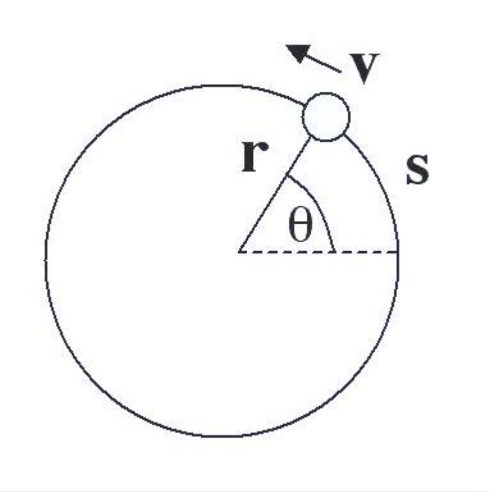
Chapter 12

Rotational Motion

Circular Motion Reviewed



θ is the **angular displacement** where $s = r\,\theta$.

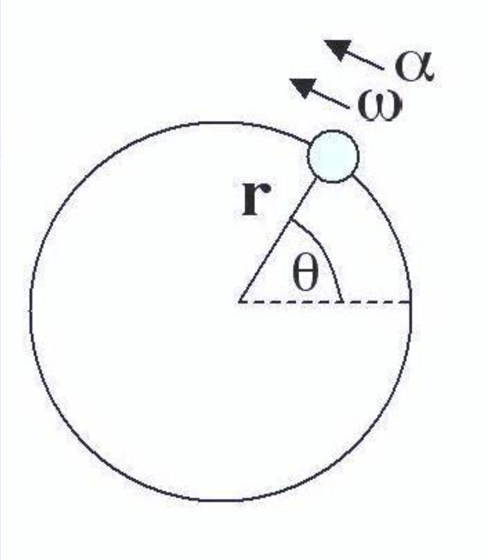
$\frac{d\theta}{dt}$ is the **angular velocity** ω where $v = r\omega$.

$\frac{dv}{dt}$ is the **instantaneous acceleration** a, which has radial component $\frac{v^2}{r} = r\omega^2$

$\frac{d\omega}{dt}$ is the **angular acceleration** α which is related to the tangential instantaneous acceleration by $a_t = r\alpha$

We studied the motion along the arc length $s = r\theta$ and how the forces and motion are solved. In Chapter 12 we begin to study the **energy contained** in the **circular motion**, and how Newton's laws and Energy Conservation apply.

Equations of Angular Motion



For an abject rotating in a circle with **constant** angular acceleration α by definition

$$\boxed{\omega_f = \omega_i + \alpha t}$$

Also by definition, we can use the average angular velocity to find the final angle after time t:

$$\theta_f = \theta_i + \omega_{ave} t = \theta_i + \frac{\omega_f + \omega_i}{2} t$$

Substituting in for ω_f $\boxed{\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2}$

Substituting in for $t = (\omega_f - \omega_i)/\alpha$ **Look familiar?**

$$\boxed{\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)}$$

Relating Linear and Angular Motion

The equations if linear and angular motion are analogous.

Linear	Angular
constant a in one dimension	**constant α**
$v_f = v_i + at$	$\omega_f = \omega_i + \alpha t$
$x_f = x_i + v_i t + \frac{1}{2}at^2$	$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$
$v_f^2 = v_i^2 + 2a(x_f - x_i)$	$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$

Thus the translation is

$$x \rightarrow \theta$$
$$v \rightarrow \omega$$
$$a \rightarrow \alpha$$

Many more analogies exist including with force, energy and momentum.

Relating Linear and Angular Motion

The simplest way to understand the link between linear and angular equations of motion is through the tangential motion.

$$s = \theta r$$

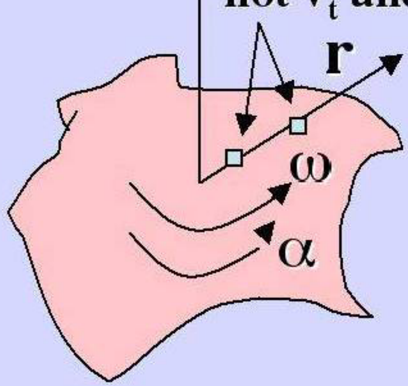
$$v_t = \omega r$$

$$a_t = \alpha r$$

S, the arc length, can be thought of as the tangential displacement.

