Physics 214 UCSD/225a UCSB

Lecture 12

- Halzen & Martin Chapter 6

Spinless vs Spin 1/2

$$\phi\left(t,\vec{x}\right) = Ne^{-ip_\mu x^\mu}$$

$$\psi\left(t,\vec{x}\right) = u(p)e^{-ip_\mu x^\mu}$$

$$J_\mu = -ie\left(\phi^*(\partial_\mu\phi) - (\partial_\mu\phi^*)\phi\right)$$

$$T_{fi} = -i\int J^\mu_{fi} A_\mu d^4x + O(e^2)$$

$$J^\mu = -e\overline{\psi}\gamma^\mu\psi$$

$$T_{fi} = -i\int J^\mu_{fi} A_\mu d^4x + O(e^2)$$

We basically make a substitution:

$$\left(p_f + p_i\right)_\mu \longrightarrow \overline{u}_f\gamma_\mu u_i$$

And all else in calculating |M|² remains the same.

Example: e$^-$ e$^-$ scattering

For Spinless (i.e. bosons) we showed:

$$M = -e^2 \left(\frac{(p_A + p_C)^\mu (p_B + p_D)_\mu}{\left(p_A - p_C\right)^2} + \frac{(p_A + p_D)^\mu (p_B + p_C)_\mu}{\left(p_A - p_D\right)^2} \right)$$

For Spin 1/2 we thus get:

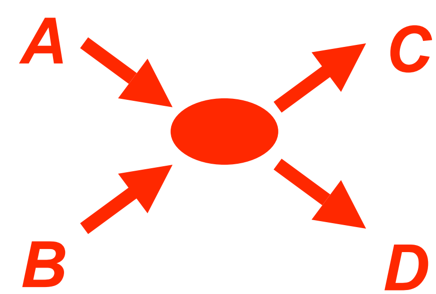
$$M = -e^2 \left(\frac{(\overline{u}_c \gamma^\mu u_A)(\overline{u}_D \gamma_\mu u_B)}{\left(p_A - p_C\right)^2} - \frac{(\overline{u}_D \gamma^\mu u_A)(\overline{u}_C \gamma_\mu u_B)}{\left(p_A - p_D\right)^2} \right)$$

Minus sign comes from fermion exchange !!!

Spin Averaging

- The M from previous page includes spinors in initial and final state.
- In many experimental situations, in particular in hadron collissions, you neither fix initial nor final state spins.
- We thus need to form a spin averaged amplitude squared before we can compare with experiment:

$$\overline{|M|^2} = \frac{1}{(2s_A+1)(2s_B+1)}\sum_{spin}|M|^2 = \frac{1}{4}\sum_{spin}|M|^2$$



Spin Averaging in non-relativistic limit

- Incoming e$^-$:

$$u^{(s=+1/2)} = \sqrt{2m}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

- Outgoing e$^-$:

$$\overline{u}^{(s=-1/2)} = \sqrt{2m}\begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}$$

Reminder:

$$\gamma^0 = \begin{pmatrix}I & 0\\0 & -I\end{pmatrix}$$

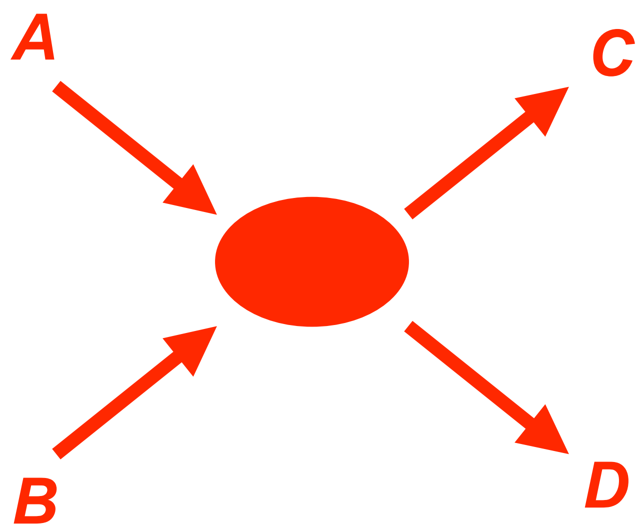
$$\gamma^{j=1,2,3} = \begin{pmatrix}0 & \sigma_{j=1,2,3}\\-\sigma_{j=1,2,3} & 0\end{pmatrix}$$

$$\overline{u}_f\gamma_\mu u_i = \begin{cases}2m & if \quad (\mu=0)\wedge s_i = s_f\\ 0 & otherwise\end{cases}$$

