpoints. This is true for *any* path, and not just piecewise linear paths of the type depicted in fig. 2.1. The reason for this is Stokes' theorem:

$$\oint_{\partial\mathcal{C}} d\boldsymbol{\ell} \cdot \mathbf{F} = \int_{\mathcal{C}} dS \hat{\mathbf{n}} \cdot \nabla \times \mathbf{F} . \quad (2.11)$$

Here, \mathcal{C} is a connected region in three-dimensional space, $\partial\mathcal{C}$ is mathematical notation for the boundary of \mathcal{C} , which is a closed path¹, dS is the scalar differential area element, $\hat{\mathbf{n}}$ is the unit normal to that differential area element, and $\nabla \times \mathbf{F}$ is the curl of \mathbf{F} :

$$\begin{aligned} \nabla \times \mathbf{F} &= \det \begin{pmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{pmatrix} \\ &= \left(\frac{\partial F_y}{\partial z} - \frac{\partial F_z}{\partial y} \right) \hat{\mathbf{x}} + \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} \right) \hat{\mathbf{y}} + \left(\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} \right) \hat{\mathbf{z}} . \end{aligned} \quad (2.12)$$

For the force under consideration, $\mathbf{F}(\mathbf{r}) = K_1 y \hat{\mathbf{x}} + K_2 x \hat{\mathbf{y}}$, the curl is

$$\nabla \times \mathbf{F} = (K_2 - K_1) \hat{\mathbf{z}} , \quad (2.13)$$

¹If \mathcal{C} is multiply connected, then $\partial\mathcal{C}$ is a set of closed paths. For example, if \mathcal{C} is an annulus, $\partial\mathcal{C}$ is two circles, corresponding to the inner and outer boundaries of the annulus.

which is a constant. The RHS of eqn. 2.11 is then simply proportional to the area enclosed by \mathcal{C} . When we compute the work difference in eqn. 2.10, we evaluate the integral $\oint_{\mathcal{C}} d\boldsymbol{\ell} \cdot \mathbf{F}$ along the path $\gamma_{\text{II}}^{-1} \circ \gamma_{\text{I}}$, which is to say path I followed by the inverse of path II. In this case, $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ and the integral of $\hat{\mathbf{n}} \cdot \nabla \times \mathbf{F}$ over the rectangle \mathcal{C} is given by the RHS of eqn. 2.10.

When $\nabla \times \mathbf{F} = 0$ everywhere in space, we can always write $\mathbf{F} = -\nabla U$, where $U(\mathbf{r})$ is the *potential energy*. Such forces are called *conservative forces* because the *total energy* of the system, $E = T + U$, is then conserved during its motion. We can see this by evaluating the work done,

$$\begin{aligned} W^{(\text{A} \rightarrow \text{B})} &= \int_{\mathbf{r}_\text{A}}^{\mathbf{r}_\text{B}} d\mathbf{r} \cdot \mathbf{F}(\mathbf{r}) \\ &= - \int_{\mathbf{r}_\text{A}}^{\mathbf{r}_\text{B}} d\mathbf{r} \cdot \nabla U \\ &= U(\mathbf{r}_\text{A}) - U(\mathbf{r}_\text{B}) . \end{aligned} \tag{2.14}$$

The work-energy theorem then gives

$$T^{(\text{B})} - T^{(\text{A})} = U(\mathbf{r}_\text{A}) - U(\mathbf{r}_\text{B}) , \tag{2.15}$$

which says

$$E^{(\text{B})} = T^{(\text{B})} + U(\mathbf{r}_\text{B}) = T^{(\text{A})} + U(\mathbf{r}_\text{A}) = E^{(\text{A})} . \tag{2.16}$$

Thus, the total energy $E = T + U$ is conserved.