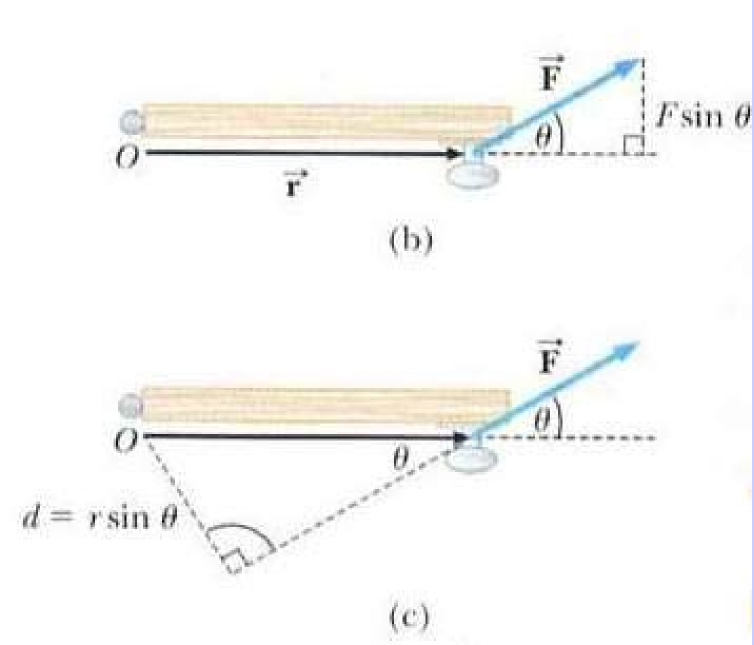
For a rigid rotating body, every point will have the same ω and α, but not the same tangential variables v_t and a_t because of r.

These two points will have the same ω and α, not v_t and a_t



Ok, but how does Force work?

Torque: the Angular Analog of Force



Torque can be thought of as an "angular force" which causes a change in angular motion. It is defined as:

$$\tau = rF\sin\theta$$

In (b) ***F*sin**θ is the component of the force perpendicular to the door.
In (c) **rsin**θ is the component of the moment arm perpendicular to the force, defined as the lever arm.
The choice is often determined by the particular application or problem.

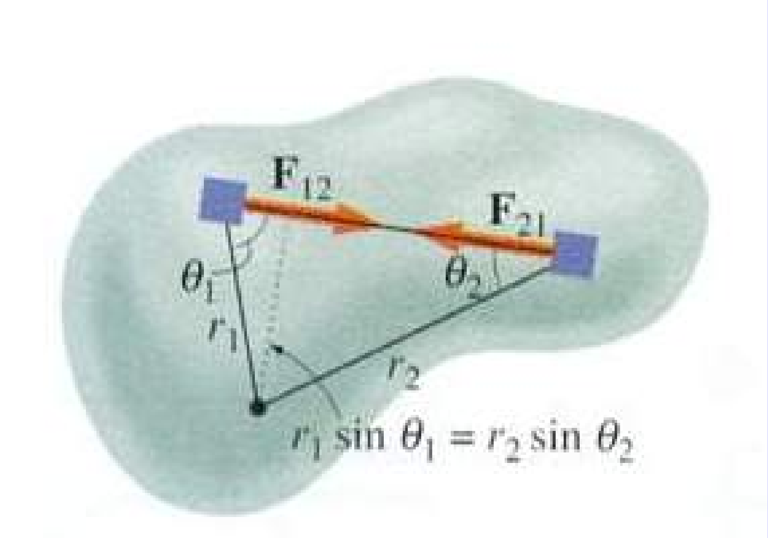
It should be noted that:

Torque depends on the choice of the origin

In the next chapter we will define torque as a vector via the vector product:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Internal Torques Cancel



Consider an arbitrary object rotating about an axis (the bold point in the figure). Two mass elements exert equal and opposite forces on each other. The torques about the axis are:

$$\tau_{12} = F_{12} r_1 \sin\theta_1 \text{ and } \tau_{21} = F_{21} r_2 \sin\theta_2$$

