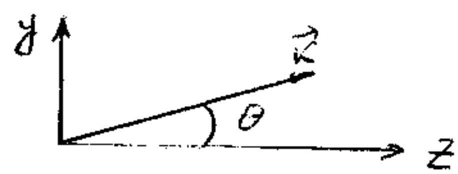
LECTURE 4

Penetration of electromagnetic wave into plasma. Transformation into plasma oscillations.

Since the plasma permittivity is negative for $\omega < \omega_{pe}$, electromagnetic waves cannot penetrate into such a plasma deeper than the skin depth c/ω_{pe}. In the vicinity of the skin layer, some unusual phenomena are developing. To understand them, let us consider non-homogeneous plasma with density that increases monotonically in z–direction. Let an electromagnetic wave propagates in such a plasma at some angle θ towards direction of non-homogeneity of plasma (z–direction) (Figure 1).

Fig. 1



If the plasma density changes sufficiently slow, i.e. $\omega c/L >> 1$, where L is the characteristic spatial scale of change of the plasma density, the problem of the wave propagation can be solved in approximation of geometric optics. In this approximation, the wave field can be found as a traveling wave with amplitude and wave number in the z-direction changing slowly with z:

$$E, B \approx \frac{const}{k_z^{1/2}} \exp\{i\int k_z dz + ik_y y - i\omega t\} \qquad (1)$$

The change of k_z with z can be found from usual dispersion relation of electromagnetic wave (see 3.17)

$$\frac{(k_z^2 + k_y^2)}{\omega^2} = \varepsilon \qquad (2)$$

We assumed for simplicity that the wave vector is in yz plane (incidence plane).

On the left from the region occupied by plasma i.e. in the vacuum this equation has the form

$$\frac{k^2 c^2}{\omega^2} = 1$$

Since $k_y = k \sin\theta$, the dispersion equation (2) can be written in the following form:

$$k_z^2 = \frac{\omega^2}{c^2}(\varepsilon - \sin^2\theta) \qquad (3)$$

The point where $\varepsilon = \sin^2\theta$ is called the turning point. At this point k_z^2 changes its sign and the wave is reflected there. The field structure is shown in the Fig. 2.

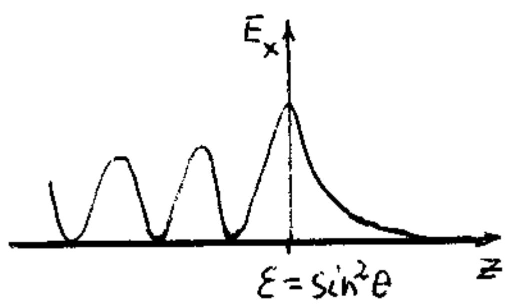


Fig. 2

On the left from the turning point the oscillatory structure of the standing electromagnetic wave is created by incident and reflected electromagnetic waves. On the right from the turning point, the field decreases exponentially.

More detail structure can be found using Maxwell's equations

$$\nabla \times \vec{E} = i\frac{\omega}{c}\vec{B}$$

$$\nabla \times \vec{B} = -i\frac{\omega}{c}\varepsilon\vec{E} \tag{4}$$

Here ε is a function of the coordinate z. Substituting $\vec{B}$ from the first of equations (4) into the second one we obtain the following equation for $\vec{E}$:

$$\nabla^2\vec{E} - \nabla div\vec{E} + \varepsilon\frac{\omega^2}{c^2}\vec{E} = 0 \tag{5}$$

Similarly excluding $\vec{E}$, we have the equation for $\vec{B}$:

$$\nabla^2\vec{B} + \varepsilon\frac{\omega^2}{c^2}\vec{B} + \frac{1}{\varepsilon}(\nabla\varepsilon \times (\nabla \times \vec{B})) = 0 \tag{6}$$

There are two possible cases of the wave polarization: S-polarization when the electric vector is perpendicular the incident plane, e.g. in the x-direction; and P-polarization when the electric field vector is in the incident plane.