Invariant variables s,t,u

Example: e- e- -> e- e-

- $s = (p_A + p_B)^2$
- $= 4 (k^2 + m^2)$
- $t = (p_A - p_C)^2$
- $= -2 k^2 (1 - \cos\theta)$
- $u = (p_A - p_D)^2$
- $= -2 k^2 (1 + \cos\theta)$



$k = |k_i| = |k_f|$ $m = m_e$ θ = scattering angle, all in com frame.

M for Different spin combos

$$M = -e^2\left(\frac{(\bar{u}_c\gamma^\mu u_A)(\bar{u}_D\gamma_\mu u_B)}{t} - \frac{(\bar{u}_D\gamma^\mu u_A)(\bar{u}_C\gamma_\mu u_B)}{u}\right)$$

$$\left[(\bar{u}_c\gamma^\mu u_A)(\bar{u}_D\gamma_\mu u_B)\right]_{\downarrow\uparrow\rightarrow\downarrow\uparrow} = 4m^2$$

$$\left[(\bar{u}_D\gamma^\mu u_A)(\bar{u}_C\gamma_\mu u_B)\right]_{\uparrow\downarrow\rightarrow\downarrow\uparrow} = 4m^2$$

$$\left[(\bar{u}_c\gamma^\mu u_A)(\bar{u}_D\gamma_\mu u_B)\right]_{\uparrow\downarrow\rightarrow\downarrow\uparrow} = 0$$

$etc.$

$$\overline{|M|^2} = \frac{1}{4}\left(4m^2e^2\right)^2 2\left[\left(\frac{1}{t} - \frac{1}{u}\right)^2 + \frac{1}{t^2} + \frac{1}{u^2}\right]$$

Let's do one relativistically

- Pick the easiest one: e- mu- -> e- mu-
  - Easiest because it has only one diagram !!!

$$M = -e^2 \frac{\left[\bar{u}_c(k')\gamma^\mu u_A(k)\right]\left[\bar{u}_D(p')\gamma_\mu u_B(p)\right]}{t}$$

$$|M|^2 = \frac{e^4}{t^2}["k"]^\mu["p"]_\mu\left(["k"]^\nu["p"]_\nu\right)^* = \frac{e^4}{t^2}["k"]^{\mu\nu}["p"]_{\mu\nu}$$

This is a sensible way to proceed because it groups electron and muon parts together into one tensor each. We then worry only about spin averaging those tensors. After all, only same flavor spinors appear at the same vertex, and all reductions of spin averages happen at a vertex.

$$M = -e^2 \frac{\left[\bar{u}_c(k')\gamma^{\mu} u_A(k)\right]\left[\bar{u}_D(p')\gamma_{\mu} u_B(p)\right]}{t}$$

$$\overline{|M|^2} = \frac{e^4}{t^2} L_{electron}{}^{\mu\nu} L^{muon}{}_{\mu\nu}$$

$$L_{electron}{}^{\mu\nu} = \frac{1}{2} \sum_{e-spins} \left[\bar{u}_c(k')\gamma^{\mu} u_A(k)\right]\left[\bar{u}_c(k')\gamma^{\nu} u_A(k)\right]^*$$

$$L_{muon}{}^{\mu\nu} = \frac{1}{2} \sum_{muon-spins} \left[\bar{u}_D(p')\gamma^{\mu} u_B(p)\right]\left[\bar{u}_D(p')\gamma^{\nu} u_B(p)\right]^*$$

We will now look at the electron tensor only, and try to simplify!

Note the structure of this expression:

$$\left[\bar{u}_D(p')\gamma^{\nu}u_B(p)\right]^{*} = \left[\left(\qquad\qquad\qquad\right)\left(4x4\right)\begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}\right]^{\nu} = \left(1x1\right)^{\nu}$$

The complex conjugate of this object is thus identical to the hermitian conjugate of this !!!

