Chapter 12

Rotational Motion II

Relating Linear and Angular Motion

The equations if linear and angular motion are analogous.

Linear	**Angular**
constant a in one dimension	**constant α**
$v_f = v_i + at$	$\omega_f = \omega_i + \alpha t$
$x_f = x_i + v_i t + \frac{1}{2}at^2$	$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$
$v_f^2 = v_i^2 + 2a(x_f - x_i)$	$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$

Thus the translation is

$$x \to \theta$$
$$v \to \omega$$
$$a \to \alpha$$

Many more analogies exist including with force, energy and momentum.

Relating Linear and Angular Motion

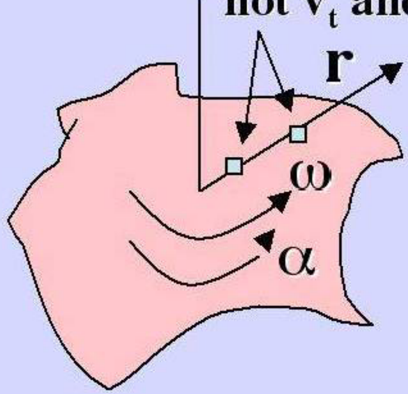
The simplest way to understand the link between linear and angular equations of motion is through the tangential motion.

$$s = \theta r$$
$$v_t = \omega r$$
$$a_t = \alpha r$$

S, the arc length, can be thought of as the tangential displacement.

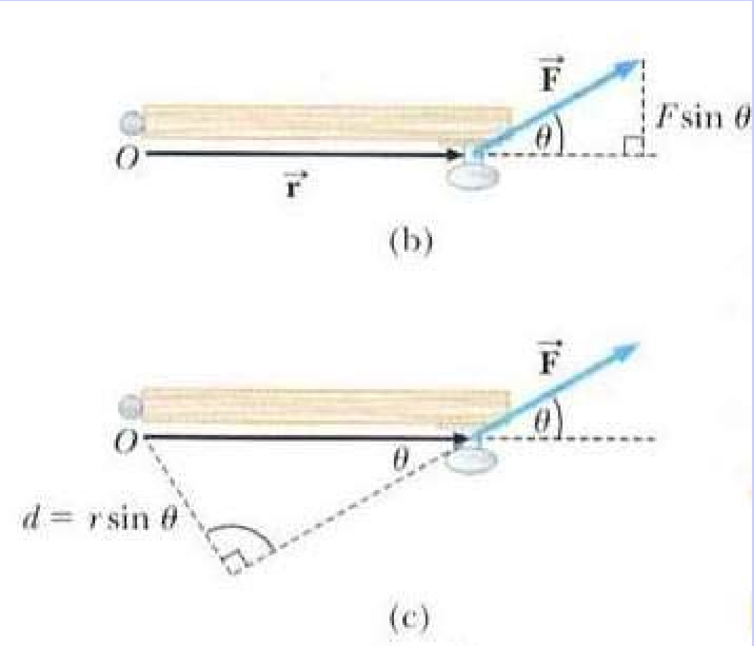
These two points will have the same ω and α, not v_t and a_t

For a rigid rotating body, **every point** will have the **same ω and α**, but **not the same** tangential variables **v_t and a_t** because of r.



Ok, but how does Force work?

Torque: the Angular Analog of Force



Torque can be thought of as an "angular force" which causes a change in angular motion. It is defined as:

$$\tau = rF\sin\theta$$

In (b) $\boldsymbol{F}\mathbf{sin}\theta$ is the component of the force perpendicular to the door.
In (c) $\mathbf{rsin}\theta$ is the component of the moment arm perpendicular to the force, defined as the lever arm.
The choice is often determined by the particular application or problem.

It should be noted that:

Torque depends on the choice of the origin

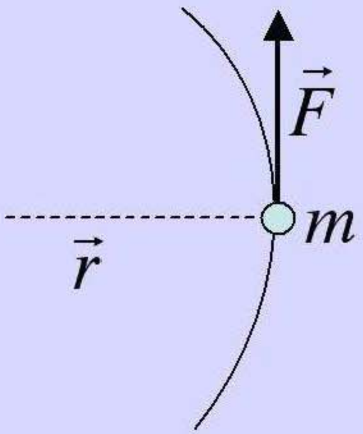
In the next chapter we will define torque as a vector via the vector product:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Moment of Inertia and Torque

What is the analog of Newton's second law F=ma in circular motion?

