Physics 110A: Problem Set #1

`Reading: Review MT chapter 1; read chapter 2, 3.4, 4.3 and class notes pp. 1-30 and 61-86`

[1] Consider the equation $\dot{x} = e^x - \cos x$.

(a) Sketch the vector field $\dot{x}$ on the real line.

(b) Graphically identify all the fixed points and classify them as stable or unstable.

(c) Sketch the function $x(t)$ for several initial conditions.

[2] Sketch $V(x)$ and the phase curves for one-dimensional motion in the following potentials:

$$\text{(a)} \quad V(x) = V_0\,(a^2 - x^2)^2$$

$$\text{(b)} \quad V(x) = V_0 \left[\left(\frac{a}{x}\right)^4 - \left(\frac{a}{x}\right)^2 \right] .$$

[3] In class, we integrated the logistic equation

$$\dot{N} = rN \left(1 - \frac{N}{K}\right) .$$

Consider here the modified logistic equation,

$$\dot{N} = rN \left(1 - \frac{N^2}{K^2}\right) ,$$

subject to the initial condition $N(0) = N_0$. Sketch the one-dimensional phase flows along the entire real line (including negative N) for both positive and negative r. Integrate the equation and show explicitly how N flows to a stable fixed point. Show that for $r < 0$ and $|N_0| > K$ that $N(t)$ flows to $\pm\infty$ in a *finite* time. Find the time $t(N_0, K)$ it takes for N to become infinite.

You may use any method you know of to integrate the ODE. The following identity, which you should derive, proves useful:

$$\frac{1}{(N-A)(N-B)(N-C)} = \frac{\alpha}{N-A} + \frac{\beta}{N-B} + \frac{\gamma}{N-C} .$$

where

$$\alpha = \frac{1}{(A-B)(A-C)}$$

$$\beta = \frac{1}{(B-A)(B-C)}$$

$$\gamma = \frac{1}{(C-A)(C-B)} .$$

[4] Sketch the phase curves for the second order equation

$$\ddot{\theta} + b\,\dot{\theta} + \sin\theta = 0\ .$$

What is a mechanical analog for this equation? Identify all fixed points and classify their stability.

[5] The Lorentz force exerted on a charged particle moving in an electric field $\vec{E}$ and a magnetic field $\vec{B}$ is given by

$$\vec{F} = q\vec{E} + \frac{q}{c}\,\vec{v}\times\vec{B}$$

where q is the particle's charge, $\vec{v}$ its velocity, and c the speed of light. Let $\vec{B} = B\hat{\mathbf{z}}$ and $\vec{E} = E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}$.

(a) Show that the motion in the $\hat{\mathbf{z}}$ direction is ballistic with uniform acceleration $a = qE_z/m$. Integrate to find $z(t)$.

(b) Find the equations for motion of the coordinates x and y. Show that the velocity $v_y = \dot{y}$ obeys the oscillator equation $\ddot{v}_y + \omega_c^2 v_y = 0$, where $\omega_c = qB/mc$ is the *cyclotron frequency*. Compute the cyclotron frequency for an electron moving in a $B = 6\,\mathrm{T}$ field. Show that $v_x = \dot{x}$ obeys the equation of a displaced oscillator.

(c) Integrate the equations from part (b) to obtain the complete solution of the problem. How many constants of integration are there?