We can use the latter in order to rearrange terms, while ignoring the lorentz index for the moment.

$$\left[\overline{u}_c \gamma^\nu u_A\right]^{T*} = \left[u_C^{T*}\gamma^0\gamma^\nu u_A\right]^{T*} = \left[u_A^{T*}\gamma^{\nu T*}\gamma^0 u_C\right] = \left[\overline{u}_A(k)\gamma^\nu u_C(k')\right]$$

Where in the last step , we used the commutation properties.

At this point we get the electron tensor:

$$L_{electron}{}^{\mu\nu} = \frac{1}{2}\sum_{s,s'}\left[\overline{u}_c(k')\gamma^\mu u_A(k)\right]\left[\overline{u}_A(k)\gamma^\nu u_C(k')\right]$$

Next, we are going to make all summations explicit, by writing out the gamma-matrices and spinor-vectors as components.

$$L_{electron}{}^{\mu\nu} = \frac{1}{2}\sum_{s,s'}\left[\overline{u}_c(k')\gamma^{\mu}u_A(k)\right]\left[\overline{u}_A(k)\gamma^{\nu}u_C(k')\right]$$

$$= \frac{1}{2}\sum_{s,s'}\sum_{ijlm}\left[\overline{u}_i^{s'}(k_C')\gamma_{ij}^{\mu}u_j^{s}(k_A)\right]\left[\overline{u}_l^{s}(k_A)\gamma_{lm}^{\nu}u_m^{s'}(k_C')\right]$$

$$= \frac{1}{2}\sum_{ijlm}\gamma_{ij}^{\mu}\gamma_{lm}^{\nu}\sum_{s'}\left[\overline{u}_i^{s'}(k_C')u_m^{s'}(k_C')\right]\sum_{s}\left[\overline{u}_l^{s}(k_A)u_j^{s}(k_A)\right]$$

Here we now apply the completeness relations:

$$\sum_{s}u^{s}(p)\overline{u}^{s}(p) = p_{\mu}\gamma^{\mu} + m = \left(4x4\right)$$

See H&M exercise 5.9 for more detail on completeness relation.

Mathematical aside:

Let A,B,C,D be 4 matrices.

$$\sum_{ijlm} A_{ij} B_{jl} C_{lm} D_{mi} \equiv Tr\left(A \cdot B \cdot C \cdot D\right)$$

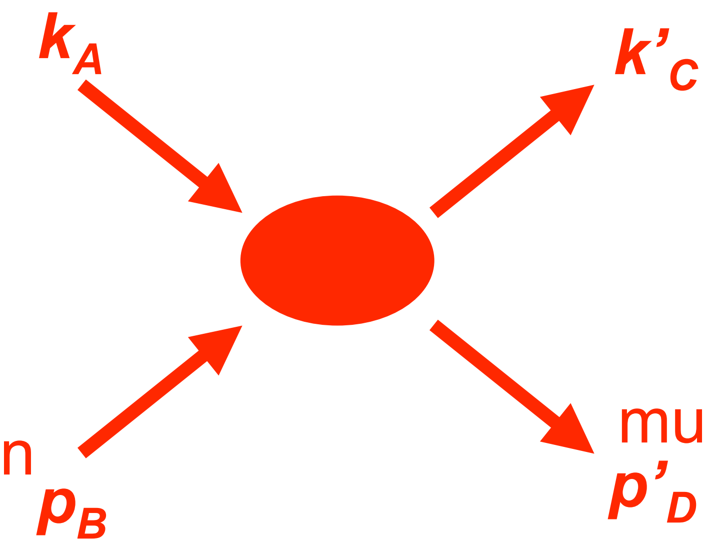
This weird sum is thus nothing more than the trace of the product of matrices !!!

I won’t prove this here, but please feel free to convince yourself.

$$L_{electron}{}^{\mu\nu} = \frac{1}{2} Tr\left[\left(k'_\alpha \gamma^\alpha + m\right)\gamma^\mu \left(k_\beta \gamma^\beta + m\right)\gamma^\nu\right]$$

$$L_{muon}{}^{\mu\nu} = \frac{1}{2} Tr\left[\left(p'_\alpha \gamma^\alpha + m\right)\gamma^\mu \left(p_\beta \gamma^\beta + m\right)\gamma^\nu\right]$$

$$\overline{|M|^2} = \frac{e^4}{t^2} L_{electron}{}^{\mu\nu} L^{muon}{}_{\mu\nu}$$