2.2.1 Example : Integrating $\mathbf{F} = -\nabla U$

If $\nabla \times \mathbf{F} = 0$, we can compute $U(\mathbf{r})$ by integrating, *viz.*

$$U(\mathbf{r}) = U(\mathbf{0}) - \int_{\mathbf{0}}^{\mathbf{r}} d\mathbf{r}' \cdot \mathbf{F}(\mathbf{r}') . \tag{2.17}$$

The integral does not depend on the path chosen connecting $\mathbf{0}$ and \mathbf{r} . For example, we can take

$$U(x, y, z) = U(0, 0, 0) - \int_{(0,0,0)}^{(x,0,0)} dx' F_x(x', 0, 0) - \int_{(x,0,0)}^{(x,y,0)} dy' F_y(x, y', 0) - \int_{(z,y,0)}^{(x,y,z)} dz' F_z(x, y, z') . \tag{2.18}$$

The constant $U(0, 0, 0)$ is arbitrary and impossible to determine from \mathbf{F} alone.

As an example, consider the force

$$\mathbf{F}(\mathbf{r}) = -ky \hat{\mathbf{x}} - kx \hat{\mathbf{y}} - 4bz^3 \hat{\mathbf{z}} , \quad (2.19)$$

where k and b are constants. We have

$$(\nabla \times \mathbf{F})_x = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) = 0 \quad (2.20)$$

$$(\nabla \times \mathbf{F})_y = \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) = 0 \quad (2.21)$$

$$(\nabla \times \mathbf{F})_z = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) = 0 , \quad (2.22)$$

so $\nabla \times \mathbf{F} = 0$ and \mathbf{F} must be expressible as $\mathbf{F} = -\nabla U$. Integrating using eqn. 2.18, we have

$$U(x, y, z) = U(0, 0, 0) + \int_{(0,0,0)}^{(x,0,0)} dx' k \cdot 0 + \int_{(x,0,0)}^{(x,y,0)} dy' kxy' + \int_{(z,y,0)}^{(x,y,z)} dz' 4bz'^3 \quad (2.23)$$

$$= U(0, 0, 0) + kxy + bz^4 . \quad (2.24)$$

Another approach is to integrate the partial differential equation $\nabla U = -\mathbf{F}$. This is in fact three equations, and we shall need all of them to obtain the correct answer. We start with the $\hat{\mathbf{x}}$ -component,

$$\frac{\partial U}{\partial x} = ky . \quad (2.25)$$

Integrating, we obtain

$$U(x, y, z) = kxy + f(y, z) , \quad (2.26)$$

where $f(y, z)$ is at this point an *arbitrary function* of y and z . The important thing is that it has no x -dependence, so $\partial f / \partial x = 0$. Next, we have

$$\frac{\partial U}{\partial y} = kx \implies U(x, y, z) = kxy + g(x, z) . \quad (2.27)$$

Finally, the z -component integrates to yield

$$\frac{\partial U}{\partial z} = 4bz^3 \implies U(x, y, z) = bz^4 + h(x, y) . \quad (2.28)$$

We now equate the first two expressions:

$$kxy + f(y, z) = kxy + g(x, z) . \quad (2.29)$$

Subtracting kxy from each side, we obtain the equation $f(y, z) = g(x, z)$. Since the LHS is independent of x and the RHS is independent of y , we must have

$$f(y, z) = g(x, z) = q(z) , \quad (2.30)$$

where $q(z)$ is some unknown function of z . But now we invoke the final equation, to obtain

$$bz^4 + h(x, y) = kxy + q(z) . \quad (2.31)$$

The only possible solution is $h(x, y) = C + kxy$ and $q(z) = C + bz^4$, where C is a constant. Therefore,

$$U(x, y, z) = C + kxy + bz^4 . \quad (2.32)$$

Note that it would be *very wrong* to integrate $\partial U/\partial x = ky$ and obtain $U(x, y, z) = kxy + C'$, where C' is a constant. As we've seen, the 'constant of integration' we obtain upon integrating this first order PDE is in fact a *function* of y and z . The fact that $f(y, z)$ carries no explicit x dependence means that $\partial f/\partial x = 0$, so by construction $U = kxy + f(y, z)$ is a solution to the PDE $\partial U/\partial x = ky$, for any arbitrary function $f(y, z)$.

