Physics 110A: Midterm Exam #1

[**1**] A damped harmonic oscillator obeys the dimensionless equation of motion,

$$\ddot{x} + 8\,\dot{x} + 36\,x = 0 \ .$$

(**a**) Write down the solution $x(t)$ under the conditions $x(0) = 1$ and $\dot{x}(0) = -3$.

(**b**) Let the oscillator now be forced at (dimensionless) angular frequency ω:

$$\ddot{x} + 8\,\dot{x} + 36\,x = \cos(\omega t) \ .$$

Find the frequency ω_r at which the amplitude of the oscillations is maximized. You may assume that t is great enough that the homogeneous component has decayed to zero (*i.e.* ignore the homogeneous component). Find the amplitude of oscillation (*i.e.* the maximum value of the displacement) at this frequency. Also find the phase angle by which the response lags the forcing.

[**2**] A particle of mass m moves in one dimension under the influence of the potential

$$U(x) = U_0 \left\{ \frac{a^2}{x^2} - \frac{a}{x} \right\} \ .$$

(**a**) Sketch $U(x)$, identifying any minima and/or maxima.

(**b**) Sketch a set of (at least four, say) phase curves at various representative energies. Make sure to identify any fixed points and separatrices, and find their energies.

(**c**) For the phase curve with energy $E = -\frac{1}{8}U_0$, derive an integral expression for the period T. Your expression should take the form $T = \tau \cdot \mathcal{I}$, where $\tau = \sqrt{ma^2/U_0}$ is the natural time scale for this system (the only quantity with dimensions of time which can be formed from a, U_0, and m), and $\mathcal{I}$ is a dimensionless integral.

MIDTERM 1, PHYSICS 110A

1. $\ddot{x} + 8\dot{x} + 36x = 0$

(a) THIS IS JUST AN EQUATION OF THE FORM $\ddot{x} + 2\beta\dot{x} + 36x = 0$.

SO, I READ OFF $\beta = 4$, $\omega_0 = 6$. SINCE $\omega_0^2 > \beta^2$, I

HAVE AN UNDERDAMPED SOLUTION:

$$\underline{x(t) = A e^{-\beta t} \cos \nu t + B e^{-\beta t} \sin \nu t,}$$

WHERE $\nu = \sqrt{\omega_0^2 - \beta^2} = 2\sqrt{5}$

NOW, TO FIND A AND B, WE NEED TO LOOK AT $x(0) \equiv X_0$

AND $\dot{x}(0) \equiv \dot{X}_0$.

$x(0) = A$ $\quad \therefore \underline{A = X_0}$

$\dot{x}(t) = \left(-A\beta + B\nu\right) e^{-\beta t} \cos \nu t - \left(A\nu + B\beta\right) e^{-\beta t} \sin \nu t$

$\dot{x}(0) = \left(-A\beta + B\nu\right) = \dot{X}_0$

$\therefore B = \dfrac{\dot{X}_0 + A\beta}{\nu}$

SINCE $X_0 = 1$, AND $\dot{X}_0 = -3$, WE HAVE $A = 1$, $B = \dfrac{1}{2\sqrt{5}}$

$\therefore \underline{x(t) = e^{-4t} \cos 2\sqrt{5}t + \dfrac{1}{2\sqrt{5}} e^{-4t} \sin 2\sqrt{5}t}$

(b) $\ddot{x} + 8\dot{x} + 36x = \cos\omega t$

WE KNOW THAT OUR PARTICULAR SOLUTION WILL LOOK

LIKE $\underline{x(t) = a(\omega)\cos(\omega t - \delta)}$; $a(\omega) = \left[(\omega_0^2 - \omega^3)^2 + 4\beta^2\omega^2\right]^{-\frac{1}{2}}$

$$\delta(\omega) = TAN^{-1}\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right)$$

I ALSO REQUIRE $\frac{da}{d\omega} = 0$

$$\frac{da}{d\omega} = -\frac{1}{2}\left[(\omega_0^2 - \omega^3)^2 + 4\beta^2\omega^3\right]^{-\frac{3}{2}}\left[2(\omega_0^2 - \omega^3)(-2\omega) + 8\beta^2\omega\right] = 0$$