Consider a point mass on a circular track and a force perpendicular to the radius of that track.



Substitute the tangential acceleration in terms of the analog of linear acceleration, the angular acceleration

$$F = ma_t = m(\alpha r)$$

This force is the linear motion force and is always changing direction. The torque (sinθ=1) is:

$$\tau = rF$$

Torque is the analog of force in circular motion. Substituting the force into the torque equation:

$$\boxed{\tau = mr^2\alpha = I\alpha}$$

Simple form in point mass case, not always

$I = mr^2$, **the moment of inertia, is the analog of mass in rotational motion.**

Calculating the Moment of Inertia

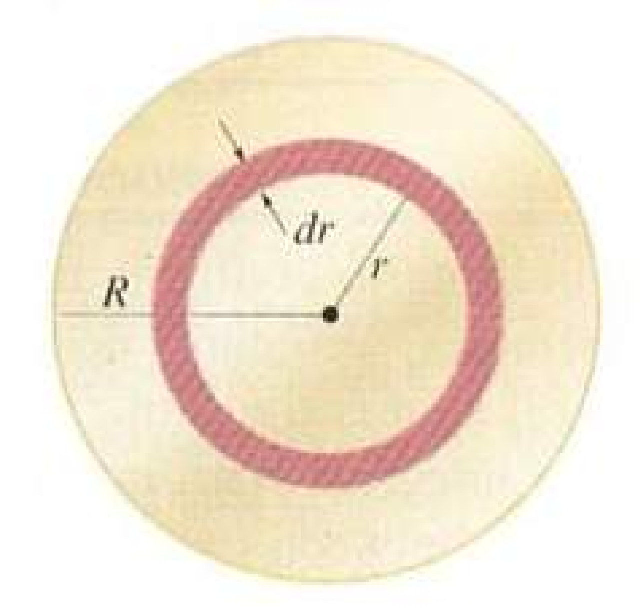
For point masses and continuous media the moment of inertia is calculated in a similar way as the center of mass, but with r^2 instead of r (and we do **not** divide by the total mass).

For a collection of point masses	$I = \sum_i r_i^2 m_i$
For a continuous medium of mass	$I = \int r^2 dm$

Similarly to the center of mass, the moment of inertia depends on the choice of origin through r.

Note that *r* is the distance to the axis of rotation, and *I* is **not** equal to $M\, r^2_{cm}$.

Example: Moment of Inertia of a Thin Disk about its Center



Due to circular symmetry we only have to integrate along the radial direction. The differential mass element is:

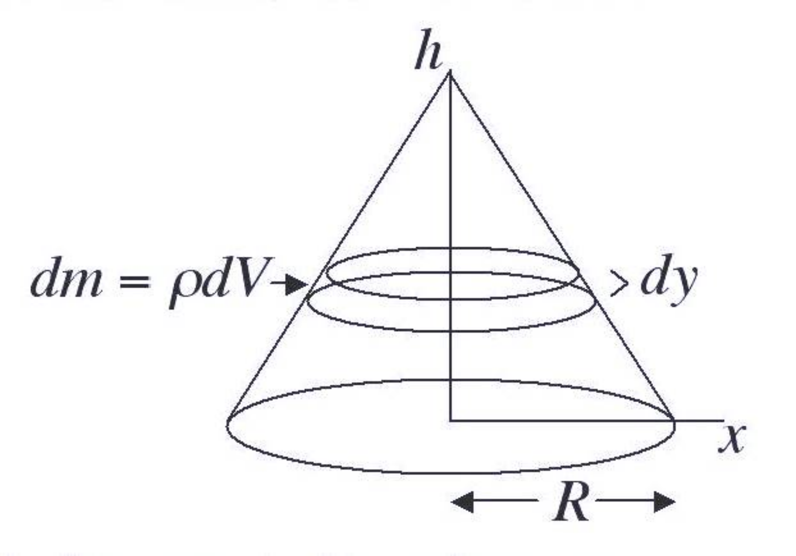
$$dm = \frac{M}{A} 2\pi r dr$$

Performing the moment of inertia integral:

$$I = \int r^2 dm = \int_0^R \frac{M}{A} 2\pi r^3 dr = \frac{2\pi M}{\pi R^2} \frac{R^4}{4} = \frac{1}{2} M R^2$$

This result can be used to sum thin disks for any objects with circular symmetry such as a sphere or disk etc.