2.3 Conservative Forces in Many Particle Systems

$$T = \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 \quad (2.33)$$

$$U = \sum_i V(\mathbf{r}_i) + \sum_{i < j} v(|\mathbf{r}_i - \mathbf{r}_j|) . \quad (2.34)$$

Here, $V(\mathbf{r})$ is the *external* (or one-body) potential, and $v(\mathbf{r} - \mathbf{r}')$ is the *interparticle* potential, which we assume to be central, depending only on the distance between any pair of particles. The equations of motion are

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i^{(\text{ext})} + \mathbf{F}_i^{(\text{int})} , \quad (2.35)$$

with

$$\mathbf{F}_i^{(\text{ext})} = - \frac{\partial V(\mathbf{r}_i)}{\partial \mathbf{r}_i} \quad (2.36)$$

$$\mathbf{F}_i^{(\text{int})} = - \sum_j \frac{\partial v(|\mathbf{r}_i - \mathbf{r}_j|)}{\mathbf{r}_i} \equiv \sum_j \mathbf{F}_{ij}^{(\text{int})} . \quad (2.37)$$

Here, $\mathbf{F}_{ij}^{(\text{int})}$ is the force exerted on particle i by particle j :

$$\mathbf{F}_{ij}^{(\text{int})} = \frac{\partial v(|\mathbf{r}_i - \mathbf{r}_j|)}{\partial \mathbf{r}_i} = -\frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|} v'(|\mathbf{r}_i - \mathbf{r}_j|). \quad (2.38)$$

Note that $\mathbf{F}_{ij}^{(\text{int})} = -\mathbf{F}_{ji}^{(\text{int})}$, otherwise known as Newton's Third Law. It is convenient to abbreviate $\mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j$, in which case we may write the interparticle force as $\mathbf{F}_{ij}^{(\text{int})} = -\hat{\mathbf{r}}_{ij} v'(r_{ij})$.

2.4 Linear and Angular Momentum

Consider now the total momentum of the system, $\mathbf{P} = \sum_i \mathbf{p}_i$. Its rate of change is

$$\frac{d\mathbf{P}}{dt} = \sum_i \dot{\mathbf{p}}_i = \sum_i \mathbf{F}_i^{(\text{ext})} + \overbrace{\sum_{i \neq j} \mathbf{F}_{ij}^{(\text{int})}}^{\mathbf{F}_{ij}^{(\text{int})} + \mathbf{F}_{ji}^{(\text{int})} = 0} = \mathbf{F}_{\text{tot}}^{(\text{ext})}, \quad (2.39)$$

since the sum over all internal forces cancels as a result of Newton's Third Law. We write

$$\mathbf{P} = \sum_i m_i \dot{\mathbf{r}}_i = M \dot{\mathbf{R}} \quad (2.40)$$

$$M = \sum_i m_i \quad (\text{total mass}) \quad (2.41)$$

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} \quad (\text{center-of-mass}). \quad (2.42)$$

Next, consider the total angular momentum,

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i. \quad (2.43)$$

The rate of change of \mathbf{L} is then

$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= \sum_i \{m_i \dot{\mathbf{r}}_i \times \dot{\mathbf{r}}_i + m_i \mathbf{r}_i \times \ddot{\mathbf{r}}_i\} \\ &= \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(\text{ext})} + \sum_{i \neq j} \mathbf{r}_i \times \mathbf{F}_{ij}^{(\text{int})} \\ &= \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(\text{ext})} + \overbrace{\frac{1}{2} \sum_{i \neq j} (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij}^{(\text{int})}}^{\mathbf{r}_{ij} \times \mathbf{F}_{ij}^{(\text{int})} = 0} \\ &= \mathbf{N}_{\text{tot}}^{(\text{ext})}. \end{aligned} \quad (2.44)$$

Finally, it is useful to establish the result

$$T = \frac{1}{2} \sum_i m_i \dot{\mathbf{r}}_i^2 = \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} \sum_i m_i (\dot{\mathbf{r}}_i - \dot{\mathbf{R}})^2 , \quad (2.45)$$

which says that the kinetic energy may be written as a sum of two terms, those being the kinetic energy of the center-of-mass motion, and the kinetic energy of the particles relative to the center-of-mass.

