Lecture 6

Hydrodynamic description of plasma

Hydrodynamical model of plasma: Plasma is mixture of electron and ion fluids.

Hydrodynamical model is applicable for plasma where

$$L >> l(mfp), \tau_p >> \tau(mfp) \tag{1}$$

L, τ_p are characteristic spatial and temporal scales of plasma processes

$$\frac{1}{L} \sim \frac{1}{f}\frac{\partial f}{\partial x}, \qquad \frac{1}{\tau_p} \sim \frac{1}{f}\frac{\partial f}{\partial t}$$

l, τ are mean free path of plasma particles time between two collisions, correspondingly.

In kinetic approach each component of plasma is described by Vlasov equation with collision operator

$$\frac{df^\alpha}{dt} = \frac{\partial f^\alpha}{\partial t} + \vec{v}\frac{\partial f^\alpha}{\partial \vec{r}} + \frac{q^\alpha}{m_\alpha}(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B})\frac{\partial f^\alpha}{\partial \vec{v}} = \left(\frac{df^\alpha}{dt}\right)_{coll} \tag{2}$$

where $\vec{E}$ and $\vec{B}$ are self-consistent fields that can be found from Maxwell's equations:

$$\nabla\cdot\vec{E} = 4\pi\rho \qquad \nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

$$div\vec{B} = 0 \qquad \nabla\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\vec{j} \tag{3}$$

$$\rho = \sum_\alpha q^\alpha \int f^\alpha d\vec{v} \qquad \vec{j} = \sum_\alpha q^\alpha \int \vec{v} f^\alpha d\vec{v} \tag{4}$$

As was shown before (see lecture 5), the distribution function for each component can in this case be decomposed over small parameters l/L or τ/τ_p. In "zero" approximation

$$f^\alpha = f_0^\alpha = \left(\frac{m_\alpha}{2\pi T_\alpha}\right)^{3/2} n_\alpha \exp\left[-\frac{m_\alpha(\vec{v}-\vec{u}_\alpha)^2}{2T_\alpha}\right] \tag{5}$$

where $n = n(\vec{r},t), T = T(\vec{r},t), \vec{u} = \vec{u}(\vec{r},t)$ are density, temperature and average velocity.

By substituting into kinetic equation (2) $f = f_0 + f_1 + \ldots$, we obtain in zero approximation

$$\frac{\partial f_0^\alpha}{\partial t} + \vec{v}\frac{\partial f_0^\alpha}{\partial \vec{r}} + \frac{q^\alpha}{m_\alpha}(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B})\frac{\partial f_0^\alpha}{\partial \vec{v}} = \left(\frac{df^\alpha}{dt}\right)_{coll} \qquad (6)$$

In the r.h. side, we wrote the collision operator for total distribution function f because $\left(df_0/dt\right)_{coll}$=0. We would like to note that $\left(df/dt\right)_{coll}$ is not an additive function, and therefore it cannot be written in the form $\left(df/dt\right)_{coll} = \left(df_0/dt\right)_{coll} + \left(df_1/dt\right)_{coll} + ...$

In hydrodinamical model, total information about features of any component of plasma is contained in three quantities: density, temperature, and average velocity which are unknown functions of coordinates and time. Our goal is to derive equations for these quantities using equation (6) or (2).

We will use more general equation (2) to obtain equations for $n, u,$ and T which are just zero, first, and second moments of the distribution function $f(t,\vec{r},\vec{v})$. Let us introduce:

$$n_\alpha(t,\vec{r}) = \int f^\alpha(t,\vec{r},\vec{v})d\vec{v}$$

$$\vec{u}_\alpha(t,\vec{r}) = \frac{1}{n_\alpha}\int \vec{v} f^\alpha(t,\vec{r},\vec{v})d\vec{v} \qquad (7)$$

$$P_{ij}^\alpha(t,\vec{r}) = \int (v_i - u_i)(v_j - u_j) f^\alpha(t,\vec{r},\vec{v})d\vec{v}$$

that are correspondingly the density, velocity and pressure tensor of α-component of plasma. We will use only the diagonal form of the pressure tensor: $P_{ij}^\alpha(t,\vec{r}) = P^\alpha \delta_{ij}$, $P^\alpha = n_\alpha T_\alpha$.

We will not simplify the collision operator in Eq.(2). We will see later on, we don't need to know the exact form of it. It is possible to use some its general features – conservation laws. We wil not take into account the recombination and ionization processes that change the number of charged particles.