Example: Moment of Inertia of a Cone



Find the moment of inertia of a uniform solid cone of height ***h***, density ***ρ***, about its central axis.

The differential element for the moment of inertia for a disk of thickness ***dy*** is:

$$dI = \frac{1}{2}r^2 dm = \frac{\rho}{2}r^2(\pi r^2)dy$$

The dependence of ***r*** on ***y*** is: $r(y) = \frac{R}{h}(h-y)$ Performing the integral:

$$I = \frac{\pi\rho}{2}\left(\frac{R}{h}\right)^4 \int_0^h (h-y)^4 dy = \frac{\pi\rho}{2}\left(\frac{R}{h}\right)^4 \frac{h^5}{5}$$

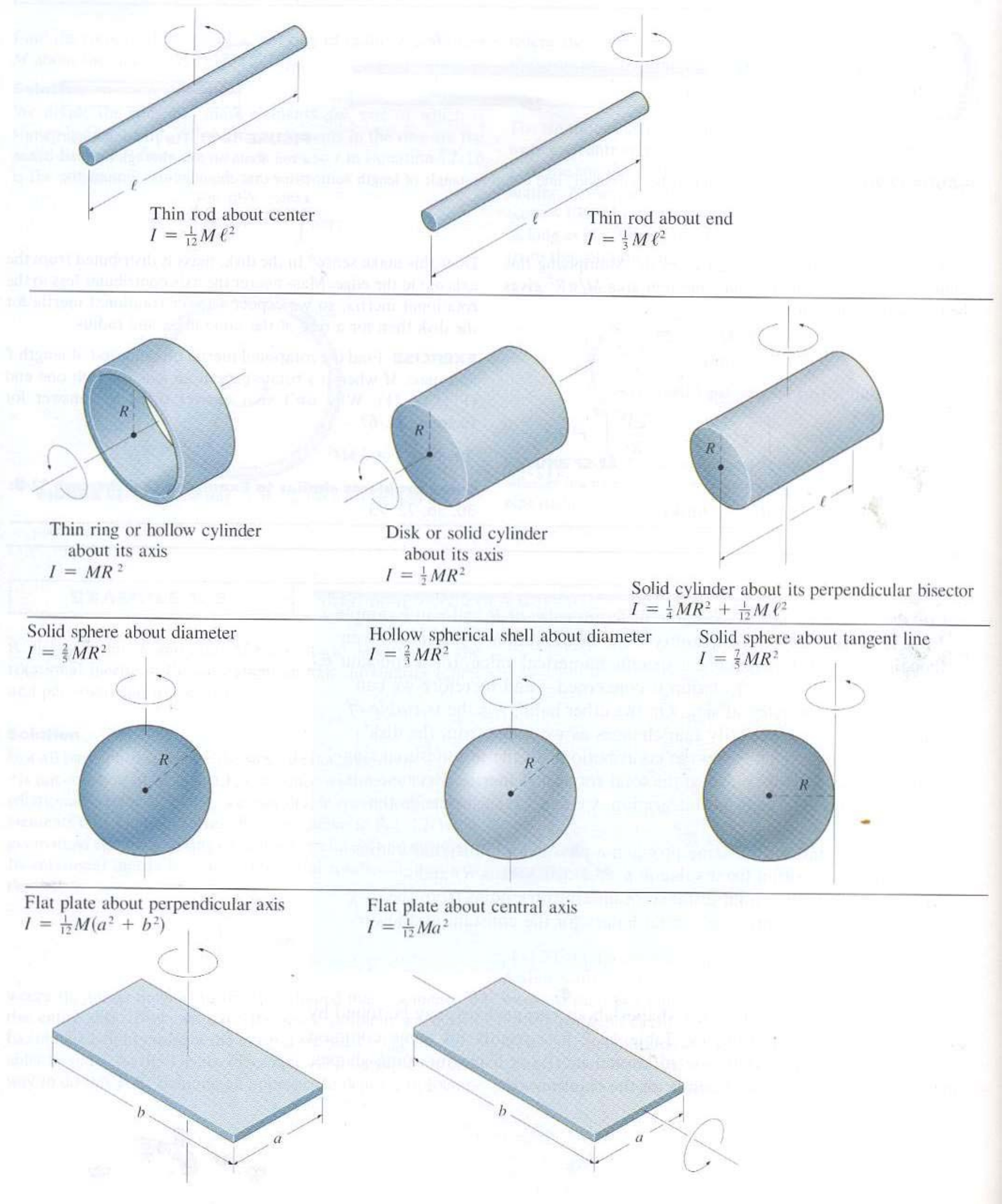
$$I = \frac{\pi M}{2\pi R^2 h/3}\left(\frac{R}{h}\right)^4 \frac{h^5}{5} = \frac{3}{10}MR^2$$