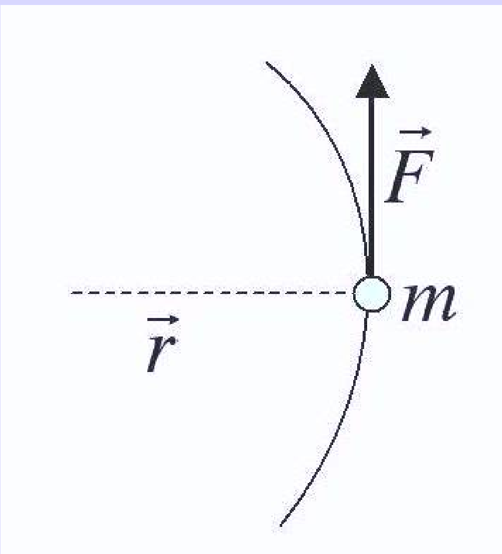
Since $\boldsymbol{F_{12} = -F_{21}}$ and $\mathbf{r_1 \sin\theta_1 = r_2 \sin\theta_2}$ the internal torques cancel!

Internal torques cancel in pairs just as the internal forces do.
The net torque is the sum of external torques!

Moment of Inertia and Torque

What is the analog of Newton's second law F=ma in circular motion?

Consider a point mass on a circular track and a force perpendicular to the radius of that track.



Substitute the tangential acceleration in terms of the analog of linear acceleration, the angular acceleration

$$F = ma_t = m(\alpha r)$$

This force is the linear motion force and is always changing direction. The torque (sinθ=1) is:

$$\tau = rF$$

Torque is the analog of force in circular motion. Substituting the force into the torque equation:

$$\boxed{\tau = mr^2\alpha = I\alpha}$$

Simple form in point mass case, not always

$I = mr^2$, **the moment of inertia, is the analog of mass in rotational motion.**

Calculating the Moment of Inertia

For point masses and continuous media the moment of inertia is calculated in a similar way as the center of mass, but with r^2 instead of r (and we do **not** divide by the total mass).

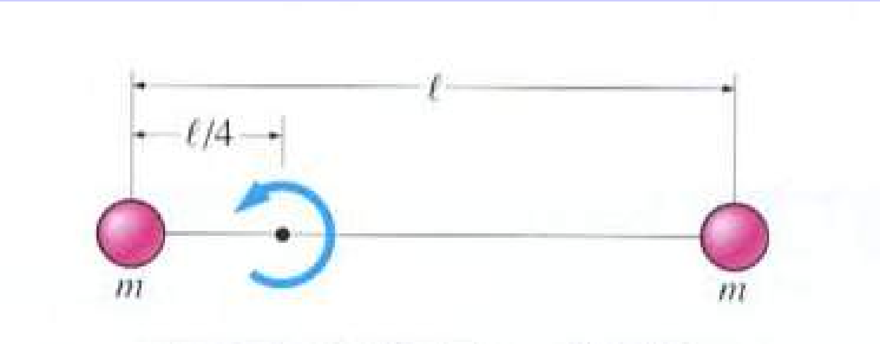
For a collection of point masses $$I = \sum_i r_i^2 m_i$$

For a continuous medium of mass $$I = \int r^2 dm$$

Similarly to the center of mass, the moment of inertia depends on the choice of origin through r.

Note that r is the distance to the axis of rotation, and I is **not** equal to $M\, r^2_{cm}$.

Example: Moment of Inertia of Two Point Masses



A dumbbell shaped object of two equal masses a distance ***l*** apart is subjected to a torque, ***τ***, about an axis ¼ of the way between the two masses. How long will it take for the system to rotate through an angle ***θ*** ?

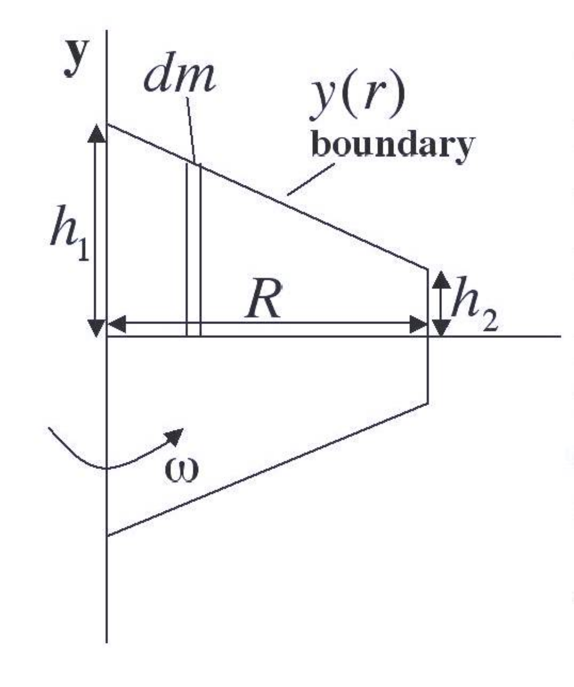
First we need to find the moment of inertia for the system:

$$I = m\left(\frac{l}{4}\right)^2 + m\left(\frac{3l}{4}\right)^2 = \frac{5}{8}ml^2$$