To obtain first equation of the hydrodynamical model, let us integrate (2) over total velocity space. Integrals of first two terms of (2) can be written in the following forms:

$$\int \frac{\partial f^\alpha}{\partial t} d\vec{v} = \frac{\partial}{\partial t}\int f^\alpha d\vec{v} = \frac{\partial n^\alpha}{\partial t}$$

$$\int \vec{v}\frac{\partial f^{\alpha}}{\partial \vec{r}}d\vec{v} = \frac{\partial}{\partial \vec{r}}\int \vec{v} f^{\alpha} d\vec{v} = div(n_{\alpha}\vec{u}_{\alpha}) \tag{8}$$

In (8) we changed the order of differentiation over t and $\vec{r}$ correspondingly and integration over $\vec{v}$, since t, $\vec{r}$, $\vec{v}$ are independent variables in kinetic theory. By integrating the third term in (2) by parts, we obtain zero

$$\int (\vec{E} + \frac{1}{c}\vec{v}\times\vec{B})\frac{\partial f^{\alpha}}{\partial \vec{v}}d\vec{v} = 0 \tag{9}$$

In absence of ionization and recombination, the total number of particles conserves, hence integral of collision operator is equal zero.

As a result, we obtain the continuity equation:

$$\frac{\partial n_{\alpha}}{\partial t} + div(n_{\alpha}\vec{u}_{\alpha}) = 0 \tag{10}$$

By multiplying Eq. (2) by $m_{\alpha}\vec{v}$ and integrating over $\vec{v}$, we obtain

$$m_{\alpha}\frac{\partial}{\partial t}(n_{\alpha}u_{i}^{\alpha}) = \frac{\partial}{\partial x_{j}}\left(m_{\alpha}\int v_{i}v_{j}f^{\alpha}d\vec{v}\right) = q_{\alpha}\left(\vec{E} + \frac{1}{c}\vec{u}_{\alpha}\times\vec{B}\right)_{i} n_{\alpha} + \left(\frac{dp_{i}^{\alpha}}{dt}\right)_{coll} \tag{11}$$

Last term in r.h. side of (11) is change the α -component's momentum due to collisions with another component of plasma.

If the average velocities for both plasma components are equal, there is no momentum transfer from one component to another and $\left(dp_{i}^{\alpha}/dt\right)_{coll} = 0$.

Since

$$m_{\alpha}\int v_{i}v_{j}f^{\alpha}(t,\vec{r},\vec{v})d\vec{v} = P_{ij}^{\alpha}(t,\vec{r}) + m_{\alpha}n_{\alpha}u_{i}^{\alpha}u_{j}^{\alpha},$$

then using the continuity equation (10), we obtain from (11) the following equation:

$$m_{\alpha}n_{\alpha}\left(\frac{\partial}{\partial t} + (\vec{u}^{\alpha}\cdot\nabla)\right)u_{i}^{\alpha} = q_{\alpha}n_{\alpha}\left(\vec{E} + \frac{1}{c}\vec{u}_{\alpha}\times\vec{B}\right)_{i} + \frac{\partial P_{\alpha}}{\partial x_{j}} + \left(\vec{F}^{\alpha\beta}\right) \tag{12}$$

Here $P_{\alpha} = n_{\alpha}T_{\alpha}$ is gas kinetic pressure of α -component, $\vec{F}^{\alpha\beta}$ is the force acting on particles of α -component in unit volume due to collisions with another component (friction force).

Equation (12) is the Euler equation for charged fluid. We can see that change of zero moment n in Eq. (10) is expressed through the first moment $\vec{u}$; change of first moment $\vec{u}$ in equation (12) is expressed through the second moment P. Then the change of the pressure P (the second moment) will be expressed through the energy flux (the third moment) and so on. To obtain the closed system of equations, we truncate the chain by putting certain high-order moment equal to zero. Conventional approximation is to put third moment of the distribution function equal to zero.

We will obtain the equation for the second moment of the distribution function in 1-d case assuming that

$$\int (v-u_\alpha)^3 f^\alpha dv = 0 \tag{13}$$

Multiplying equation (2) by $m_\alpha (v-u_\alpha)^2$ and integrating over velocity we obtain

$$\frac{\partial}{\partial t}(n_\alpha T_\alpha) + m_\alpha \int v(v-u_\alpha)^2 \frac{\partial f^\alpha}{\partial x} dv = m_\alpha \int (v-u_\alpha)^2 \left(\frac{df^\alpha}{dt}\right)_{coll} \tag{14}$$