"All" that's left to do is apply trace theorems.
(There's a whole bunch of them in H&M p.123)

$$L_{electron}{}^{\mu\nu} = \frac{1}{2} Tr\left[\left(k'_\alpha \gamma^\alpha + m\right)\gamma^\mu \left(k_\beta \gamma^\beta + m\right)\gamma^\nu\right]$$

$$= \frac{1}{2} Tr\left[k'_\alpha \gamma^\alpha \gamma^\mu k_\beta \gamma^\beta \gamma^\nu + k'_\alpha \gamma^\alpha \gamma^\mu \gamma^\nu m + m\gamma^\mu k_\beta \gamma^\beta \gamma^\nu + m^2 \gamma^\mu \gamma^\nu\right]$$

Trace of product of any 3 gamma matrices is zero!

$$L_{electron}{}^{\mu\nu} = \frac{1}{2} Tr\left[k'_\alpha \gamma^\alpha \gamma^\mu k_\beta \gamma^\beta \gamma^\nu\right] + \frac{m^2}{2} Tr\left[\gamma^\mu \gamma^\nu\right]$$

Next, define unit vectors a,b for μ and ν coordinate:

$$a_\iota \gamma^\iota \equiv \gamma^\mu$$

$$b_\kappa \gamma^\kappa \equiv \gamma^\nu$$

This will allow us to use trace theorems to evaluate the remaining traces.

2 Trace Theorems to use:

(1) (2) (3) (4)

$$Tr\left[(k'_\alpha \gamma^\alpha)(a_\iota \gamma^\iota)(k_\beta \gamma^\beta)(b_\kappa \gamma^\kappa)\right] =$$

(1x2) (3x4) - (1x3) (2x4) + (1x4) (2x3)

$$= 4\left[(k' \cdot a)(k \cdot b) - (k' \cdot k)(a \cdot b) + (k' \cdot b)(k \cdot a)\right]$$

$$= 4\left[k'^\mu k^\nu - (k' \cdot k) g^{\mu\nu} + k^\mu k'^\nu\right]$$

$$Tr\left[(a_\iota \gamma^\iota)(b_\kappa \gamma^\kappa)\right] = 4 a \cdot b = 4 g^{\mu\nu}$$

Now put it all together …

Electron - Muon scattering

$$L^{\mu\nu}_{electron} = 2\left[k'^{\mu}k^{\nu} + (m^2 - k'\cdot k)g^{\mu\nu} + k^{\mu}k'^{\nu}\right]$$

$$L^{muon}_{\mu\nu} = 2\left[p'_{\mu}p_{\nu} + (M^2 - p'\cdot p)g_{\mu\nu} + p_{\mu}p'_{\nu}\right]$$

$$\overline{|M|^2} = \frac{e^4}{t^2} L_{electron}{}^{\mu\nu} L^{muon}{}_{\mu\nu}$$

$$= \frac{8e^4}{t^2}\left[(k'p')(kp) + (k'p)(kp') - m^2 pp' - M^2 kk' + 2M^2 m^2\right]$$

This is the "exact" form. Next look at relativistic approx.

Relativistic approx. for e-mu scattering:

$$\overline{|M|^2} = \frac{8e^4}{t^2}\left[(k'p')(kp) + (k'p)(kp')\right]$$

Let's use the invariant variables:

s = (k+p)^2 ~ 2kp ~ 2k'p'

u = (k - p')^2 ~ -2kp' ~ -2k'p

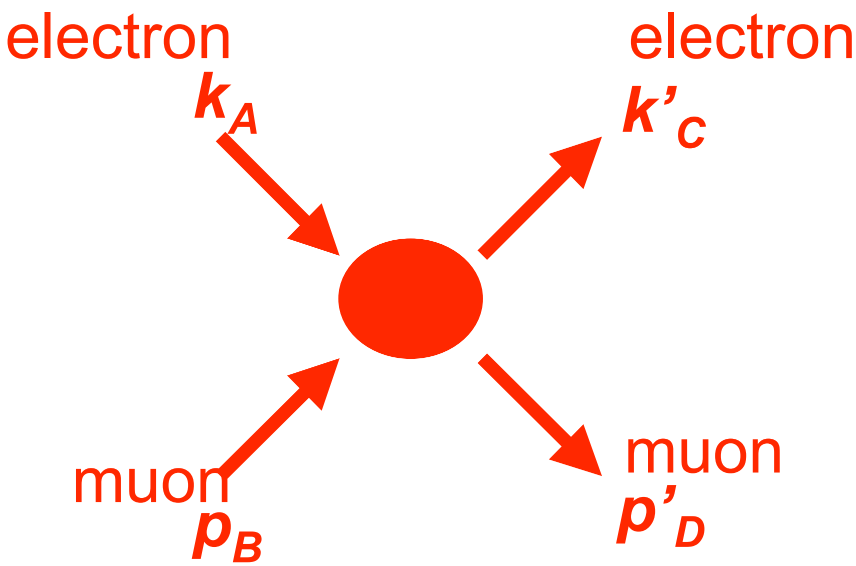
$$\boxed{\overline{|M|^2} = 2e^4\,\frac{\left(s^2 + u^2\right)}{t^2}}$$