The angular acceleration is: $\alpha = \tau/I = 8\tau/(5ml^2)$

From the kinematic relations: $\theta = \frac{1}{2}\alpha t^2 = \frac{4\tau}{5ml^2}t^2$

Example: Moment of Inertia for a Flat Plate



Find the moment of inertia of the flat plate of mass ***M*** with the dimensions shown in the figure.

From symmetry about the ***y* = 0** axis we can find the momentum of inertia for the upper half and multiply by 2. The expression for the upper boundary is:

$$y(r) = h_1 + \frac{h_2 - h_1}{R} r$$

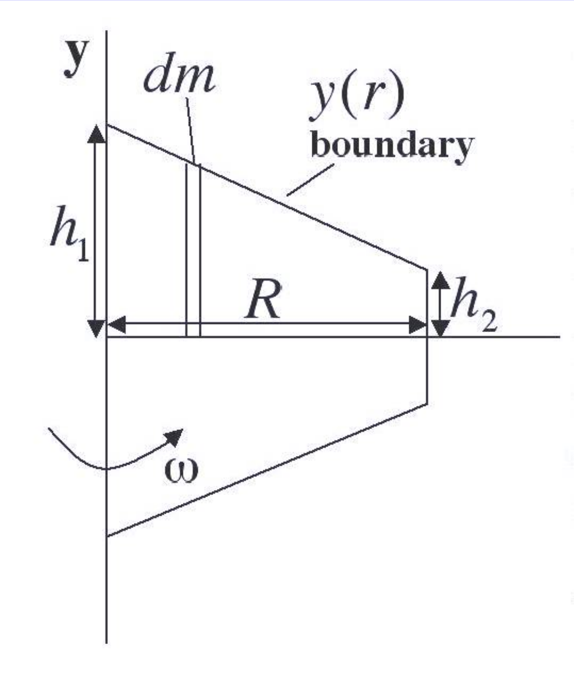
The differential mass element is:

$$dm = \frac{M/2}{A} y dr = \frac{M}{(h_1 + h_2)R} y dr$$

The integral for the momentum of inertia of the entire plate (remember the factor of 2) :

$$I = \int r^2 dm = 2\int_0^R r^2 \frac{M}{(h_1 + h_2)R} y(r) dr$$

Example: Moment of Inertia for a Flat Plate



Find the moment of inertia of the flat plate of mass ***M*** with the dimensions shown in the figure.

Substituting in the expression for ***y(r)*** and integrating:

$$I = \int r^2 dm = 2\int_0^R r^2 \frac{M}{(h_1+h_2)R} y(r)dr$$

$$I = \frac{2M}{(h_1+h_2)R}\int_0^R r^2\left(h_1 + \frac{h_2-h_1}{R}r\right)dr$$

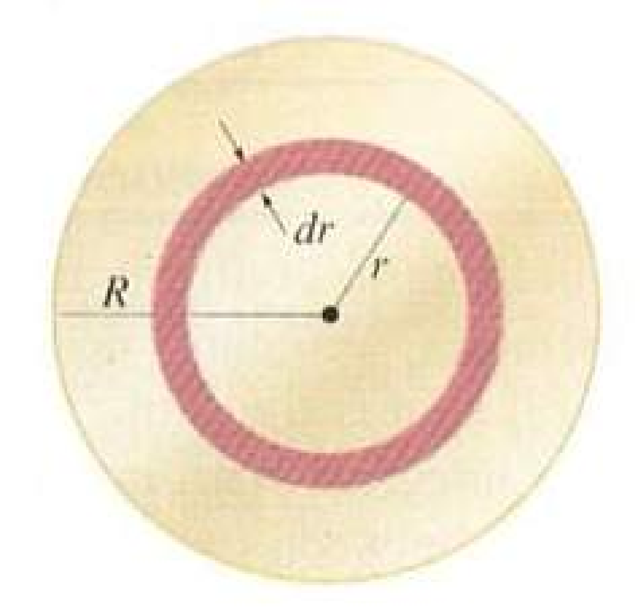
$$I = \frac{2M}{(h_1+h_2)R}\left(h_1\frac{R^3}{3} + (h_2-h_1)\frac{R^3}{4}\right)$$

Algebraically simplifying:

$$I = \frac{1}{6}\frac{(h_1+3h_2)}{(h_1+h_2)}MR^2$$

If $\boldsymbol{h_1 = h_2}$ then $\boldsymbol{I}$ reduces to $\boldsymbol{I = 1/3\ M\ R^2}$, the correct result for a flat plate.

