Today’s Lecture

Lecture 16: **Course Review;**
Kinematics,
Newton’s Laws of Motion,
Conservation of Energy

Chapter 11, Rockets

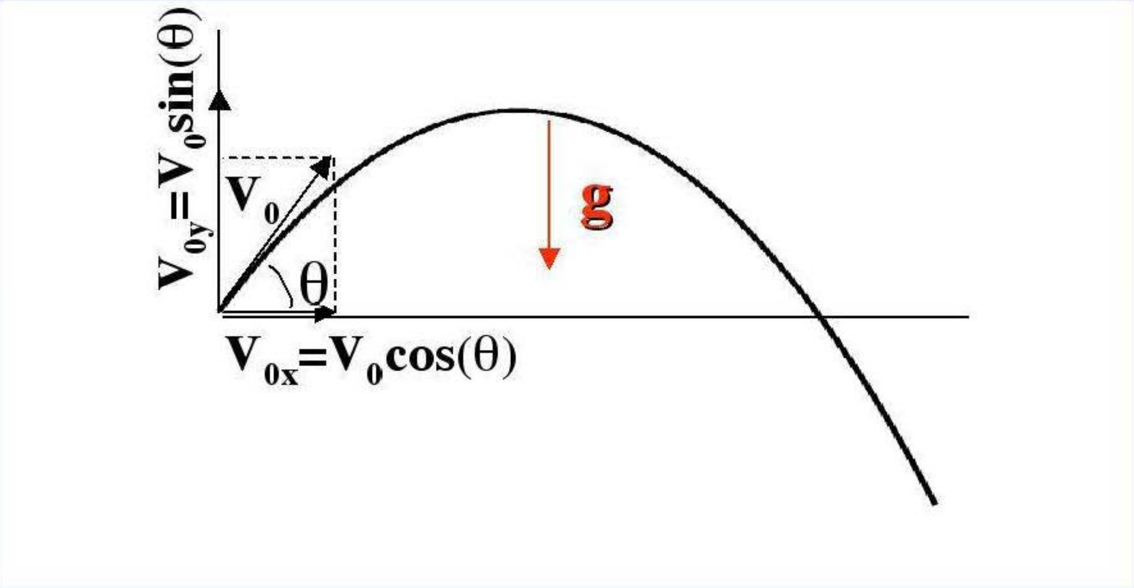
Review
Kinematics

Projectile Trajectory is Parabolic

We can show this by eliminating time from the problem and deriving the function y(x). First, solve for the time from the position equation in x

$$x = V_0 \cos(\theta) t$$

$$t = \frac{x}{V_0 \cos(\theta)}$$



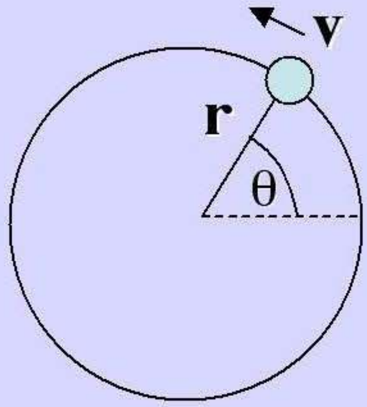
Then substitute in for t in position equation for y

$$y = V_0 \sin(\theta)\left(\frac{x}{V_0 \cos(\theta)}\right) - \frac{1}{2} g \left(\frac{x}{V_0 \cos(\theta)}\right)^2$$