Integration is a bit easier if we invert the cone. Also, **does this result make sense?**

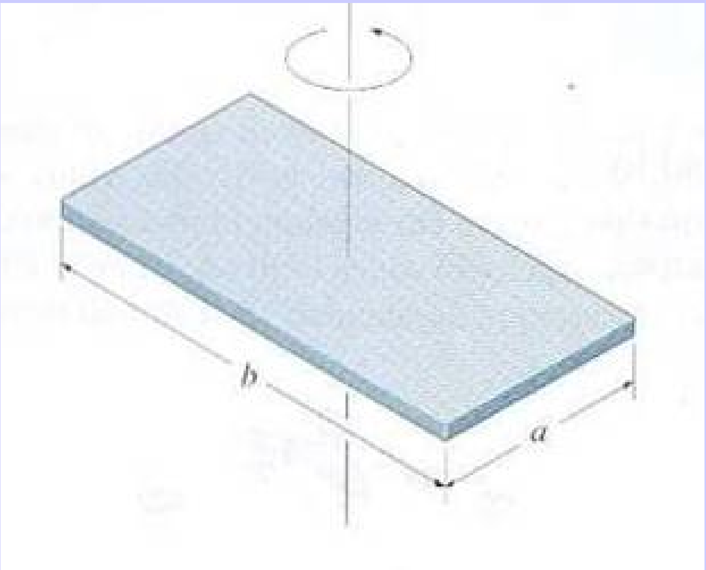
Moment of Inertia Several Examples

Note that the moment of inertial is always of the form, $\boldsymbol{I = \alpha MR^2}$, where $\boldsymbol{\alpha < 1}$. Why is that?

What about the moment of inertia of a ring or hollow cylinder?



Moment of Inertia for a Rectangle About an Axis Perpendicular to its Center



A rectangle is does not have circular symmetry about its axis of rotation. Does this change things? Not really!

The differential mass element ***dm = M/A dxdy.***
The distance to an arbitrary mass element from the axis of rotation is $\boldsymbol{r^2 = x^2 + y^2}$. Hence:

$$I = \frac{M}{A}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}(x^2+y^2)dxdy = \frac{M}{A}\int_{-a/2}^{a/2}\left[\frac{x^3}{3}+y^2x\right]_{-b/2}^{b/2}dy$$

$$I = \frac{M}{ab}\int_{-a/2}^{a/2}\left[\frac{b^3}{12}+by^2\right]dy = \frac{M}{a}\left[\frac{b^2}{12}y+\frac{y^3}{3}\right]_{-a/2}^{a/2}$$