Example: Moment of Inertia of a Thin Disk about its Center



Due to circular symmetry we only have to integrate along the radial direction. The differential mass element is:

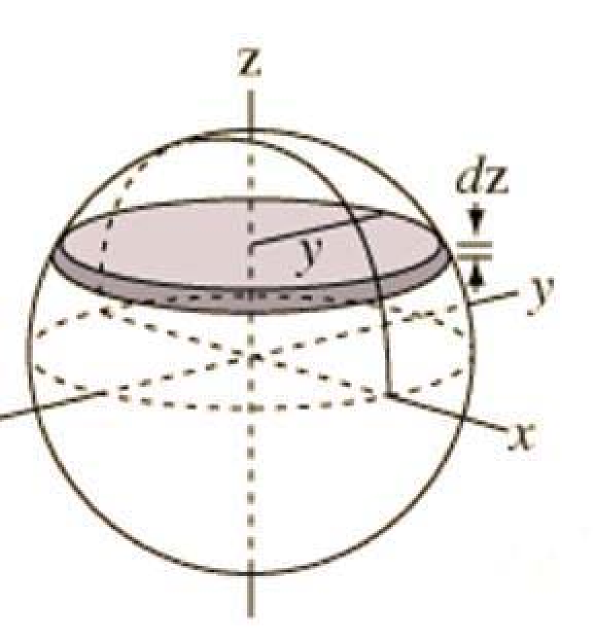
$$dm = \frac{M}{A} 2\pi r dr$$

Performing the moment of inertia integral:

$$I = \int r^2 dm = \int_0^R \frac{M}{A} 2\pi r^3 dr = \frac{2\pi M}{\pi R^2} \frac{R^4}{4} = \frac{1}{2} M R^2$$

This result can be used to sum thin disks for any objects with circular symmetry such as a sphere or disk etc.

Example: Moment of Inertia for a Sphere about its Axis



For a sphere of radius $\boldsymbol{R}$, we can sum (integrate) the moments of inertia for series of infinitesimal disks.

The momentum of inertia for an infinitesimal disk is:

$$dI = y^2 dm = y^2 \rho dV = y^2 \frac{\rho}{2} \pi y^2 dz$$

The radius of these disks, $\boldsymbol{y(z)}$, is found from:

$$y^2 = R^2 - z^2$$

Performing the integral for $\boldsymbol{I}$ yields:

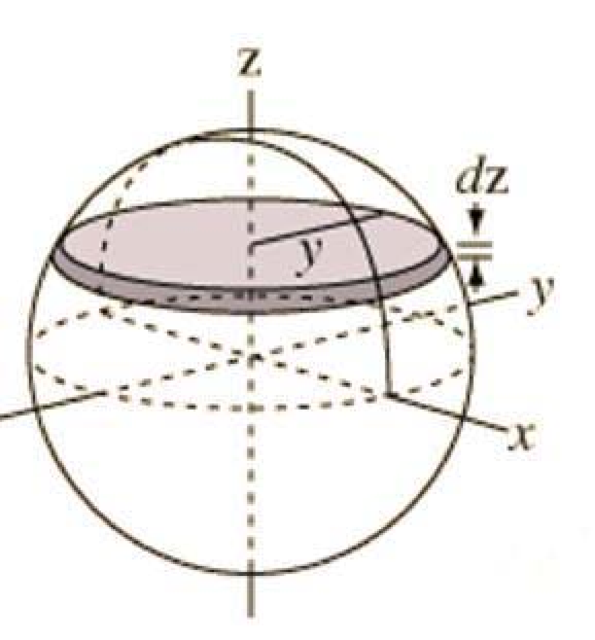
$$I = \int_{-R}^{R} (R^2 - z^2) \frac{\rho}{2} \pi (R^2 - z^2) dz = \frac{\pi M}{2V} \int_{-R}^{R} (R^2 - z^2)^2 dz$$

$$I = \frac{\pi M}{8\pi R^3/3} (2R^5 - 4R^5/3 + 2R^5/5) = \frac{3}{8} MR^2 (2 - 4/3 + 2/5)$$

$$I = \frac{1}{8} MR^2 (2 + 6/5) = \frac{2}{5} MR^2$$

$\boldsymbol{I}$ could also have been found from a spherical volume element.