$$\boxed{y = \tan(\theta) x - \frac{g}{2V_0^2 \cos^2(\theta)} x^2}$$

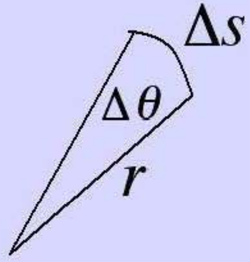
Trajectory is always parabolic in x

Uniform Circular Motion



A body moving in a circle of radius r
with uniform speed v

In a time interval Δt the arc length traversed is



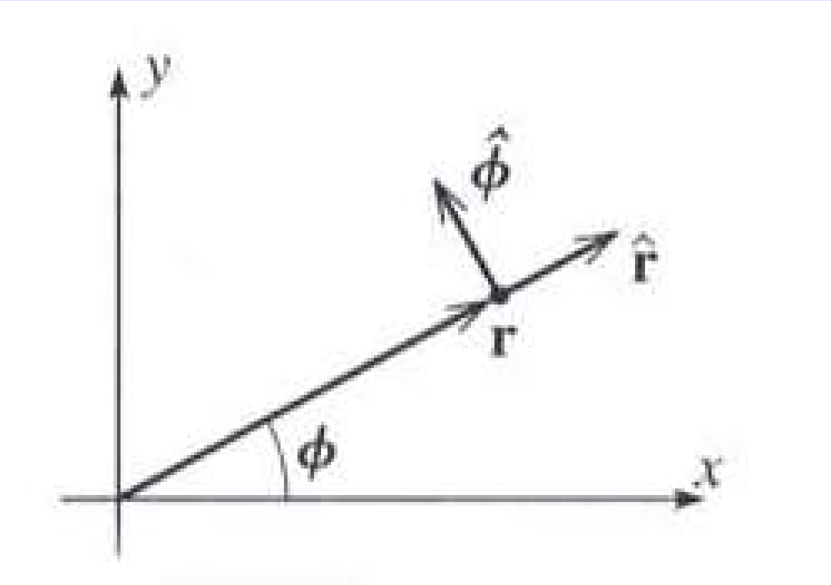
$\Delta s = r\Delta\theta$ θ in radians $0 \rightarrow 2\pi$

The limit as that time interval goes to zero is v

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = r\frac{d\theta}{dt}$$

$\frac{d\theta}{dt}$ is called the angular velocity ω where $v = r\omega$

Circular Motion and Acceleration



Consider the two unit vectors that are convenient for angular motion:

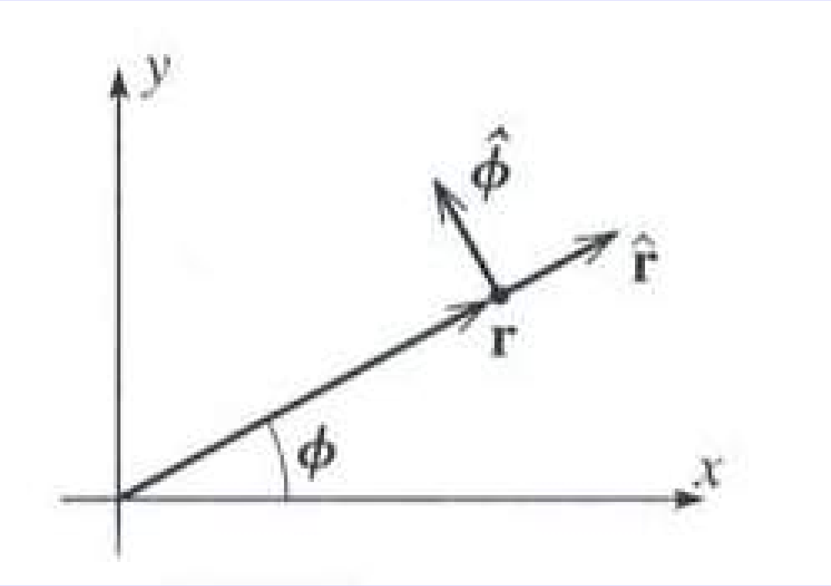
$$\hat{r} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$
$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

The position vector can now be written

$$\vec{r} = r\hat{r} = r\left(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}\right) = x\hat{i} + y\hat{j}$$

The velocity vector is simply the time derivative of the position vector. For circular motion the radius of curvature, ***r***, is constant. However, $\boldsymbol{\phi = \phi(t)}$, is time dependent. This means to find the velocity vector we must be careful when taking the derivative of the position vector – chain rule!

