Lecture 9 Kinetic theory of waves in plasma (Part II)

Let us obtain the Landau damping decrement by Conservation of energy and Landau damping

$$\frac{\partial \vec{E}}{\partial t} + 4\pi \vec{j} = 0$$

$$\frac{1}{8\pi}\frac{\partial \langle E^2 \rangle}{\partial t} = -\langle \vec{j}\cdot\vec{E} \rangle \qquad (1)$$

$\langle \ \rangle$ is averaging over period of oscillations

R.h. side in (1) is the work in unit time performed by plasma particles on wave fields.

Consider high frequency electrostatic plasma waves

$$\varphi = \frac{1}{2}\varphi e^{i\vec{k}\vec{r} - i\omega t} + c.c. \qquad (2)$$

$$\vec{E} = \frac{1}{2}\left(-i\vec{k}\varphi e^{i\vec{k}\vec{r} - i\omega t} + c.c.\right)$$

$$\vec{j} = -e\int \vec{v} f\, d\vec{v} = -e\int \vec{v}\,\delta f\, d\vec{v}$$

$$\delta f = -\frac{1}{2}\frac{e}{m}\frac{\varphi}{k v_\parallel - \omega}\vec{k}\frac{\partial f_0}{\partial \vec{v}} e^{i\vec{k}\vec{r} - i\omega t} + c.c.$$

$$\langle \vec{j}\cdot E \rangle = \frac{1}{4}\frac{e^2}{m}\left\langle \left[\int \frac{k v_\parallel}{k v_\parallel - \omega}\frac{\partial F_0}{\partial v_\parallel} dv_\parallel\, \varphi e^{i\vec{k}\vec{r} - i\omega t} + c.c.\right] * \left[-ik\varphi e^{i\vec{k}\vec{r} - i\omega t} + c.c.\right]\right\rangle$$

$$= \frac{1}{4}\frac{e^2}{m}\left\langle \left[\underset{\text{nonres.}}{P.V.\int_{-\infty}^{\infty}} \frac{kv_\parallel}{kv_\parallel-\omega}\frac{\partial F_0}{\partial v_\parallel} dv_\parallel \, \varphi e^{i\vec{k}\vec{r}-i\omega t} + \right. \right.$$

$$\left. + i\pi \frac{\omega}{k}\frac{\partial F_0}{\partial v_\parallel}\left(\frac{\omega}{k}\right)\varphi e^{i\vec{k}\vec{r}-i\omega t} + c.c. \right]$$

$$\left. * \left[-ik\varphi e^{i\vec{k}\vec{r}-i\omega t} + c.c. \right] \right\rangle ;$$

$$\frac{1}{8\pi}\frac{\partial \langle E^2 \rangle}{\partial t} \equiv \frac{k^2}{16\pi} 2\gamma |\varphi|^2$$

$$\underline{\omega = \omega_R + i\gamma}$$

$$\frac{i}{kv_\parallel - \omega_R - i\gamma} = \frac{i(kv_\parallel - \omega_R) - \gamma}{(kv_\parallel-\omega_R)^2 + \gamma^2}$$

Eq-n (1) can be written in the form:

$$\frac{k^2}{8\pi}\gamma|\varphi|^2 = \frac{e^2}{2m}\gamma|\varphi|^2 \underset{\text{non-res.}}{P.V.\int_{-\infty}^{\infty}} \frac{k^2 v_\parallel}{(kv_\parallel-\omega)^2+\gamma^2}\frac{\partial F_0}{\partial v_\parallel} dv_\parallel$$

$$\underbrace{+\frac{\pi}{2}\frac{e^2}{m}|\varphi|^2 \omega \frac{\partial F_0}{\partial v_\parallel}\left(\frac{\omega}{k}\right)}_{= \langle \vec{j}_{res.} \cdot \vec{E} \rangle}$$

$\vec{j}_{res}$ – current density of resonant particles.