Example: Thin Spherical Shell



For a thin spherical shell the mass element is:

$$dm = \frac{M}{A} 2\pi y (R d\theta) = \frac{M}{A} 2\pi R^2 \sin\theta d\theta$$

Finding the mass element was the hard part.
The integral for the moment of inertia is:

$$I = \int r^2 dm = \int y^2 dm = \frac{2\pi M}{A} R^4 \int_0^{\pi} \sin^3\theta d\theta$$

$$I = \frac{2\pi M}{4\pi R^2} R^4 \int_0^{\pi} (1 - \cos^2\theta) \sin\theta d\theta = \frac{1}{2} MR^2 \left(2 - \frac{2}{3}\right)$$

$$I = \frac{2}{3} MR^2$$

Could we have used the moment of inertia for a solid sphere of radius R and subtracted the moment of inertia for a solid sphere (same density) of slightly smaller radius, $\boldsymbol{R\text{-}\delta R}$?

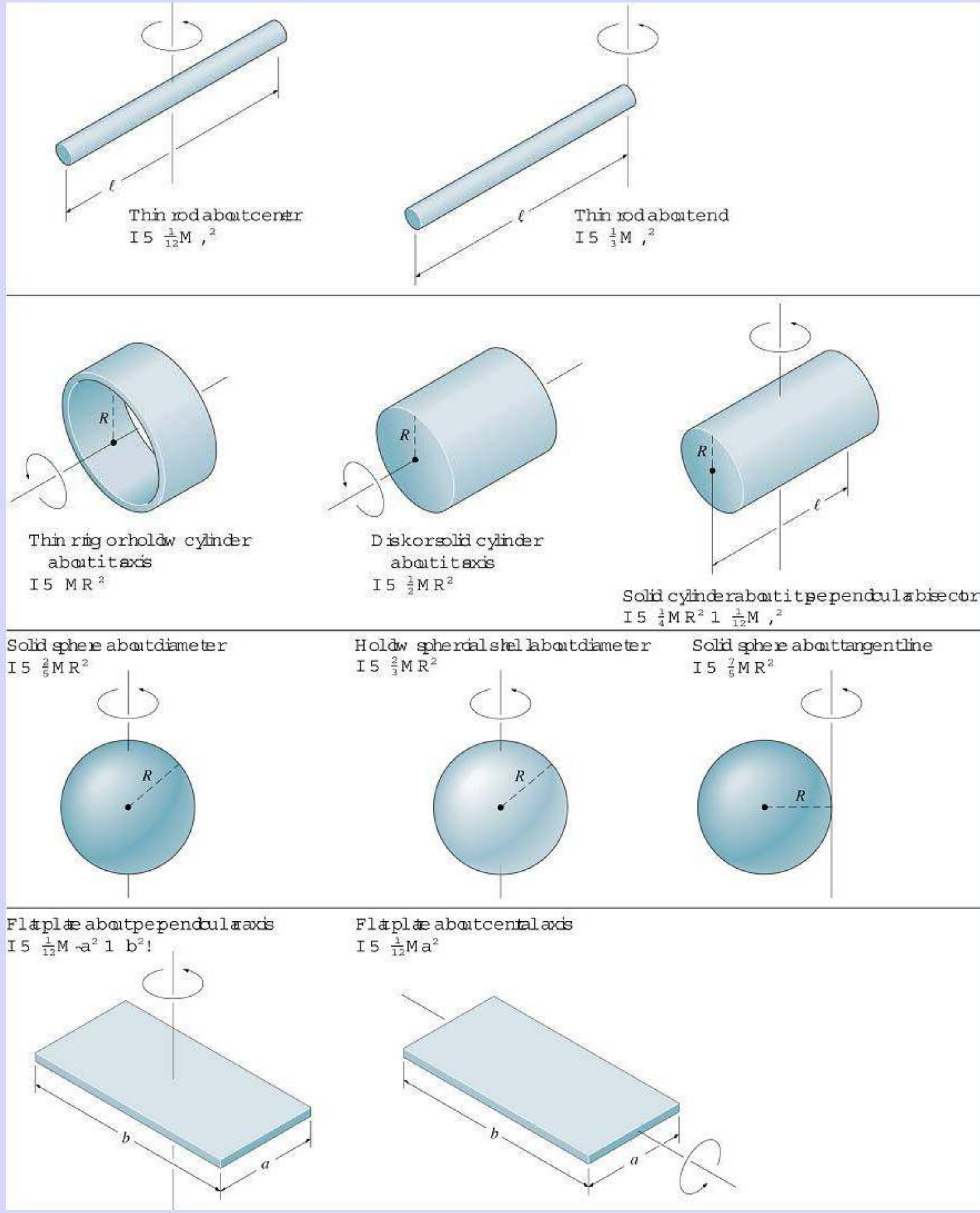
Absolutely, give it a try!