Circular Motion and Acceleration



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$$\hat{r} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$
$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

The velocity vector can now be expressed:

$$\vec{v} = \frac{d\vec{r}}{dt} = r\left(\frac{d\cos\phi}{dt}\hat{i} + \frac{d\sin\phi}{dt}\hat{j}\right)$$
$$\vec{v} = r\left(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}\right)\frac{d\phi}{dt} = \omega r\hat{\phi} = v\hat{\phi}$$

Taking the derivative of the velocity yields the acceleration. Assuming a **constant rate of rotation,** ω:

$$\vec{a} = \frac{d\vec{v}}{dt} = \omega r\left(-\frac{d\sin\phi}{dt}\hat{i} + \frac{d\cos\phi}{dt}\hat{j}\right)$$
$$\vec{a} = -\omega r\left(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}\right)\frac{d\phi}{dt} = -\omega^2 r\hat{r} = -\frac{v^2}{r}\hat{r}$$

The radial acceleration points radially **inward**! What happens when there is angular acceleration? Stay tuned!

Review
Dynamics

Kinematics vs. Dynamics

Kinematics are a description of motion

- What we have studied so far
- Displacement, Velocity, Acceleration

Dynamics are a description of the cause of change in motion

- Require New Concepts: **Force** and **Mass**
- Newton's Three Laws of Motion
  - Obtained from first principles
  - Fundamental
  - Based on experimental observations of motion
  - Not a derivation

Newton's Laws of Motion

1. **The Law of Inertia:** The velocity of a body will not change unless a net force acts on that body.

2. **The Law of Force, Mass and Acceleration:** The force on a body is equal to its change in momentum with time. For constant mass, this is the mass times the acceleration.

3. **Law of Action and Reaction:** For each force acting on body B from body A, there is an equal and opposite force acting on body A from body B.

Newton's Second Law

$\sum \vec{F}$ is the net force

This is the vector sum of all the forces acting on the object

Newton's Second Law, ***F=ma***, can be expressed in terms of components:

$\Sigma F_x = m\, a_x$

$\Sigma F_y = m\, a_y$

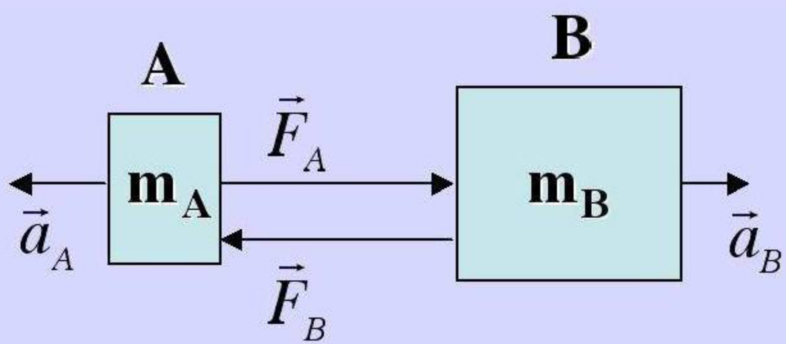
$\Sigma F_z = m\, a_z$

Most philosopher's of science consider Newton's 2nd to be the definition of a force.

Newton's Third Law of Motion

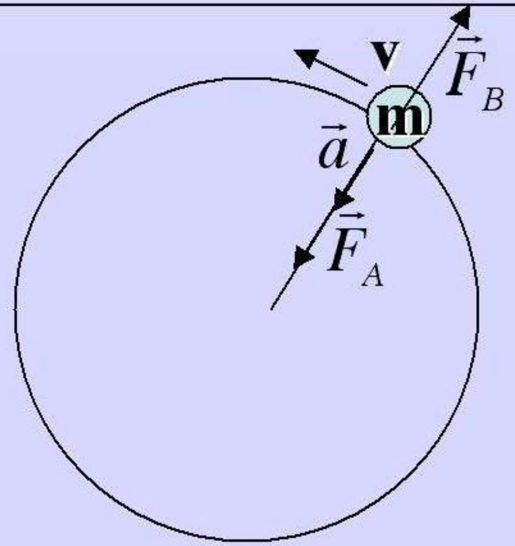
Law of Action and Reaction: **For each force acting on body B from body A, there is an equal and opposite force acting on body A from body B.**

$$\vec{F}_A = -\vec{F}_B$$



$$\vec{F} = m\vec{a}$$

Both bodies move, the lighter mass accelerates faster.