$$\frac{1}{8\pi}\gamma|\varphi|^2\kappa^2\underbrace{\left[1-\frac{4\pi e^2}{m\kappa^2}P.V\int_{-\infty}^{\infty}\frac{\kappa^2 v_\parallel}{(\kappa v_\parallel-\omega)^2}\frac{\partial F_0}{\partial v_\parallel}dv_\parallel\right]}_{\frac{\partial}{\partial\omega}(\omega\varepsilon_R)} \tag{3}$$

$$= \frac{\pi}{2}\frac{e^2}{m}|\varphi|^2\omega\frac{\partial F_0}{\partial v_\parallel}\left(\frac{\omega}{\kappa}\right)$$

$$\varepsilon(\kappa,\omega)=\underbrace{\left(1-\frac{4\pi e^2}{m\kappa^2}P.V.\int_{-\infty}^{\infty}dv_\parallel\frac{\kappa}{\kappa v_\parallel-\omega}\frac{\partial F_0}{\partial v_\parallel}\right)}_{\varepsilon_R}$$

$$-i\underbrace{\frac{4\pi^2e^2}{m\kappa^2}\frac{\partial F_0}{\partial v_\parallel}\left(\frac{\omega}{\kappa}\right)}_{\varepsilon_I}$$

$$\frac{\partial}{\partial\omega}\left[\omega\varepsilon_R\right]=1-\frac{4\pi e^2}{m\kappa^2}P.V.\int dv_\parallel\,\kappa\frac{\partial F_0}{\partial v_\parallel}\frac{\partial}{\partial\omega}\left(\frac{\omega}{\kappa v_\parallel-\omega}\right)=$$

$$\frac{\partial}{\partial\omega}\left[\frac{\omega}{\kappa v_\parallel-\omega}\right]=\frac{\kappa v_\parallel}{(\kappa v_\parallel-\omega)^2}, \tag{4}$$

From (3), (4)

$$\gamma\frac{1}{8\pi}\kappa^2|\varphi|^2\frac{\partial}{\partial\omega}(\omega\varepsilon_R)=\frac{\pi}{2}\frac{e^2}{m}|\varphi|^2\omega\frac{\partial F_0}{\partial v_\parallel}\left(\frac{\omega}{\kappa}\right) \tag{5}$$

$$\gamma=\pi\frac{\omega_{pe}^2}{\kappa^2}\frac{1}{n_0}\frac{\omega}{\frac{\partial}{\partial\omega}(\omega\varepsilon_R)}\frac{\partial F_0}{\partial v_\parallel}\left(\frac{\omega}{\kappa}\right) \tag{6}$$

In case of electron plasma oscillations

$$\mathcal{E}_R = 1 - \frac{\omega_{pe}^2}{\omega^2} \; ; \quad \omega \approx \omega_{pe}$$

$$\frac{\partial}{\partial\omega}(\omega\mathcal{E}_R) = \frac{\partial}{\partial\omega}\left(\omega - \frac{\omega_{pe}^2}{\omega}\right) \approx 2$$

From (6):

$$\gamma_L = \frac{\pi}{2}\omega_{pe}\frac{\omega_{pe}^2}{k^2}\frac{1}{n_0}\frac{\partial F_0}{\partial v_\parallel}\left(\frac{\omega}{k}\right) \qquad (7)$$

The same as (8.31)

Equation (5) can be written in the form

$$\frac{d}{dt}\bar{W}_k = 2\gamma_L|E|^2 \qquad (8)$$

where $\bar{W}_k = \frac{|E|^2}{8\pi}\frac{\partial}{\partial\omega}(\omega\mathcal{E}_R)$ – (9)

– is the density of the wave energy.

It consists of potential energy of the wave $\frac{|E|^2}{8\pi}$ and kinetic energy of non-resonant particles.

Plasma dispersion function

$$\varepsilon(k,\omega) = 1 - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2} \frac{1}{n_{o\alpha}} \int \frac{\frac{\partial F_{o\alpha}}{\partial v_{\parallel}}}{v_{\parallel} - \omega/k} \, dv_{\parallel} \quad (10)$$

For Maxwellian distribution

$$F_{o\alpha}(v_{\parallel}) = \frac{n_{o\alpha}}{(2\pi)^{1/2} v_{T\alpha}} e^{-\frac{m v_{\parallel}^2}{2T_{\alpha}}} \quad (11)$$

$$\varepsilon(k,\omega) = 1 + \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2 v_{T\alpha}^2} \frac{1}{(2\pi)^{1/2} v_{T\alpha}} \int \frac{v_{\parallel} - \omega/k + \omega/k}{v_{\parallel} - \omega/k} e^{-\frac{v_{\parallel}^2}{2v_{T\alpha}^2}} dv_{\parallel}$$