Some Examples of Moments of Inertia

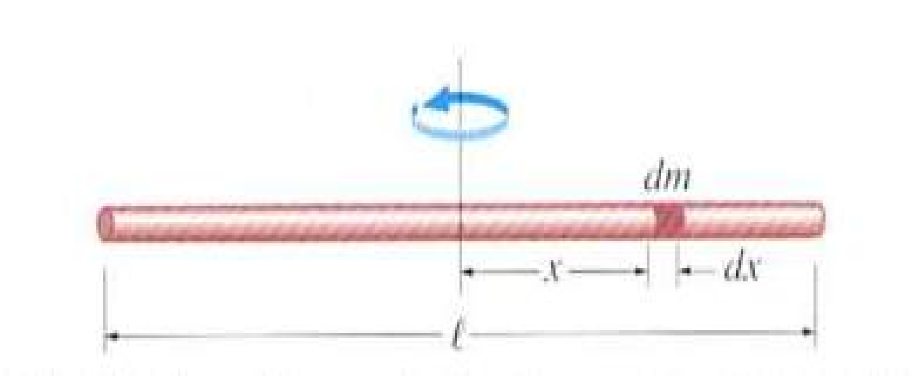
How can we do all these about ANY axis??

Good news: you don't have to. **We can use what's called the Parallel axis theorem to get the I of any object about any axis parallel to one we know.**

Bad news: You still need to do it for one axis.



Example: Moment of Inertia of a Rod



Find the moment of inertia for a rod of mass ***m*** and length ℓ about an axis perpendicular to the rod through its (a) center and its (b) end.

(a) The integral for *I* with an axis through its center is straightforward

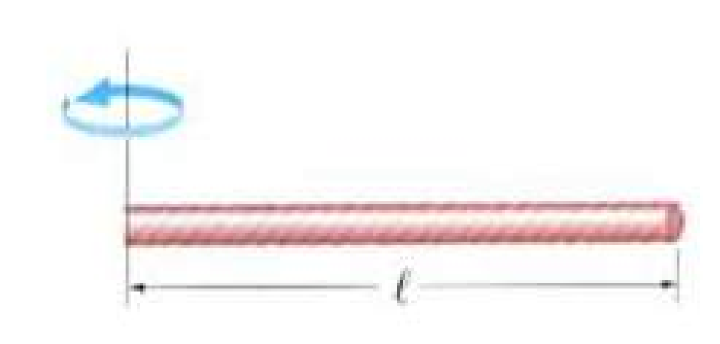
$$I = \int_{-\ell/2}^{\ell/2} x^2 dm = \int_{-\ell/2}^{\ell/2} x^2 \mu dx = \frac{m}{\ell} \int_{-\ell/2}^{\ell/2} x^2 dx$$

with the result:

$$I = \frac{m}{\ell} \frac{2(\ell/2)^3}{3} = \frac{1}{12} m\ell^2$$

What is particularly interesting about this result is to note what happens when we consider its momentum of inertia about an axis through its end.

Example: Moment of Inertia of a Rod



Find the moment of inertia for a rod of mass ***m*** and length ℓ about an axis perpendicular to the rod through its (a) center and its (b) end.