Uniform circular motion F_B is outward on the string.

Review
Conservation of Energy
and the
Work-KE Theorem

Conservation of Energy

We previously (Chapter 7) described the change in the kinetic energy as being equal to the work done on an object.

$$\Delta K = W = m\int d\vec{v}\cdot\frac{d\vec{r}}{dt} = m\int d\vec{v}\cdot\vec{v} = m[\frac{1}{2}v_f^2 - \frac{1}{2}v_i^2]$$

If we consider separately the work done by conservative and nonconservative forces: $\Delta K = W_c + W_{nc}$

We have defined the Potential Energy as the negative of the work done by Conservative forces: $\Delta K = -\Delta U + W_{nc}$

Thus:

$$\boxed{\Delta K + \Delta U = W_{nc}}$$

When $W_{nc} = 0$ **the total energy is conserved, and:**

$K + U = K_0 + U_0$ **total energy is always equal to initial energy**

Example: Incline Plane with Friction

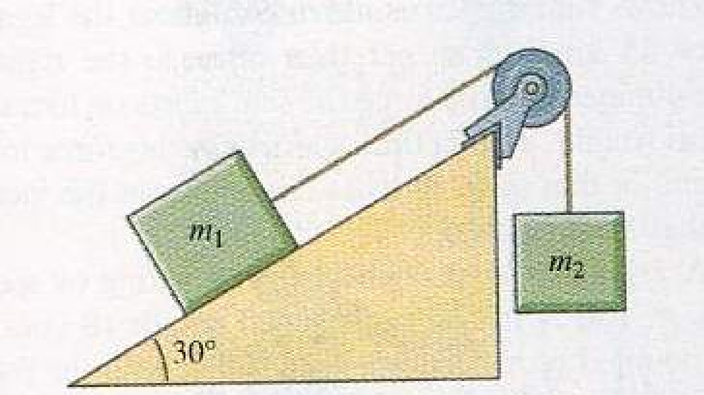


FIGURE 6-76 Problems 67, 68.

Consider the incline plane in the figure with a coefficient of kinetic friction, μ_k. Using the conservation of energy determine (a) the value of m_2 necessary for it to sink and (b) its velocity after it drops Δy when starting from rest.

From the work-energy theorem:

$$\Delta K + \Delta U = W_{nc}$$

After m_2 drops a distance Δy the work-energy theorem for this system is expressed:

$$\frac{1}{2}m_1v^2 + \frac{1}{2}m_2v^2 + m_1g\Delta y\sin\theta - m_2g\Delta y = -\mu_k m_1 g\cos\theta\Delta y$$

Solving for v^2 yields:

$$v^2 = (m_2 - m_1\sin\theta - \mu_k m_1\cos\theta)2g\Delta y/(m_1 + m_2)$$

Since v^2 must be greater than zero the criteria for m_2 to sink is:

$$m_2 > m_1\sin\theta + \mu_k m_1\cos\theta$$

Example: Incline Plane with Friction

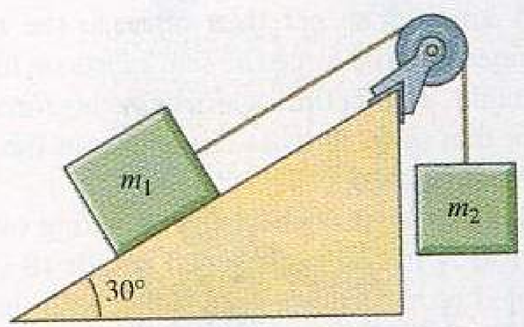


FIGURE 6-76 Problems 67, 68.

Consider the incline plane in the figure with a coefficient of kinetic friction, $\mu_{\mathbf{k}}$. Using the conservation of energy determine (a) its velocity after it drops $\boldsymbol{\Delta y}$ when starting from rest and (b) the value of $\boldsymbol{m_2}$ necessary for it to sink.