S-polarization

Only $E_x \neq 0$ and $div\bar{E} = 0$. Substituting $E_x = E_x(z)\exp(-i\omega t + ik_y y)$ into equation (5), we obtain for $E_x(z)$

$$\frac{d^2 E_x}{dz^2} + \frac{\omega^2}{c^2}(\varepsilon - \sin^2\theta)E_x = 0 \tag{7}$$

Rather far from the turning point $\varepsilon = \sin^2\theta$, solution of this equation can be obtained in approximation of geometric optics, it coincides with formula (1). The field structure is shown in Figure 2. To obtain solution near the turning point, we assume that in this region the plasma density is changing linearly with distance $n = n_0(1 + z/L)$ (here n_0 is the electron density at the point where the electron plasma frequency is equal to the wave frequency. Equation (7) has the form of the Airy equation

$$\frac{d^2 E_x}{dz^2} - \frac{\omega^2}{c^2}\frac{z'}{L}E_x = 0 \tag{8}$$

here $z' = z + L\sin^2\theta$. Solution of this equations is expressed in terms of The Airy function. Rather far from the point $z' = 0$, this solution is described by geometric optics solution

$$E_x(z) \sim \frac{1}{k_z^{1/2}} e^{i\int k_z dz}; \qquad k_z^2 = -\frac{\omega^2}{c^2}\frac{z'}{L}$$

We get this oscillation solution on the left from the point z'=0 and exponential decay takes place on the right from the turning point.

P-polarization (the wave electric field is in the incident plane)

The physical picture is more complicated. The main feature is the transformation of electromagnetic wave into plasma oscillations in the vicinity of the point z=0, where $\varepsilon = 0$. Since for P-polarization $E_z \neq 0$, this component of the wave electric field can drive the charge separation in the direction of non-homogeneity and excite plasma oscillations at this point. Such type of transformation of one type of oscillation into another one occurs in plasma when their dispersion curves intersect (see Figure 3).

Let us consider the behavior of the wave amplitude near the point z=0. z-component of the second Maxwell's equation from (4):

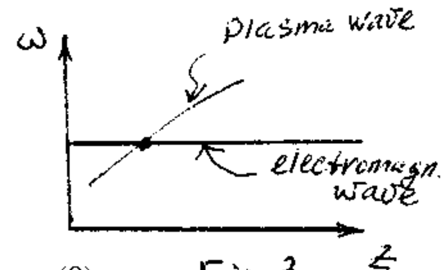


Fig. 3

$$-\frac{\partial B_x}{\partial y} = -i\frac{\omega}{c}E_z \quad \Rightarrow \quad \varepsilon E_z \approx B_x(0)\sin\theta \qquad (9)$$

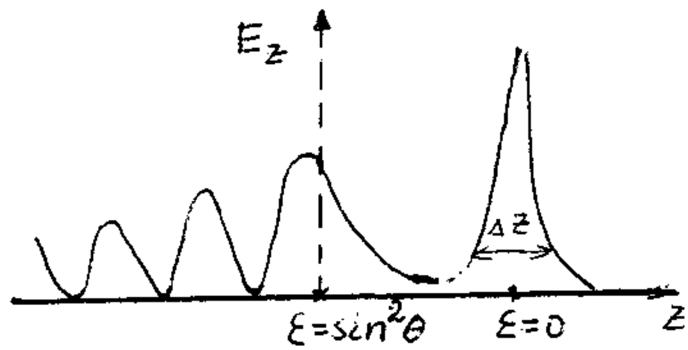
Analysis of the wave magnetic field that is parallel to the x-axis can be done using Equation (6)

$$\frac{\partial^2 B_x}{\partial z^2} + \frac{\omega^2}{c^2}(\varepsilon - \sin^2\theta)B_x - \frac{1}{\varepsilon}\frac{\partial \varepsilon}{\partial z}\frac{\partial B_x}{\partial z} = 0 \qquad (10)$$

It follows from this equation that the reflection point for the wave with P-polarization is also determined from condition $z = -L\sin^2\theta$. After the reflection point the wave magnetic field exponentially decays with the characteristic scale on the order of $\dfrac{c}{\omega\sin\theta}$.

The exact solution of the equation (10) shows that the wave magnetic field doesn't have any anomaly at the point of the plasma resonance, so the magnetic field in formula (9) can be considered as a constant. Then the electric field will have a singularity $1/\varepsilon$ at the point z=0 . (see Figure 4).