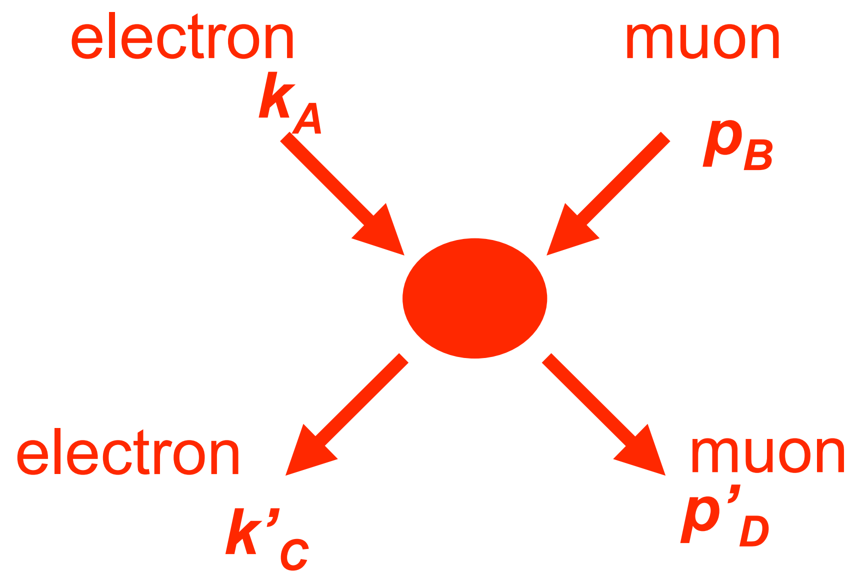
Next look at ee -> mumu, and get it via crossing.

emu -> emu => ee -> mumu

via crossing

$$k'_C \longleftrightarrow -p_B$$





$s = (k+p)^2$
$t = (k' - k)^2$

$$s \longleftrightarrow t$$

$t = (k+p)^2$
$s = (k' - k)^2$

e-mu- -> e-mu- => e+e- -> mu+mu-

$$\overline{|M|^2} = 2e^4 \frac{\left(s^2 + u^2\right)}{t^2}$$

$$\overline{|M|^2} = 2e^4 \frac{\left(t^2 + u^2\right)}{s^2}$$

Recall exercise 4.2 from H&M:

$$\left.\frac{d\sigma}{d\Omega}\right|_{cm} = \frac{|M|^2}{64\pi^2 s} \frac{p_f}{p_i}$$

We will now use this for relativistic e+e- -> mu+mu- scattering.

$p_f \sim p_i$ **=>** $$\left.\frac{d\sigma}{d\Omega}\right|_{cm} \approx \frac{|M|^2}{64\pi^2 s}$$

$t = -2\,k^2\,(1 - \cos\theta)$

$u = -2\,k^2\,(1 + \cos\theta)$ **=>** $$\overline{|M|^2} = 2e^4\,\frac{8k^4\left(1+\cos^2\theta\right)}{16k^4}$$

$s \sim 4k^2$

$\alpha = e^2/4\pi$

$$\boxed{\begin{aligned} \left.\frac{d\sigma}{d\Omega}\right|_{cm} &\approx \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right) \\ \sigma(e^+e^- \to \mu^+\mu^-) &\approx \frac{4\pi\alpha^2}{3s} \end{aligned}}$$

Relativistic limit

Result worthy of discussion

1. $\sigma \propto 1/s$ must be so on dimensional grounds
2. $\sigma \propto \alpha^2$ two vertices!
3. At higher energies, Z-propagator also contributes:

More discussion

4. Calculation of e+e- -> q qbar is identical as long as sqrt(s) >> Mass of quark.

$$\sigma(e^+e^- \rightarrow q\bar{q}) = \underbrace{3}_{\text{\# of flavors}} \cdot \underbrace{\sum_{q-flavor}}_{\text{Sum over quark flavors}} \underbrace{e_q^2}_{\text{Charge of quark}} \sigma(e^+e^- \rightarrow \mu^+\mu^-)$$

Measurement of this cross section was very important !!!