Solving for ***v*** yields:

$$v = \sqrt{(m_2 - m_1 \sin\theta - \mu_k m_1 \cos\theta)\frac{2g\Delta y}{m_1 + m_2}}$$

These results also arise from Newton's 2nd:

$$m_2 g - m_1 g \sin\theta - \mu_k m_1 g \cos\theta = (m_1 + m_2)a$$

And:

$$v = \sqrt{2a\Delta y} = \sqrt{(m_2 - m_1 \sin\theta - \mu_k m_1 \cos\theta)\frac{2g\Delta y}{m_1 + m_2}}$$

Force and Potential Energy

Consider a force pushing a body along the x axis. The work being done by the force is:

$$W = -\Delta U$$

We also know that

$$W = F_x \Delta x$$

Combining these two we can write:

$$F_x = -\frac{\Delta U}{\Delta x}$$

This applies to 3D motion in general:

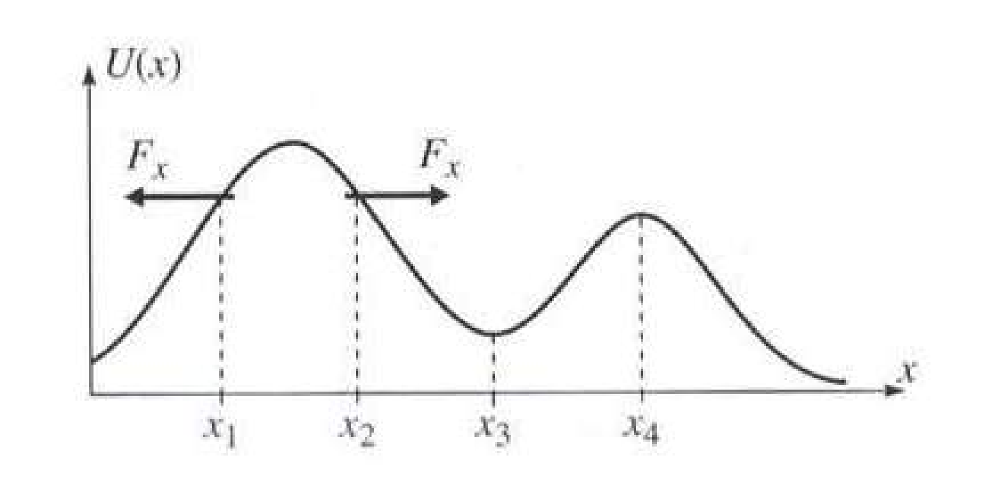
$$\vec{F} = -\left(\frac{dU}{dx}\hat{i} + \frac{dU}{dy}\hat{j} + \frac{dU}{dz}\hat{k}\right)$$

Or, in gradient notation:

$$\vec{F} = -\vec{\nabla} U$$

Here, $\vec{\nabla}$ is a vector differential operator.

Force and Potential Energy



Again think of the potential energy plot as a picture of a roller coaster. The force

$$F_x = -\frac{dU}{dx}$$

tends to push the object downhill as shown in the plot at $\boldsymbol{x=x_1}$ and $\boldsymbol{x=x_2}$.

Note that at the points x_3 and x_4 where $dU/dx = 0$, U is a minimum or a maximum. The object is in equilibrium as the net force vanishes at those points. However x_3 is a point of stable equilibrium (why?) and x_4 is a point of unstable equilibrium (why?).

For example consider the potential energy for a spring:

$$U(x) = \frac{1}{2}kx^2 \;\rightarrow\; F = -\frac{dU}{dx} = -kx$$

Power is the Rate at Which Energy is Changed, or Work is Done

The amount of work done itself does not depend on how long it takes to do the work. The rate of change in Energy is the Power that is being imparted. The average power can be written:

$$P = \frac{\Delta W}{\Delta t}$$ **While in the differential limit:** $$P = \frac{dW}{dt}$$