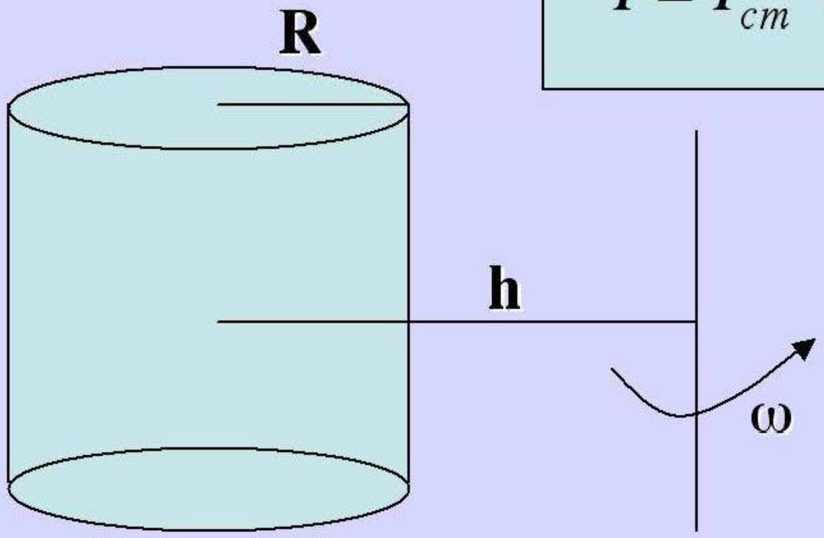
$$I = \frac{1}{12}M(a^2+b^2)$$

The Parallel Axis Theorem

Given the moment of inertia about an axis through the center of mass of an object, the moment of inertia about any axis parallel to that axis can be written as:

$$\boxed{I = I_{cm} + Mh^2}$$

Where h is the distance to center of mass (or axis of the moment of inertia) of the object.

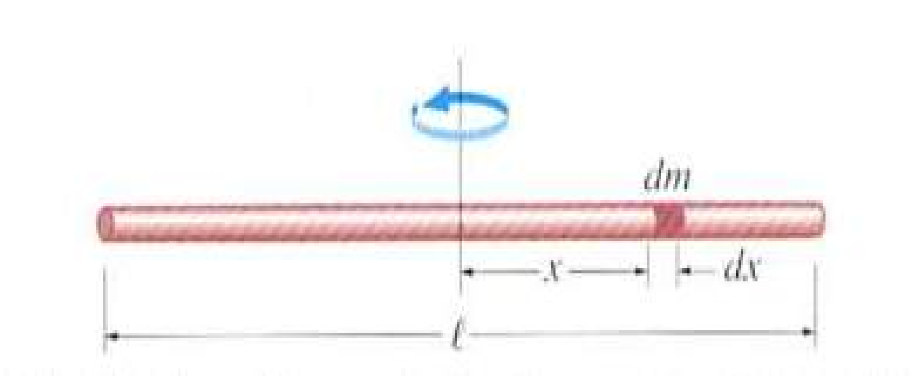


For the cylinder off axis:

$$I = \frac{1}{2}MR^2 + Mh^2$$

This can simplify the solution of rotational motions about axes and also differences between rotational motion around different axes.

Example: Moment of Inertia of a Rod



Find the moment of inertia for a rod of mass m and length ℓ about an axis perpendicular to the rod through its (a) center and its (b) end.

(a) The integral for I with an axis through its center is straightforward

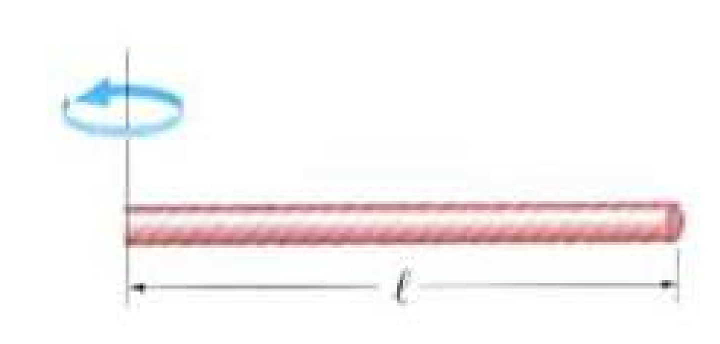
$$I = \int_{-\ell/2}^{\ell/2} x^2 dm = \int_{-\ell/2}^{\ell/2} x^2 \mu dx = \frac{m}{\ell} \int_{-\ell/2}^{\ell/2} x^2 dx$$

with the result:

$$I = \frac{m}{\ell} \frac{2(\ell/2)^3}{3} = \frac{1}{12} m\ell^2$$

What is particularly interesting about this result is to note what happens when we consider its momentum of inertia about an axis through its end.