Fig. 4



The physical reason for this is the following. The layer in the vicinity of the point works like a capacitor where the wave energy is accumulated. The level of the energy goes to infinity if we neglect dissipation (collisions) and put plasma temperature to be equal zero, then $v_g = 0$, i.e. there is no outflow of the plasma wave energy out of the layer.

This effect can be sufficient only in conditions when the reflection point and the point of the plasma resonance are not very far from each other. The wave amplitude drops exponentially after the turning point with

$$\operatorname{Im} k_z \approx \frac{\omega}{c}\sin\theta \tag{11}$$

The distance between reflection point and the plasma resonance point ($z = 0$) is

$$|\Delta z| = L\sin^2\theta$$

For effective transformation of electromagnetic waves into plasma waves the condition

$$\operatorname{Im} k_z\,|\Delta z| \leq 1 \qquad \text{or} \qquad \frac{\omega L}{c}\sin\theta \leq 1 \tag{12}$$

should be fulfilled. This means that effective transformation takes place for rather small angles

$$\theta \approx \left(\frac{c}{\omega L}\right)^{1/3} \tag{13}$$

Please note that θ cannot be zero since in this case $E_z = 0$.

There is no singularity of the electric field E_z at the point z=0 if collisional dissipation and/or the outflow of plasma wave energy from the point of plasma resonance ($\nu \neq 0, T \neq 0$) are taken into account.

1. Collisional dissipation

Consider first the problem when the singularity at the resonance point is removed by collisions. Using the expression for the permittivity of plasma with collisions (see (3.20), (3.21))

$$\varepsilon = 1 - \frac{\omega_{pe}^2}{\omega(\omega + i\nu)} \approx 1 - \frac{\omega_{pe}^2}{\omega^2} + i\frac{\omega_{pe}^2 \nu}{\omega^3}, \qquad \frac{\nu}{\omega} << 1$$

we find that the amplitude of the electric field along the non-homogeneity direction is determined by the following expression:

$$E_z = B_x(0)\sin\theta \frac{1}{-\frac{z}{L} + i\frac{\nu}{\omega}} \tag{14}$$

It follows from (14) that dissipation limits the longitudinal electric field at the level

$$E_{z\,\max} \approx B_x(0)\sin\theta \frac{\omega}{\nu} \tag{15}$$

Such a value is reached in the region width

$$\Delta z \approx L\frac{\nu}{\omega} \tag{16}$$

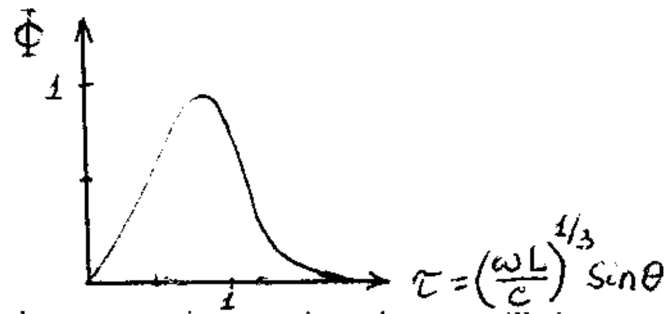
In all above formulas $B_x(0)$ is the wave magnetic field in the vicinity of the singularity.

To find it, we should solve Equation (10). The answer is:

$$B_x(0) = B_x\left(\frac{c}{\omega L}\right)^{1/2}\Phi(\tau)\frac{1}{(2\pi)^{1/2}} \tag{17}$$

here $\tau = (\omega L/c)^{1/3}\sin\theta$. The plot of Φ is shown in Figure 5. For large τ, Φ decreases exponentially.

Fig. 5



Coefficient of transformation of electromagnetic waves into plasma oscillations

Is the ratio of the total dissipation power to the energy flux in the incident wave.