Recall the “work-energy theorem” for conservative systems,

$$\begin{aligned} 0 &= \int_{\text{initial}}^{\text{final}} dE = \int_{\text{initial}}^{\text{final}} dT + \int_{\text{initial}}^{\text{final}} dU \\ &= T^{(\text{B})} - T^{(\text{A})} - \sum_i \int d\mathbf{r}_i \cdot \mathbf{F}_i \end{aligned} \quad (2.46)$$

$$\Delta T = T^{(\text{B})} - T^{(\text{A})} = \sum_i \int d\mathbf{r}_i \cdot \mathbf{F}_i . \quad (2.47)$$

Note that for continuous systems, we replace

$$\sum_i m_i \phi(\mathbf{r}_i) \longrightarrow \int d^3r \rho(\mathbf{r}) \phi(\mathbf{r}) , \quad (2.48)$$

where $\rho(\mathbf{r})$ is the mass density, and $\phi(\mathbf{r})$ is any function.

2.5 Scaling of Solutions for Homogeneous Potentials

2.5.1 Euler’s Theorem for Homogeneous Functions

In certain cases of interest, the potential is a homogeneous function of the coordinates. This means

$$U(\lambda \mathbf{r}_1, \dots, \lambda \mathbf{r}_N) = \lambda^k U(\mathbf{r}_1, \dots, \mathbf{r}_N) . \quad (2.49)$$

Here, k is the *degree of homogeneity* of U . Familiar examples include gravity,

$$U(\mathbf{r}_1, \dots, \mathbf{r}_N) = -G \sum_{i < j} \frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|} ; \quad k = -1 , \quad (2.50)$$

and the harmonic oscillator,

$$U(q_1, \dots, q_n) = \frac{1}{2} \sum_{\sigma, \sigma'} V_{\sigma\sigma'} q_\sigma q_{\sigma'} \quad ; \quad k = +2 . \quad (2.51)$$

The sum of two homogeneous functions is itself homogeneous only if the component functions themselves are of the same degree of homogeneity. Homogeneous functions obey a special result known as *Euler's Theorem*, which we now prove. Suppose a multivariable function $H(x_1, \dots, x_n)$ is homogeneous:

$$H(\lambda x_1, \dots, \lambda x_n) = \lambda^k H(x_1, \dots, x_n) . \quad (2.52)$$

Then

$$\boxed{\left. \frac{d}{d\lambda} \right|_{\lambda=1} H(\lambda x_1, \dots, \lambda x_n) = \sum_{i=1}^n x_i \frac{\partial H}{\partial x_i} = k H} \quad (2.53)$$

2.5.2 Scaled Equations of Motion

Now suppose the we rescale distances and times, defining

$$\mathbf{r}_i = \alpha \tilde{\mathbf{r}}_i \quad , \quad t = \beta \tilde{t} . \quad (2.54)$$

Then

$$\frac{d\mathbf{r}_i}{dt} = \frac{\alpha}{\beta} \frac{d\tilde{\mathbf{r}}_i}{d\tilde{t}} \quad , \quad \frac{d^2\mathbf{r}_i}{dt^2} = \frac{\alpha}{\beta^2} \frac{d^2\tilde{\mathbf{r}}_i}{d\tilde{t}^2} . \quad (2.55)$$

The force \mathbf{F}_i is given by

$$\begin{aligned} \mathbf{F}_i &= -\frac{\partial}{\partial \mathbf{r}_i} U(\mathbf{r}_1, \dots, \mathbf{r}_N) \\ &= -\frac{\partial}{\partial (\alpha \tilde{\mathbf{r}}_i)} \alpha^k U(\tilde{\mathbf{r}}_1, \dots, \tilde{\mathbf{r}}_N) \\ &= \alpha^{k-1} \tilde{\mathbf{F}}_i . \end{aligned} \quad (2.56)$$

Thus, Newton's 2nd Law says

$$\frac{\alpha}{\beta^2} m_i \frac{d^2\tilde{\mathbf{r}}_i}{d\tilde{t}^2} = \alpha^{k-1} \tilde{\mathbf{F}}_i . \quad (2.57)$$

If we choose β such that

We now demand

$$\frac{\alpha}{\beta^2} = \alpha^{k-1} \quad \Rightarrow \quad \beta = \alpha^{1-\frac{1}{2}k} , \quad (2.58)$$

then the equation of motion is invariant under the rescaling transformation! This means that if $\mathbf{r}(t)$ is a solution to the equations of motion, then so is $\alpha \mathbf{r}(\alpha^{\frac{1}{2}k-1} t)$. This gives us an entire one-parameter family of solutions, for all real positive α .