Example: Moment of Inertia of a Rod



Find the moment of inertia for a rod of mass *m* and length ℓ about an axis perpendicular to the rod through its (a) center and its (b) end.

(b) The integral for *I* with an axis through its end is also straightforward

$$I = \int_0^{\ell} x^2 dm = \frac{m}{\ell}\int_0^{\ell} x^2 dx = \frac{1}{3} m\ell^2$$

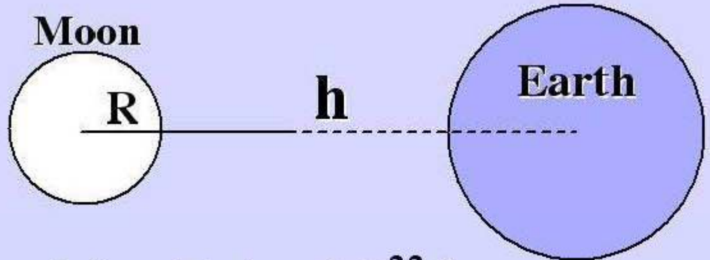
We note that the difference between this result and the moment of inertia about an axis through its center is:

$$\Delta I = I_{end} - I_{cm} = \frac{1}{4} m\ell^2 = m(\ell/2)^2$$

ΔI is *m* times the distance to the CM squared. Verification of the parallel axis theorem!

Example: Parallel Axis Theorem

What is the moment of inertia of the Moon about the Earth? What is the ratio of the moment of inertia of the actual Moon to that of the Moon as a point mass? Approximate the Moon as a sphere and ignore the rotation about the sun etc.



$$M = 7.4 \times 10^{22} kg$$

$$R = 1,738,000 m$$

$$h = 384,400,000 m$$

$$I = I_{cm} + Mh^2$$

$$I = \frac{2}{5} MR^2 + Mh^2$$

$$\boxed{I = 1.09 \times 10^{40} kgm^2}$$

The ratio is:

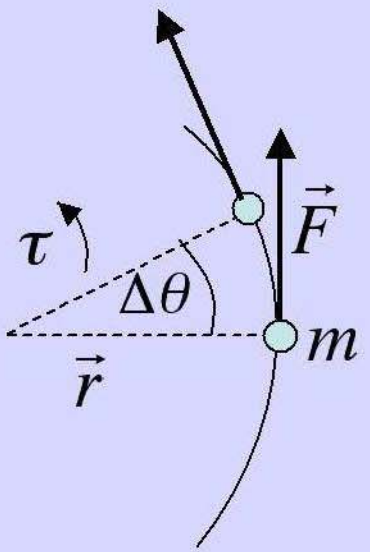
$$\frac{I_1}{I_2} = \frac{\frac{2}{5} MR^2 + Mh^2}{Mh^2} = 1 + \frac{2R^2}{5h^2} \boxed{= 1.00001 \approx 1}$$

We can treat the moon as a point mass to a degree of accuracy of 10^{-5}.

Rotational Energy

In the simplest sense we learned $\Delta W = F\Delta x$ in linear motion.

What are the work and energy in rotational motion?



Again consider a point mass on a circular track and a force perpendicular to the radius of that track (for simplicity). For a distance along the arc length:

$$\Delta W = F\Delta s$$

Substituting in for the force from the torque (sinθ=1) and the arc length in terms of r and θ:

$$\tau = rF \qquad \Delta s = r\Delta\theta$$

we find the expression for work done to apply a torque to the mass across and angle Δθ

$$\Delta W = \frac{\tau}{r}\Delta s = \frac{\tau}{r}r\Delta\theta = \tau\Delta\theta$$

More generally $\boxed{W = \int \tau d\theta}$

Rotational Energy

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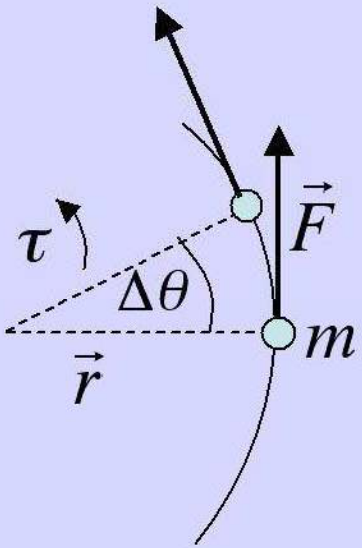