$\therefore (\omega_0^2 - \omega^2) - 2\beta^2 = 0$

$\therefore \omega' = \sqrt{\omega_0^2 - 2\beta^2} = 2$

$$a(\omega') = \left[(36-4)^2 + 16^2\right]^{-\frac{1}{2}} = \left[16^2 \cdot 5\right]^{-\frac{1}{2}} = \frac{1}{16\sqrt{5}}$$

$$\delta(\omega') = TAN^{-1}\left(\frac{2\cdot2\cdot4}{36-4}\right) = TAN^{-1}\left(\frac{1}{2}\right)$$

$$\therefore \underline{x(t) = \frac{1}{16\sqrt{5}}\cos\left(\omega t - TAN^{-1}\left(\frac{1}{2}\right)\right)}$$
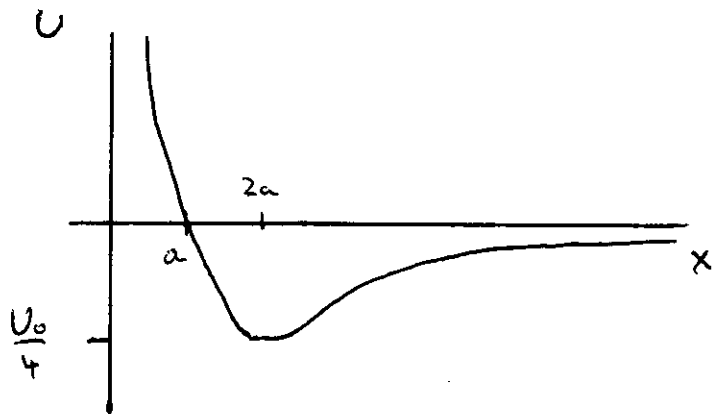
2. $U(x) = U_0 \left(\frac{a^2}{x^2} - \frac{a}{x} \right)$

$U(x \to 0) \sim \frac{a^2}{x^2} \to \infty$; $U(x \to \infty) \sim -\frac{a}{x} \to^- 0$
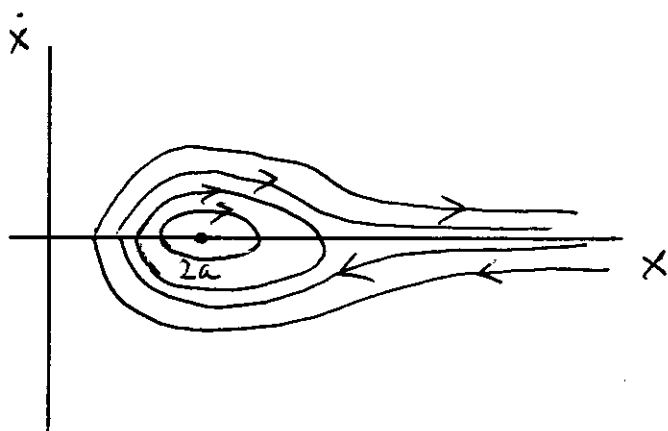
$U'(x) = U_0 \left(\frac{a}{x^2} - \frac{2a^2}{x^3} \right)$ $U'(2a) = 0$

$U(2a) = -\frac{1}{4} U_0$; $U(a) = 0$

(a)

(b)

(c) RECALL $E = \frac{1}{2}m\dot{x}^2 + U(x)$

$$dt = \sqrt{\frac{m}{2}} \frac{dx}{\sqrt{E - U(x)}}$$

NOW, $T = \sqrt{2m} \int_{-x}^{+x} \frac{dx}{\sqrt{E - U(x)}} = \sqrt{2m} \int_{-x}^{+x} \frac{dx}{\sqrt{-\frac{1}{8}U_0 - U_0 \left(\frac{a^2}{x^2} - \frac{a}{x}\right)}}$

a MAKE A SUBSTITUTION: $S \equiv x/a$

THEN, $T = a\left(\frac{2m}{U_0}\right)^{1/2} \int_{S-}^{S+} \frac{ds}{\sqrt{-\frac{1}{8} - S^{-2} + S^{-1}}}$

$$= \left(\frac{2ma^2}{U_0}\right)^{1/2} \int_{S-}^{S+} \frac{ds}{\sqrt{-\frac{1}{8} - S^{-2} + S^{-1}}}$$

SO, WHAT ARE $S^{\pm}$? WE CAN GET THESE BY

SOLVING $E = -\frac{1}{8}U_0 = U_0\left(S^{-2} - S^{-1}\right)$

OK, $\frac{1}{8}S^2 - S + 1 = 0$

AND $S^{\pm} = 2\left(2 \pm \sqrt{2}\right)$

SO $T = \left(\frac{2ma^2}{U_0}\right)^{1/2} \int_{2(2-\sqrt{2})}^{2(2+\sqrt{2})} ds \left(-\frac{1}{8} - S^{-2} + S^{-1}\right)^{-1/2}$