If $\mathbf{r}(t)$ is periodic with period T , the $\mathbf{r}_i(t; \alpha)$ is periodic with period $T' = \alpha^{1-\frac{1}{2}k} T$. Thus,

$$\left(\frac{T'}{T}\right) = \left(\frac{L'}{L}\right)^{1-\frac{1}{2}k}. \quad (2.59)$$

Here, $\alpha = L'/L$ is the ratio of length scales. Velocities, energies and angular momenta scale accordingly:

$$[v] = \frac{L}{T} \quad \Rightarrow \quad \frac{v'}{v} = \frac{L'}{L} \bigg/ \frac{T'}{T} = \alpha^{\frac{1}{2}k} \quad (2.60)$$

$$[E] = \frac{ML^2}{T^2} \quad \Rightarrow \quad \frac{E'}{E} = \left(\frac{L'}{L}\right)^2 \bigg/ \left(\frac{T'}{T}\right)^2 = \alpha^k \quad (2.61)$$

$$[\mathbf{L}] = \frac{ML^2}{T} \quad \Rightarrow \quad \frac{|\mathbf{L}'|}{|\mathbf{L}|} = \left(\frac{L'}{L}\right)^2 \bigg/ \frac{T'}{T} = \alpha^{(1+\frac{1}{2}k)}. \quad (2.62)$$

As examples, consider:

(i) *Harmonic Oscillator* : Here $k = 2$ and therefore

$$q_\sigma(t) \longrightarrow q_\sigma(t; \alpha) = \alpha q_\sigma(t). \quad (2.63)$$

Thus, rescaling lengths alone gives another solution.

(ii) *Kepler Problem* : This is gravity, for which $k = -1$. Thus,

$$\mathbf{r}(t) \longrightarrow \mathbf{r}(t; \alpha) = \alpha \mathbf{r}(\alpha^{-3/2} t). \quad (2.64)$$

Thus, $r^3 \propto t^2$, *i.e.*

$$\left(\frac{L'}{L}\right)^3 = \left(\frac{T'}{T}\right)^2, \quad (2.65)$$

also known as Kepler's Third Law.

2.6 Appendix I : Curvilinear Orthogonal Coordinates

The standard cartesian coordinates are $\{x_1, \dots, x_d\}$, where d is the dimension of space. Consider a different set of coordinates, $\{q_1, \dots, q_d\}$, which are related

to the original coordinates x_μ via the d equations

$$q_\mu = q_\mu(x_1, \dots, x_d) . \quad (2.66)$$

In general these are nonlinear equations.

Let $\hat{e}_i^0 = \hat{x}_i$ be the Cartesian set of orthonormal unit vectors, and define \hat{e}_μ to be the unit vector perpendicular to the surface $dq_\mu = 0$. A differential change in position can now be described in both coordinate systems:

$$d\mathbf{s} = \sum_{i=1}^d \hat{e}_i^0 dx_i = \sum_{\mu=1}^d \hat{e}_\mu h_\mu(q) dq_\mu , \quad (2.67)$$

where each $h_\mu(q)$ is an as yet unknown function of all the components q_ν . Finding the coefficient of dq_μ then gives

$$h_\mu(q) \hat{e}_\mu = \sum_{i=1}^d \frac{\partial x_i}{\partial q_\mu} \hat{e}_i^0 \quad \Rightarrow \quad \hat{e}_\mu = \sum_{i=1}^d M_{\mu i} \hat{e}_i^0 , \quad (2.68)$$

where

$$M_{\mu i}(q) = \frac{1}{h_\mu(q)} \frac{\partial x_i}{\partial q_\mu} . \quad (2.69)$$

The dot product of unit vectors in the new coordinate system is then

$$\hat{e}_\mu \cdot \hat{e}_\nu = (MM^t)_{\mu\nu} = \frac{1}{h_\mu(q) h_\nu(q)} \sum_{i=1}^d \frac{\partial x_i}{\partial q_\mu} \frac{\partial x_i}{\partial q_\nu} . \quad (2.70)$$