The dissipated power is equal

$$P = \nu\int\frac{|E_z|^2}{8\pi}dz = \nu\frac{B_x^2}{16\pi^2}\frac{c}{\omega L}\Phi^2\int\frac{dz}{\frac{z^2}{L^2}+\frac{\nu^2}{\omega^2}}$$
$$= c\frac{B_x^2}{16\pi^2}\frac{c}{\omega L}\Phi^2\frac{\omega^2}{\nu^2}\int_{-\infty}^{\infty}\frac{dz}{1+(\frac{\omega}{\nu L}z)^2} = c\frac{B_x^2}{16\pi}\Phi^2 \tag{18}$$

The fact why P the dissipated power doesn't depend on ν, can be explained qualitatively: The power can be approximated by relation $P \approx \nu E_{\max}^2 \Delta z$. According to (15) and (16), $E_{\max} \sim 1/\nu$ and the width of the plasma resonance is on the order of $L\nu/\omega$. As a result, the power doesn't depend on ν.

The energy flux in the incident wave is defined by the Pointing vector and is equal to $c\dfrac{B_x^2}{4\pi}$. The coefficient of transformation is equal

$$K = \frac{1}{4}\Phi^2(\tau); \qquad K_{\max} \approx 0.43 \tag{19}$$

It does not depend on ν but depends on the incidence angle. It has maximal value for very small incidence angles $\theta \approx 0.5(c/\omega L)^{1/2}$.

2. Transformation into outflowing plasma wave

Another mechanism of limitation of the electric field in the resonance region is outflow of the plasma wave energy. Equation describing the transformation of the electromagnetic wave into plasma waves is the same as equation (9), where we should take into account the "sound effect"

$$\frac{\omega^2}{k^2} = \frac{\omega_{pe}^2}{\omega^2} + 3\frac{T_e}{m_e} \qquad \Rightarrow \qquad \varepsilon = 1 - \frac{\omega_{pe}^2}{\omega^2} - 3k^2\lambda_{De}^2$$

$$\varepsilon E \;\Rightarrow\; \left(1 - \frac{\omega_{pe}^2}{\omega^2}\right)E + 3\lambda_{De}^2\frac{\partial^2 E}{\partial z^2}$$

As a result we obtain the following equation instead of (9):

$$3\lambda_{De}^2\frac{\partial^2 E_z}{\partial z^2} - \frac{z}{L}E_z = B_x(0)\sin\theta \tag{21}$$

The term with second spatial derivative eliminates singularity at z=0. We can find the characteristic width of resonance by substituting $1/(\Delta z)^2$ instead of $\partial^2/\partial z^2$:

$$\Delta z \approx (L\lambda_{De}^2)^{1/3} \tag{22}$$

Maximal value of the electric field

$$E_{z\max} \approx B_x(0)\sin\theta\left(\frac{L}{\lambda_{De}}\right)^{2/3} \tag{23}$$

Compare these two competing mechanisms. Transformation into plasma waves prevails is the characteristic spatial scale of plasma oscillations is larger than the width of the resonance due to collisions $\nu < \omega(\lambda_{De}/L)^{2/3}$. In opposite case the limitation of the longitudinal electric field in the vicinity of z=0 is connected with collisional dissipation. The flow of energy in plasma wave is equal

$$v_g \frac{|E_z|^2}{4\pi} \tag{24}$$

Here $v_g = \partial\omega/\partial k$ is the group velocity of plasma oscillations that is equal $v_g = 3k_z\lambda_{De}v_{Te}$

By the order of magnitude, $k_z \approx 1/\Delta z$ therefore the energy flux, transported by plasma waves is approximately equal

$$v_g \frac{|E_z|^2}{4\pi} = \frac{1}{4\pi}\left(B_x(0)\sin\theta\right)^2\omega_{pe}L \tag{25}$$

By substituting in this relationship the value of magnetic field at the resonance point from (17), we obtain the final expression for the energy flux in the plasma wave:

$$v_g \frac{|E_z|^2}{4\pi} = c\frac{B_x^2}{4\pi}\Phi^2(\tau)\frac{1}{2\pi} \tag{26}$$

If we introduce the coefficient of transformation of electromagnetic wave into plasma wave as a ratio of nergy fluxes in plasma and incident electromagnetic waves, we can see that for angles $\theta \approx (c/\omega L)^{1/3}$ it is of the order of unity. More precise consideration based on the solution of the Airy equation (21) shows that the transformation coefficient in this case is absolutely the same as in case of dissipation of the plasma wave energy by collisions. Therefore in both cases the same part of the energy flux of the electromagnetic wave will be lost.