Measurement of R

$$R = \frac{\sigma(e^+e^- \rightarrow q\overline{q})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = 3 \cdot \sum_{q-flavor} e_q^2$$

Below charm threshold: R = 3 [$(2/3)^2$ + $(1/3)^2$ + $(1/3)^2$] =2

Between charm and bottom: R = 2 + 3(4/9) = 10/3

Above bottom: R = 10/3 + 3(1/9) = 11/3

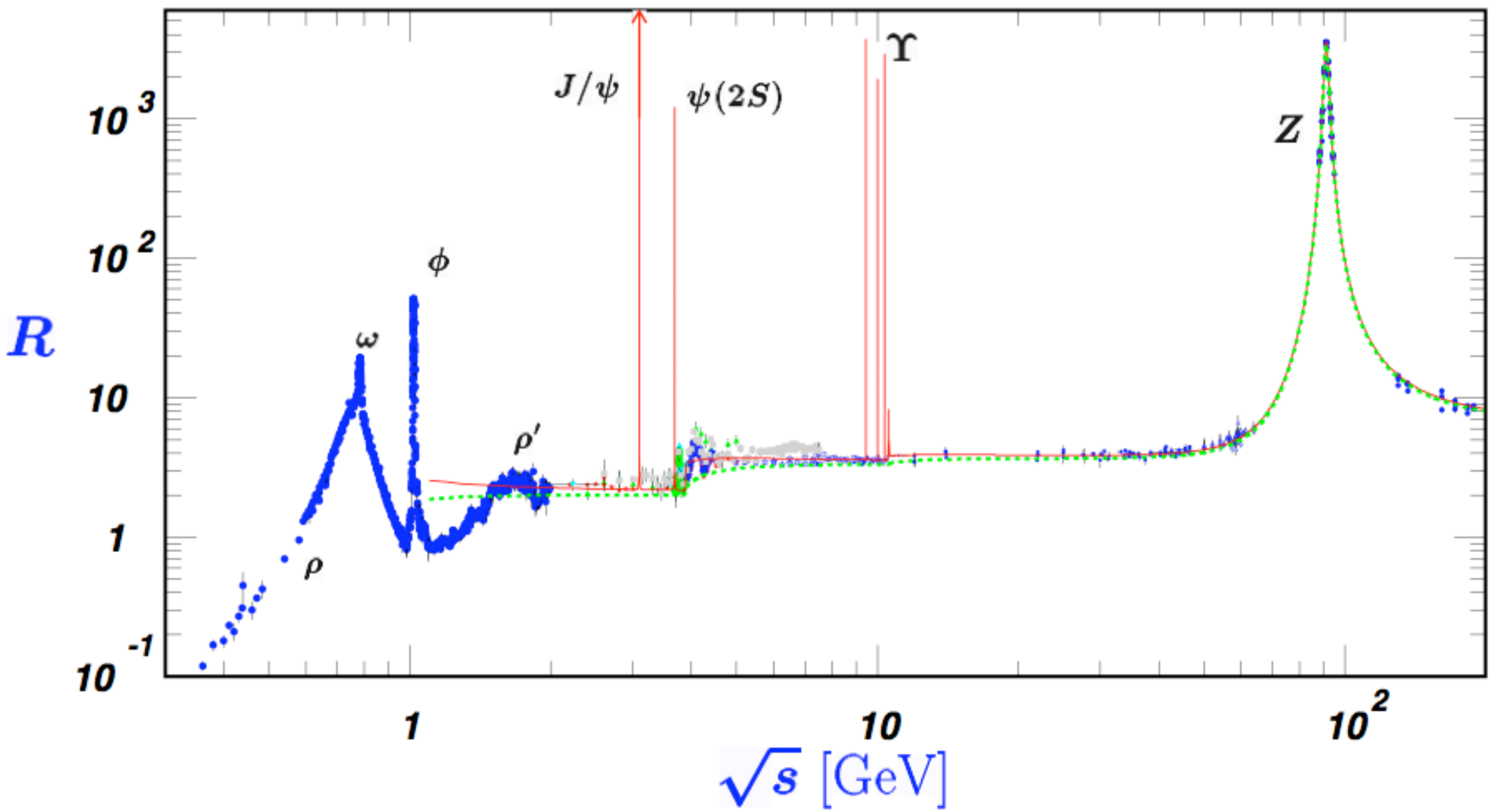
Measurement of R was crucial for:

a. Confirm that quarks have 3 colors

b. Search for additional quarks

c. Search for additional leptons

Experimental Result



Ever more discussion

5. $d\sigma/d\Omega \propto (1 + \cos^2\theta)$

5.1 θ is defined as the angle between e+ and mu+ in com. $\cos^2\theta$ means that the outgoing muons have no memory of the direction of incoming particle vs antiparticle.

Probably as expected as the e+e- annihilate before the mu+mu- is created.

5.2 Recall, phase space is flat in $\cos\theta$. $\cos^2\theta$ dependence thus implies that the initial state axis matters to the outgoing particles. Why?

Helicity Conservation in relativistic limit

- You showed as homework that u_L and u_R are helicity eigenstates in the relativistic limit, and thus:

$$\overline{u}\gamma^{\mu}u = \left(\overline{u}_L + \overline{u}_R\right)\gamma^{\mu}\left(u_L + u_R\right)$$

- We'll now show that the cross terms are zero, and helicity is thus conserved at each vertex.
- We then show how angular momentum conservation leads to the cross section we calculated.

Let’ do one cross product explicitly:

$$\bar{u}\gamma^{\mu}u = \left(\bar{u}_L + \bar{u}_R\right)\gamma^{\mu}\left(u_L + u_R\right)$$

$$\bar{u}_L = u_L^{T*}\gamma^0 = u^{T*}\frac{1}{2}\left(1-\gamma^5\right)\gamma^0 = \bar{u}\frac{1}{2}\left(1+\gamma^5\right)$$

$$u_R = \frac{1}{2}\left(1+\gamma^5\right)u$$

$$\bar{u}_L\gamma^{\mu}u_R = \bar{u}\frac{1}{4}\left(1+\gamma^5\right)\gamma^{\mu}\left(1+\gamma^5\right)u = \bar{u}\gamma^{\mu}\frac{1}{4}\left(1-\gamma^5\right)\left(1+\gamma^5\right)u = 0$$

Here we have used:

$$\gamma^5\gamma^{\mu} = -\gamma^{\mu}\gamma^5$$

$$\gamma^5 = \gamma^{5T*}$$

$$\gamma^5\gamma^5 = 1$$

Helicity conservation holds for all vector and axialvector currents as E>>m.

- $e_L^- e_R^+$ -> $mu_L^- mu_R^+$ J_z +1 -> +1
- $e_L^- e_R^+$ -> $mu_R^- mu_L^+$ J_z +1 -> -1
- $e_R^- e_L^+$ -> $mu_L^- mu_R^+$ J_z -1 -> +1
- $e_R^- e_L^+$ -> $mu_R^- mu_L^+$ J_z -1 -> -1
- *Next look at the rotation matrices:*

$$d^1_{11}(\theta) = \frac{1}{2}\left(1+\cos\theta\right) \approx \frac{-u}{s}$$

$$d^1_{-1-1}(\theta) = \frac{1}{2}\left(1+\cos\theta\right) \approx \frac{-u}{s}$$

$$d^1_{-11}(\theta) = \frac{1}{2}\left(1-\cos\theta\right) \approx \frac{-t}{s}$$

$$d^1_{1-1}(\theta) = \frac{1}{2}\left(1-\cos\theta\right) \approx \frac{-t}{s}$$

Cross products cancel in Spin average:

$$\overline{|M|^2} \propto \left(1+\cos^2\theta\right)$$

$$d^J_{1-1}$$

initial final J_z

Conclusion on relativistic limit

- Dependence on scattering angle is given entirely by angular momentum conservation !!!
- This is a generic feature for any vector or axialvector current.
- We will thus see the exact same thing also for V-A coupling of Electroweak interactions.