The condition that the new basis be orthonormal is then

$$\sum_{i=1}^d \frac{\partial x_i}{\partial q_\mu} \frac{\partial x_i}{\partial q_\nu} = h_\mu^2(q) \delta_{\mu\nu} . \quad (2.71)$$

This gives us the relation

$$h_\mu(q) = \sqrt{\sum_{i=1}^d \left(\frac{\partial x_i}{\partial q_\mu} \right)^2} . \quad (2.72)$$

Note that

$$(d\mathbf{s})^2 = \sum_{\mu=1}^d h_\mu^2(q) (dq_\mu)^2 . \quad (2.73)$$

For general coordinate systems, which are not necessarily orthogonal, we have

$$(d\mathbf{s})^2 = \sum_{\mu,\nu=1}^d g_{\mu\nu}(q) dq_\mu dq_\nu , \quad (2.74)$$

where $g_{\mu\nu}(q)$ is a real, symmetric, positive definite matrix called the *metric tensor*.

2.6.1 Example : Spherical Coordinates

Consider spherical coordinates (ρ, θ, ϕ) :

$$x = \rho \sin \theta \cos \phi \quad , \quad y = \rho \sin \theta \sin \phi \quad , \quad z = \rho \cos \theta \quad . \quad (2.75)$$

It is now a simple matter to derive the results

$$h_\rho^2 = 1 \quad , \quad h_\theta^2 = \rho^2 \quad , \quad h_\phi^2 = \rho^2 \sin^2 \theta \quad . \quad (2.76)$$

Thus,

$$ds = \hat{\rho} d\rho + \rho \hat{\theta} d\theta + \rho \sin \theta \hat{\phi} d\phi \quad . \quad (2.77)$$

2.6.2 Vector Calculus : Grad, Div, Curl

Here we restrict our attention to $d = 3$. The gradient ∇U of a function $U(q)$ is defined by

$$\begin{aligned} dU &= \frac{\partial U}{\partial q_1} dq_1 + \frac{\partial U}{\partial q_2} dq_2 + \frac{\partial U}{\partial q_3} dq_3 \\ &\equiv \nabla U \cdot ds \quad . \end{aligned} \quad (2.78)$$

Thus,

$$\nabla = \frac{\hat{e}_1}{h_1(q)} \frac{\partial}{\partial q_1} + \frac{\hat{e}_2}{h_2(q)} \frac{\partial}{\partial q_2} + \frac{\hat{e}_3}{h_3(q)} \frac{\partial}{\partial q_3} \quad . \quad (2.79)$$

For the divergence, we use the divergence theorem, and we appeal to fig. 2.2:

$$\int_{\Omega} dV \nabla \cdot \mathbf{A} = \int_{\partial\Omega} dS \hat{\mathbf{n}} \cdot \mathbf{A} \quad , \quad (2.80)$$

where Ω is a region of three-dimensional space and $\partial\Omega$ is its closed two-dimensional boundary. The LHS of this equation is

$$\text{LHS} = \nabla \cdot \mathbf{A} \cdot (h_1 dq_1) (h_2 dq_2) (h_3 dq_3) \quad . \quad (2.81)$$

The RHS is

$$\begin{aligned} \text{RHS} &= A_1 h_2 h_3 \Big|_{q_1}^{q_1+dq_1} dq_2 dq_3 + A_2 h_1 h_3 \Big|_{q_2}^{q_2+dq_2} dq_1 dq_3 + A_3 h_1 h_2 \Big|_{q_3}^{q_3+dq_3} dq_1 dq_2 \\ &= \left[\frac{\partial}{\partial q_1} (A_1 h_2 h_3) + \frac{\partial}{\partial q_2} (A_2 h_1 h_3) + \frac{\partial}{\partial q_3} (A_3 h_1 h_2) \right] dq_1 dq_2 dq_3 \quad . \end{aligned} \quad (2.82)$$