We can express the power as a function of Force and Velocity by differentiating the expression for the work done

$$P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v}$$

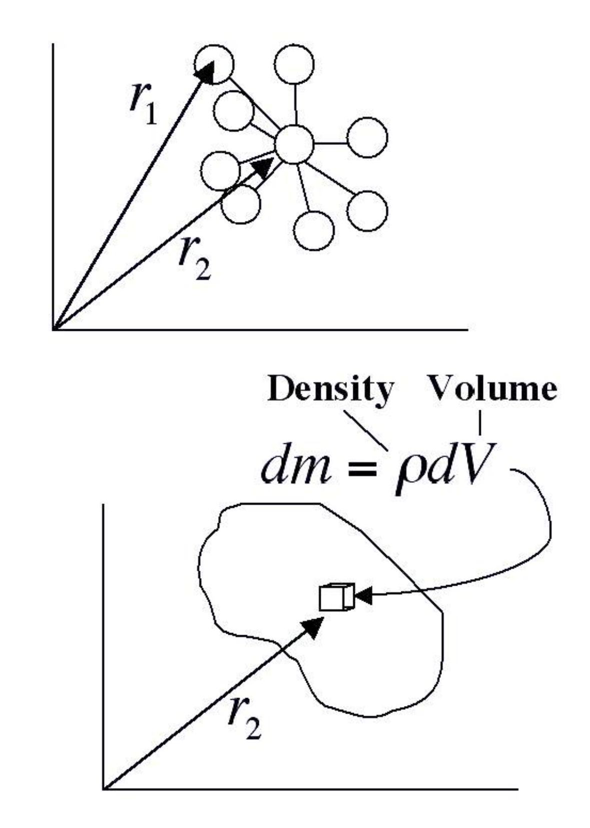
The SI units of Power are J/s=W called Watts.

Center of Mass Integrations
and
Rockets

Center of Mass

Mathematically we define the center of mass as the average of the mass weighted vector displacement of the individual particles. Defining the total mass as ***M***.

$$M = m_1 + m_2 + m_3 + \cdots = \sum_{i=1}^{N} m_i$$



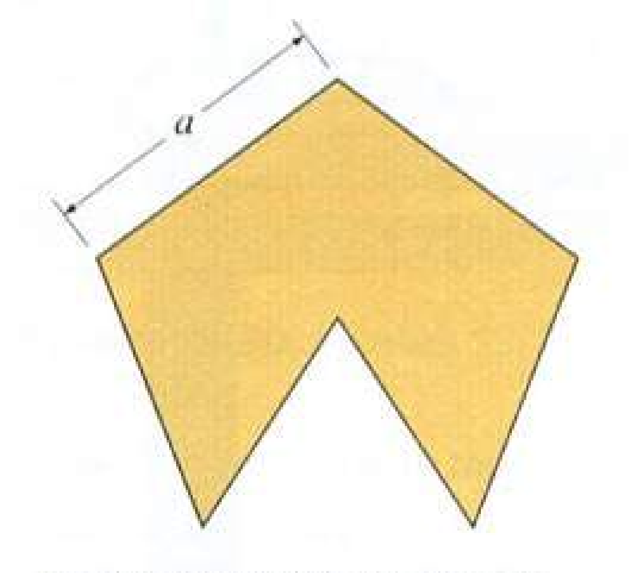
This allows us to define the center of mass as:

$$\vec{R}_{cm} = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots) = \frac{1}{M}\sum_{i=1}^{N} m_i\vec{r}_i$$

For continuous media both sums become integrals:

$$M = \int dm, \quad \text{and} \quad \vec{R}_{cm} = \frac{1}{M}\int \vec{r}\,dm$$

CM for Pentagon with One Triangle Missing



Find the center of mass of a pentagon of side ***a*** with one triangle missing.

The easiest way to do this problem is consider it as a complete pentagon with 5 triangles of mass ***m*** plus the missing triangle with mass ***-m***.