$$= 1 + \sum_{\alpha} \frac{1}{k^2 \lambda_{D\alpha}^2} \left[1 + \frac{\omega}{(2\pi)^{1/2} k v_{T\alpha}} \int \frac{e^{-\frac{v_{\parallel}^2}{2v_{T\alpha}^2}}}{v_{\parallel} - \omega/k} dv_{\parallel} \right]$$

$$t = \frac{v_{\parallel}}{\sqrt{2}\, v_{T\alpha}}$$

$$\varepsilon(k,\omega) = 1 + \sum_{\alpha} \frac{1}{k^2 \lambda_{D\alpha}^2} \left[1 + \zeta_{\alpha} Z(\zeta_{\alpha}) \right]; \quad (12)$$

Where $\zeta_{\alpha} = \dfrac{\omega}{\sqrt{2}\, k v_{T\alpha}}$;

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int \frac{e^{-t^2}}{t - \zeta} dt \quad - \quad (13)$$

- plasma dispersion function.

Power series (small argument)

$$Z(\zeta) = i\pi^{1/2} e^{-\zeta^2} - 2\zeta\left(1 - \frac{2\zeta^2}{3} + \frac{4\zeta^4}{15} \cdots\right) \quad (14)$$

Asymptotic series, $|\zeta| \gg 1$

$$Z(\zeta) = i\pi^{1/2} e^{-\zeta^2} - \frac{1}{\zeta}\left(1 + \frac{1}{2\zeta^2} + \frac{3}{4\zeta^4} + \cdots\right) \quad (15)$$

Damping in space versus damping in time

In time:

Solve equation $\varepsilon(k, \omega) = 0$ (16)

Real k specified

Solve for complex ω

In space:

Solve equation $\varepsilon(k, \omega) = 0$ (17)

Real ω specified

Solve for complex k

Relation between temporal and spatial damping rates

(for weak damping)

$$\mathcal{E}(k,\omega) = \mathcal{E}_R + i\,\mathcal{E}_I(k,\omega) \tag{18}$$

Temporal damping

We are looking for a root of $\mathcal{E}(k,\omega)=0$ in the form

$$\omega = \omega_R + i\gamma, \quad |\gamma|/\omega \ll 1 \tag{19}$$

$$\cancel{0} = \mathcal{E}(k, \omega_R + i\gamma) \approx \mathcal{E}_R(k,\omega_R) + i\gamma \frac{\partial \mathcal{E}_R}{\partial \omega_R} + i\,\mathcal{E}_I(k,\omega_R)$$

$$\mathcal{E}_R(k,\omega_R) = 0$$

$$\gamma = -\frac{\mathcal{E}_I(k,\omega_R)}{\partial \mathcal{E}_R/\partial\omega\big|_{\omega=\omega_R}} \tag{20}$$

Spatial damping

We are looking for a root of $\mathcal{E}(k,\omega)=0$ in the form

$$k = k_R - i\Gamma, \quad |\Gamma|/k_R \ll 1 \tag{21}$$

$$0 = \mathcal{E}(k - i\Gamma, \omega) \approx \mathcal{E}_R(k_R, \omega) - \left.\frac{\partial \mathcal{E}_R}{\partial k}\right|_{k=k_R} i\Gamma +$$

$$+ i\,\mathcal{E}_I(k_R, \omega)$$

$$\Gamma = \frac{\mathcal{E}_I(k_R, \omega)}{\left.\frac{\partial \mathcal{E}_R}{\partial k}\right|_{k=k_R}} \tag{22}$$

$$\Gamma = -\gamma\, \frac{\frac{\partial \mathcal{E}_R}{\partial \omega}}{\frac{\partial \mathcal{E}_R}{\partial k}}$$

Group velocity:

$$\frac{d}{dk_R}\Big|\, \mathcal{E}_R(k_R, \omega_R) = 0$$

$$\frac{\partial \mathcal{E}}{\partial k} + \frac{\partial \mathcal{E}}{\partial \omega}\frac{\partial \omega}{\partial k} = 0$$

$$v_g \equiv \frac{\partial \omega}{\partial k} = -\frac{\partial \mathcal{E}/\partial k}{\partial \mathcal{E}/\partial \omega} \tag{23}$$

group velocity

$$\Gamma = \frac{\gamma}{v_g} \tag{24}$$