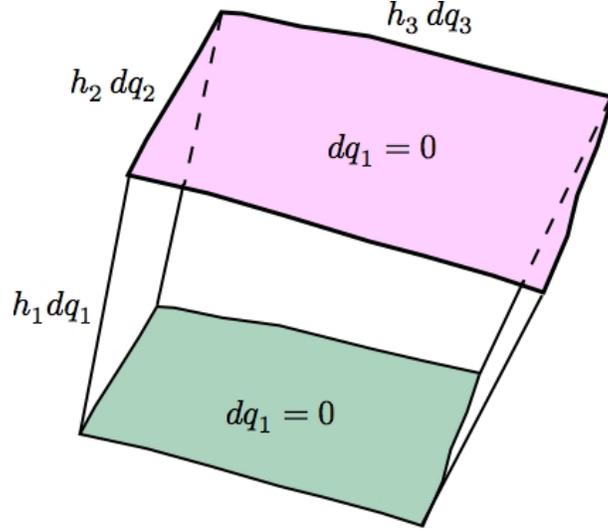


Figure 2.2: Volume element Ω for computing divergences.

We therefore conclude

$$\nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} (A_1 h_2 h_3) + \frac{\partial}{\partial q_2} (A_2 h_1 h_3) + \frac{\partial}{\partial q_3} (A_3 h_1 h_2) \right]. \quad (2.83)$$

To obtain the curl $\nabla \times \mathbf{A}$, we use Stokes' theorem again,

$$\int_{\Sigma} dS \hat{\mathbf{n}} \cdot \nabla \times \mathbf{A} = \oint_{\partial \Sigma} d\ell \cdot \mathbf{A}, \quad (2.84)$$

where Σ is a two-dimensional region of space and $\partial \Sigma$ is its one-dimensional boundary. Now consider a differential surface element satisfying $dq_1 = 0$, *i.e.* a rectangle of side lengths $h_2 dq_2$ and $h_3 dq_3$. The LHS of the above equation is

$$\text{LHS} = \hat{\mathbf{e}}_1 \cdot \nabla \times \mathbf{A} (h_2 dq_2) (h_3 dq_3). \quad (2.85)$$

The RHS is

$$\begin{aligned} \text{RHS} &= A_3 h_3 \Big|_{q_2}^{q_2+dq_2} dq_3 - A_2 h_2 \Big|_{q_3}^{q_3+dq_3} dq_2 \\ &= \left[\frac{\partial}{\partial q_2} (A_3 h_3) - \frac{\partial}{\partial q_3} (A_2 h_2) \right] dq_2 dq_3. \end{aligned} \quad (2.86)$$

Therefore

$$(\nabla \times \mathbf{A})_1 = \frac{1}{h_2 h_3} \left(\frac{\partial (h_3 A_3)}{\partial q_2} - \frac{\partial (h_2 A_2)}{\partial q_3} \right). \quad (2.87)$$

This is one component of the full result

$$\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \det \begin{pmatrix} h_1 \hat{\mathbf{e}}_1 & h_2 \hat{\mathbf{e}}_2 & h_3 \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{pmatrix}. \quad (2.88)$$

The Laplacian of a scalar function U is given by

$$\begin{aligned} \nabla^2 U &= \nabla \cdot \nabla U \\ &= \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial U}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial U}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial U}{\partial q_3} \right) \right\}. \end{aligned} \quad (2.89)$$

2.6.3 Common Curvilinear Orthogonal Systems

Rectangular Coordinates

In *rectangular* coordinates (x, y, z) , we have

$$h_x = h_y = h_z = 1. \quad (2.90)$$

Then the gradient is

$$\nabla U = \hat{\mathbf{x}} \frac{\partial U}{\partial x} + \hat{\mathbf{y}} \frac{\partial U}{\partial y} + \hat{\mathbf{z}} \frac{\partial U}{\partial z}. \quad (2.91)$$

The divergence is

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \quad (2.92)$$

The curl is

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{\mathbf{z}}. \quad (2.93)$$

The Laplacian is

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}. \quad (2.94)$$

Cylindrical Coordinates

In *cylindrical* coordinates (ρ, ϕ, z) , we have

$$\hat{\rho} = \hat{\mathbf{x}} \cos \phi + \hat{\mathbf{y}} \sin \phi \quad \hat{\mathbf{x}} = \hat{\rho} \cos \phi - \hat{\phi} \sin \phi \quad d\hat{\rho} = \hat{\phi} d\phi \quad (2.95)$$

$$\hat{\phi} = -\hat{\mathbf{x}} \sin \phi + \hat{\mathbf{y}} \cos \phi \quad \hat{\mathbf{y}} = \hat{\rho} \sin \phi + \hat{\phi} \cos \phi \quad d\hat{\phi} = -\hat{\rho} d\phi. \quad (2.96)$$