If the height of the triangles is ***h***, then the CM for the ***–m*** triangle is ***h/3***. The CM for the full pentagon is ***h*** above the base (the center of the pentagon). The expression for y_{cm} is then:

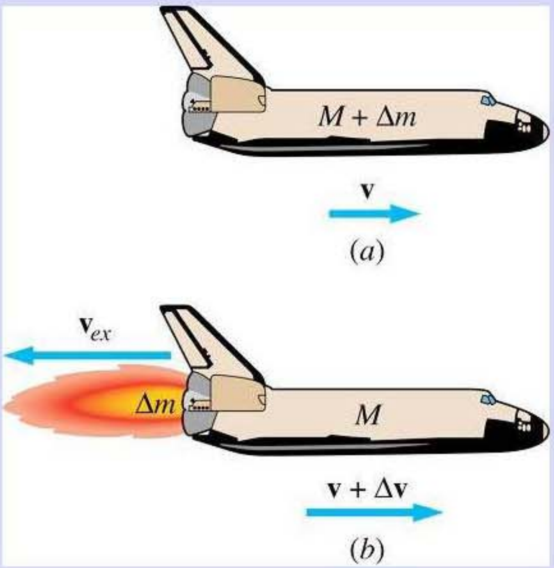
$$y_{cm} = \frac{1}{4m}(5mh - mh/3) = \frac{14}{12}h = h + \frac{1}{6}h$$

The height, ***h***, is ***h = a/2 tan54º*** and above the vertex.

$$y_{cm} = \frac{a}{12}\tan 54^\circ = .115a$$

Rockets

Rocket propulsion is achieved by the **exhaust of fuel mass** at high speed, in the process the **mass of the rocket and fuel left over reduces**. By the law of **conservation of momentum**, the momentum of the rocket and (all of the) fuel **system remains constant** as this fuel mass is expelled.



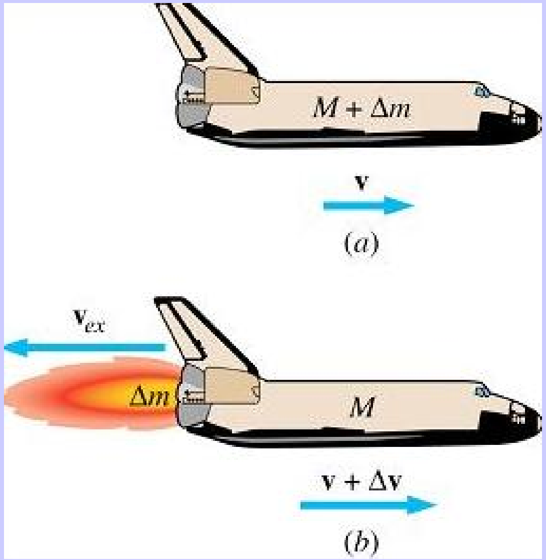
Exhausted fuel mass | Change in rocket speed | Fuel exhaust speed

$$(M + \Delta m)v = M(v + \Delta v) + \Delta m(v - v_{ex})$$

Initial momentum | New rocket momentum | Exhausted fuel momentum

Thus the change in mass of the rocket + fuel is a critical part to finding the correct or accurate thrust.

Rockets



Rewriting the equation for conservation of momentum:

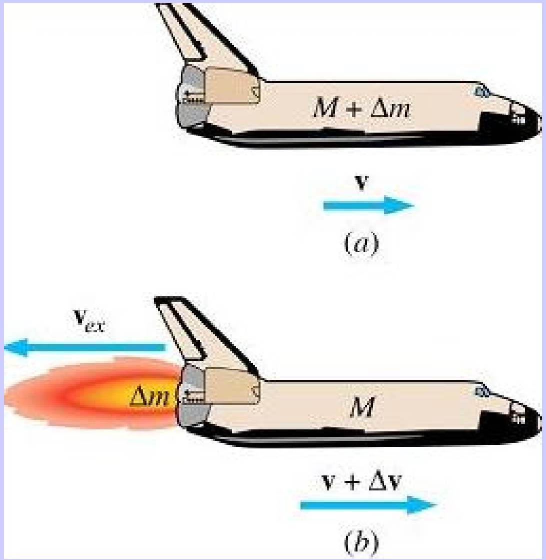
$$(M + \Delta m)v = M(v + \Delta v) + \Delta m(v - v_{ex})$$
$$0 = M\Delta v - v_{ex}\Delta m$$

The expelled mass, $\boldsymbol{\Delta m}$, represents a decrease in rocket mass, hence the change $\boldsymbol{\Delta M}$ in rocket mass is given by $\boldsymbol{\Delta M = -\Delta m}$. We now divide by $\boldsymbol{\Delta t}$ and take the limit to form differentials.

$$\boxed{F_T = M\frac{dv}{dt} = -v_{ex}\frac{dM}{dt}}$$

Here we have identified the thrust as $\boldsymbol{F_T}$ which is equal to $\boldsymbol{-v_{ex}\, dM/dt}$. This is constant. However $\boldsymbol{M}$ decreases with time, so the acceleration increases.