(b) The integral for *I* with an axis through its end is also straightforward

$$I = \int_0^{\ell} x^2 dm = \frac{m}{\ell}\int_0^{\ell} x^2 dx = \frac{1}{3} m\ell^2$$

We note that the difference between this result and the moment of inertia about an axis through its center of mass is:

$$\Delta I = I_{end} - I_{cm} = \frac{1}{4} m\ell^2 = m(\ell/2)^2$$

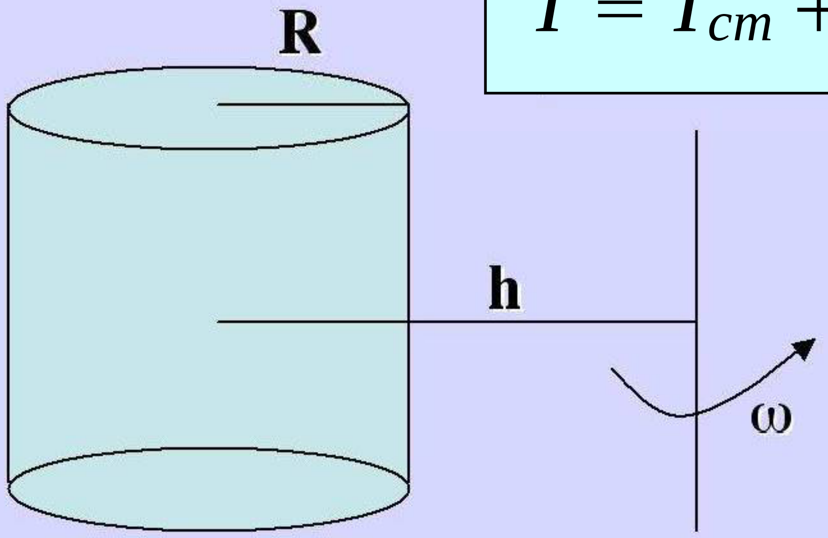
The increase in *I* just happens to be *m* times the distance to the CM squared!

The Parallel Axis Theorem

Given the moment of inertia about an axis through the center of mass of an object, the moment of inertia about any axis parallel to that axis can be written as:

$$I = I_{cm} + Mh^2$$

Where h is the distance to center of mass (or axis of the moment of inertia) of the object.

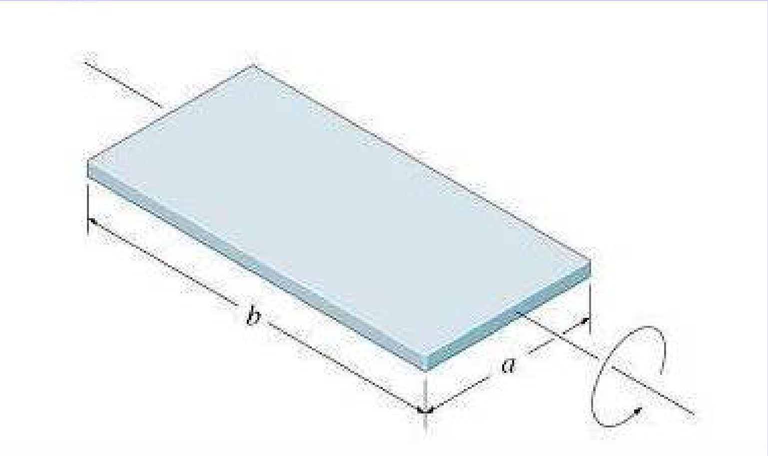


For the cylinder off axis:

$$I = \frac{1}{2}MR^2 + Mh^2$$

This can simplify the solution of rotational motions about axes and also differences between rotational motion around different axes.

Example: Parallel Axis Theorem for Flat Plate



For a flat plate the mass element is:

$$dm = \frac{M}{A} b dx$$

The moment of inertial integral with the rotation axis through the CM is:

$$I_{CM} = \int r^2 dm = \int_{-a/2}^{a/2} x^2 \frac{M}{A} b dx = \frac{2M}{3A} \frac{a^3 b}{8} = \frac{1}{12} M a^2$$

The moment of inertial integral with the rotation axis along one side is:

$$I_{side} = \int r^2 dm = \int_0^a x^2 \frac{M}{A} b dx = \frac{M}{A} \frac{a^3 b}{3} = \frac{1}{3} M a^2$$

From the Parallel Axis Theorem:

$$I_{side} = I_{CM} + M(a/2)^2 = \left(\frac{1}{12} + \frac{1}{4}\right) M a^2 = \frac{1}{3} M a^2$$