The metric is given in terms of

$$h_\rho = 1 \quad , \quad h_\phi = \rho \quad , \quad h_z = 1 . \quad (2.97)$$

Then the gradient is

$$\nabla U = \hat{\rho} \frac{\partial U}{\partial \rho} + \frac{\hat{\phi}}{\rho} \frac{\partial U}{\partial \phi} + \hat{z} \frac{\partial U}{\partial z} . \quad (2.98)$$

The divergence is

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} . \quad (2.99)$$

The curl is

$$\nabla \times \mathbf{A} = \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \hat{\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \hat{\phi} + \left(\frac{1}{\rho} \frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{1}{\rho} \frac{\partial A_\rho}{\partial \phi} \right) \hat{z} . \quad (2.100)$$

The Laplacian is

$$\nabla^2 U = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial U}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} . \quad (2.101)$$

Spherical Coordinates

In *spherical* coordinates (r, θ, ϕ) , we have

$$\hat{r} = \hat{x} \sin \theta \cos \phi + \hat{y} \sin \theta \sin \phi + \hat{z} \sin \theta \quad (2.102)$$

$$\hat{\theta} = \hat{x} \cos \theta \cos \phi + \hat{y} \cos \theta \sin \phi - \hat{z} \cos \theta \quad (2.103)$$

$$\hat{\phi} = -\hat{x} \sin \phi + \hat{y} \cos \phi , \quad (2.104)$$

for which

$$\hat{r} \times \hat{\theta} = \hat{\phi} \quad , \quad \hat{\theta} \times \hat{\phi} = \hat{r} \quad , \quad \hat{\phi} \times \hat{r} = \hat{\theta} . \quad (2.105)$$

The inverse is

$$\hat{x} = \hat{r} \sin \theta \cos \phi + \hat{\theta} \cos \theta \cos \phi - \hat{\phi} \sin \phi \quad (2.106)$$

$$\hat{y} = \hat{r} \sin \theta \sin \phi + \hat{\theta} \cos \theta \sin \phi + \hat{\phi} \cos \phi \quad (2.107)$$

$$\hat{z} = \hat{r} \cos \theta - \hat{\theta} \sin \theta . \quad (2.108)$$

The differential relations are

$$d\hat{r} = \hat{\theta} d\theta + \sin \theta \hat{\phi} d\phi \quad (2.109)$$

$$d\hat{\theta} = -\hat{r} d\theta + \cos \theta \hat{\phi} d\phi \quad (2.110)$$

$$d\hat{\phi} = -(\sin \theta \hat{r} + \cos \theta \hat{\theta}) d\phi \quad (2.111)$$

The metric is given in terms of

$$h_r = 1 \quad , \quad h_\theta = r \quad , \quad h_\phi = r \sin \theta . \quad (2.112)$$

Then the gradient is

$$\nabla U = \hat{\mathbf{r}} \frac{\partial U}{\partial \rho} + \frac{\hat{\boldsymbol{\theta}}}{r} \frac{\partial U}{\partial \theta} + \frac{\hat{\boldsymbol{\phi}}}{r \sin \theta} \frac{\partial U}{\partial \phi} . \quad (2.113)$$

The divergence is

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\sin \theta A_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} . \quad (2.114)$$

The curl is

$$\begin{aligned} \nabla \times \mathbf{A} = & \frac{1}{r \sin \theta} \left(\frac{\partial(\sin \theta A_\phi)}{\partial r} - \frac{\partial A_\theta}{\partial \phi} \right) \hat{\mathbf{r}} + \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial(r A_\phi)}{\partial r} \right) \hat{\boldsymbol{\theta}} \\ & + \frac{1}{r} \left(\frac{\partial(r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right) \hat{\boldsymbol{\phi}} . \end{aligned} \quad (2.115)$$

The Laplacian is

$$\nabla^2 U = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial U}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial U}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 U}{\partial \phi^2} . \quad (2.116)$$