It $u_e = u_i$ and $T_e = T_i$, the r.h. side of (14) is equal to zero. If $T_e \neq T_i$, there is the thermal energy exchange between plasma components and r.h. side of (14) is not zero in this case. We assume here that $T_e = T_i$. Let us calculate second term in (14):

$$m_\alpha \int v(v-u_\alpha)^2 \frac{\partial f^\alpha}{\partial x} dv = m_\alpha \frac{\partial}{\partial x}\left(\int v(v-u_\alpha)^2 f^\alpha dv\right) + 2\frac{\partial u_\alpha}{\partial x}\int v(v-u_\alpha) f^\alpha dv$$

$$= m_\alpha \frac{\partial}{\partial x}\left(\int (v-u_\alpha)^3 f^\alpha dv\right) + \frac{\partial}{\partial x}(u_\alpha n_\alpha T_\alpha) + 2\frac{\partial u_\alpha}{\partial x} n_\alpha T_\alpha$$

As a result, we obtain from (14) equation for the temperature

$$\frac{\partial T_\alpha}{\partial t} + u_\alpha \frac{\partial T_\alpha}{\partial x} + 2T_\alpha \frac{\partial u_\alpha}{\partial x} = 0 \tag{15}$$

Equation (15) is equation of 1-d adiabatic law ($\gamma = 3$). It is easy to show this, if we write the continuity equation (10) in the form

$$\frac{dn}{dt} \equiv \frac{\partial n}{\partial t} + u\frac{\partial n}{\partial x} = -n\frac{\partial u}{\partial x} \tag{16}$$

Then it follows from (15), (16)

$$\frac{1}{T}\frac{dT}{dt} = -2\frac{\partial u}{\partial x} = 2\frac{1}{n}\frac{dn}{dt} \qquad (17)$$

or $T \sim n^2$ and $P = nT = An^3; A = const$.

The reason we got adiabatic law is obvious – there is no heat flux, no energy input from collisions.

The system of equations of two-fluid hydrodynamics:

$$\frac{\partial n_i}{\partial t} + div(n_i \vec{u}_i) = 0$$

$$\frac{\partial n_e}{\partial t} + div(n_e \vec{u}_e) = 0 \qquad (18)$$

$$m_i n_i \frac{d\vec{u}_i}{dt} = -\nabla(n_i T_i) + e n_i \left(\vec{E} + \frac{1}{c}\vec{u}_i \times \vec{B} \right) + \vec{F}_{ie}$$

$$m_e n_e \frac{d\vec{u}_e}{dt} = -\nabla(n_e T_e) - e n_e \left(\vec{E} + \frac{1}{c}\vec{u}_e \times \vec{B} \right) + \vec{F}_{ei} \qquad (19)$$

The friction force is equal to momentum transferred from particles of one plasma component to those of another one in unit volume per one second. In τ-approximation

$$\vec{F}_{ei} = -\vec{F}_{ie} = -nm_e(\vec{u}_e - \vec{u}_i)\nu_{ei} \qquad (20)$$

Equation (15) in 3-d case has the form

$$\frac{\partial T_\alpha}{\partial t} + (\vec{u}_\alpha \cdot \nabla)T_\alpha + \frac{2}{3}T_\alpha div u_\alpha = 0 \qquad (21)$$

$\vec{E}$ and $\vec{B}$ in (19) are external and selfconsistent fields. Selfconsistent fields are determined from Maxwell's equations:

$$\nabla \cdot \vec{E} = 4\pi \sum_\alpha q_\alpha n_\alpha \qquad \nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$div\vec{B} = 0 \qquad \nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\sum_\alpha q_\alpha n_\alpha \vec{u}_\alpha \qquad (22)$$

It is important to note that conventional hydrodynamic model is applicable to phenomena with characteristic scales much larger than mean free path. There are so called microscopic processes with characteristic scales much smaller than m.f.p, such as oscillations in plasma, electromagnetic wave propagation, Landau damping. These microscopic processes constitute a special part of plasma physics – collisionless plasma physics that is described by the system of collisionless kinetic equations. Kinetic description is often much more complicated than hydrodynamic description.

It is possible to introduce hydrodynamic model for description of collisionless plasma. The physical reason for possibility of description of plasma by this "collisionless" hydrodynamic model is is absolutely different from that in case of conventional hydrodynamics.

Collisionless hydrodynamics is usually used for description of wave processes when there is additional temporal scale – the oscillation frequency. In collisionless case transition to hydrodynamics and disregard of thermal motion are possible if

$$\frac{v_T}{\omega} << \lambda \qquad \text{or} \qquad v_T << \frac{\omega}{k} \tag{23}$$

Here λ is the wavelength.

Hence, collisionless hydrodynamics is applicable when thermal velocity is much smaller than characteristic velocity of wave motions. The system of equations in this case consists of equations (18), (19) in which the friction force is omitted. Equation of state of plasma components are often used instead of equation (21).