Rockets



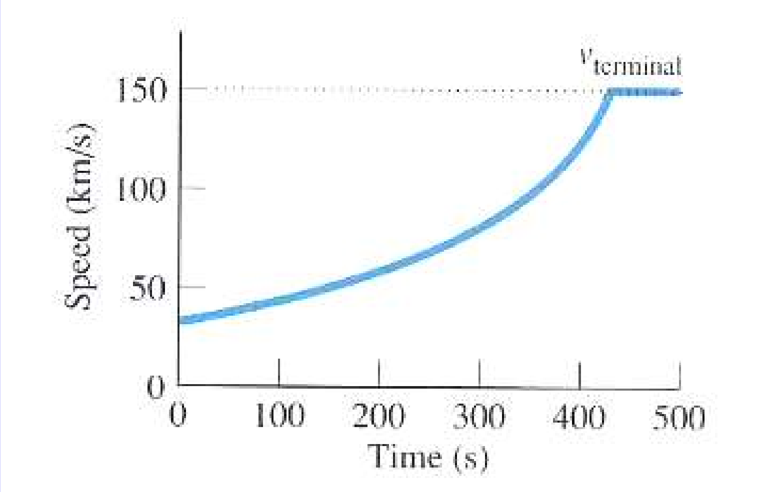
To solve for the velocity as a function of time we multiply our EOM,

$$F_T = M\frac{dv}{dt} = -v_{ex}\frac{dM}{dt}$$

by ***dt***, separate and integrate.

$$\int_i^f dv = v_f - v_i = -v_{ex}\int_i^f \frac{dM}{M} = v_{ex}\ln\frac{M_i}{M_f}$$

The velocity as a function of time is shown in the plot:



Examples: Rockets

A rocket ejects **10^5 kg** of fuel in the **90s** after launch. (a) How much thrust is developed if the fuel is ejected at **3.0km/s** with respect to the rocket? (b) What is the maximum total mass of the rocket if it is to get off the ground? (c) If the rocket has the maximum mass from (b), what is the maximum acceleration of the rocket?

(a) $$F_T = v_{ex}\frac{dM}{dt} = 3\times 10^3\frac{10^5}{90} = 3.33\times 10^6 N$$

(b) $$M_{\max} = F_T/g = 3.33\times 10^6/9.8 = 3.4\times 10^5 kg$$

(c) $$a_{\max} = (F_T - M_{\min} g)/M_{\min} = 3.33\times 10^6/(2.4\times 10^5) - 9.8$$

$$a_{\max} = 4.075 m/s^2$$

What is the terminal speed of a rocket if it exhaust **80%** of its fuel at an exhaust velocity of $\boldsymbol{v_{ex} = 2.5 km/s}$?

$$v = v_{ex}\ln\frac{M_i}{M_f} = 2.5\times 10^3 \ln 5 = 4 km/s$$

Examples: Rockets

If a rocket's exhaust speed is ***200m/s*** relative to the rocket, what fraction of its initial mass must be ejected to increase the rocket's speed by ***50m/s***?

From the rocket equation the ratio of the final mass, $\boldsymbol{M_f}$, to the initial mass, $\boldsymbol{M_i}$, is:

$$\Delta v = v_{ex} \ln \frac{M_i}{M_f}$$

$$\frac{M_f}{M_i} = e^{-\Delta v / v_{ex}} = e^{-50/200} = .78$$

The amount of mass that was ejected during this burn is:

$$\frac{M_i - M_f}{M_i} = .22 \rightarrow \Delta M = .22 M_i$$