If we integrate the torque we find the work done in the angular acceleration, generating rotational kinetic energy:

$$W = \int \tau d\theta = \int_{\theta_i}^{\theta_f} I\alpha d\theta = \int_{\theta_i}^{\theta_f} I\frac{d\omega}{dt}d\theta$$

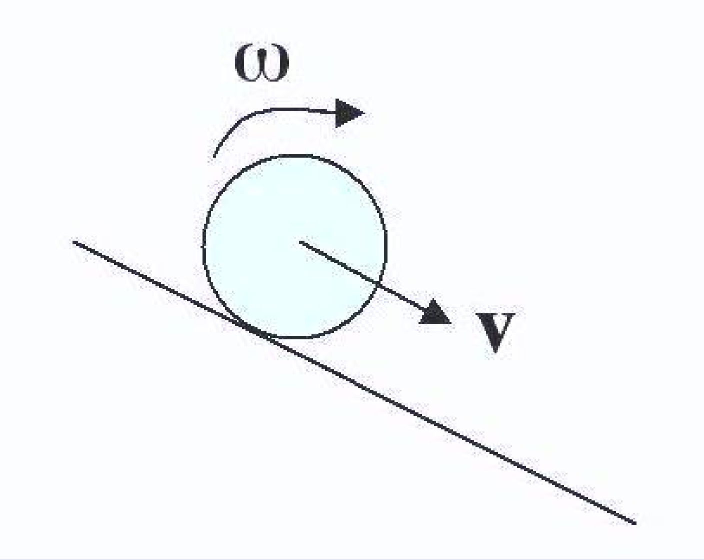
$$W = I\int_{\omega_i}^{\omega_f} \frac{d\theta}{dt}d\omega = I\int_{\omega_i}^{\omega_f} \omega d\omega = \frac{1}{2}I(\omega_f^2 - \omega_i^2)$$

The K_{rot} is as we expect, with I and ω analogs of linear motion.

$$\boxed{K_{rot} = \frac{1}{2}I\omega^2}$$

How does this effect rolling motion?

Kinetic Energy for a Rolling Object



For a rolling object the total kinetic energy is the sum of the linear and rotational kinetic energies.

$$K_{roll} = K_{lin} + K_{rot} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

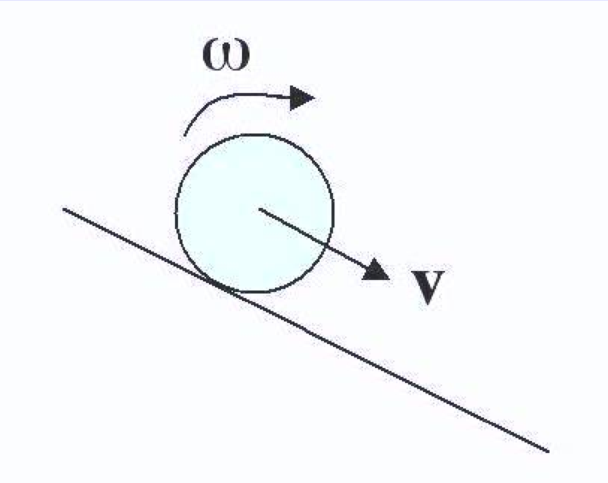
If a sphere rolls down an incline (without slipping), what fraction of its kinetic energy is rotational?

The non-slip condition is $\boldsymbol{v = R\omega}$. Hence the ratio of the KE's is:

$$\frac{K_{rot}}{K_{lin} + K_{rot}} = \frac{I\omega^2}{mv^2 + I\omega^2} = \frac{(2/5)mR^2\omega^2}{mv^2 + (2/5)mR^2\omega^2}$$

$$\frac{K_{rot}}{K_{lin} + K_{rot}} = \frac{2v^2}{5v^2 + 2v^2} = \frac{2}{7}$$

Example: A Ball Rolling Down an Incline



After rolling down a height ***h***, what is the speed of the ball after starting from rest?

Even in the presence of friction, if the ball rolls and does not slip, the total mechanical energy is conserved. (Without friction the ball will only slide.)

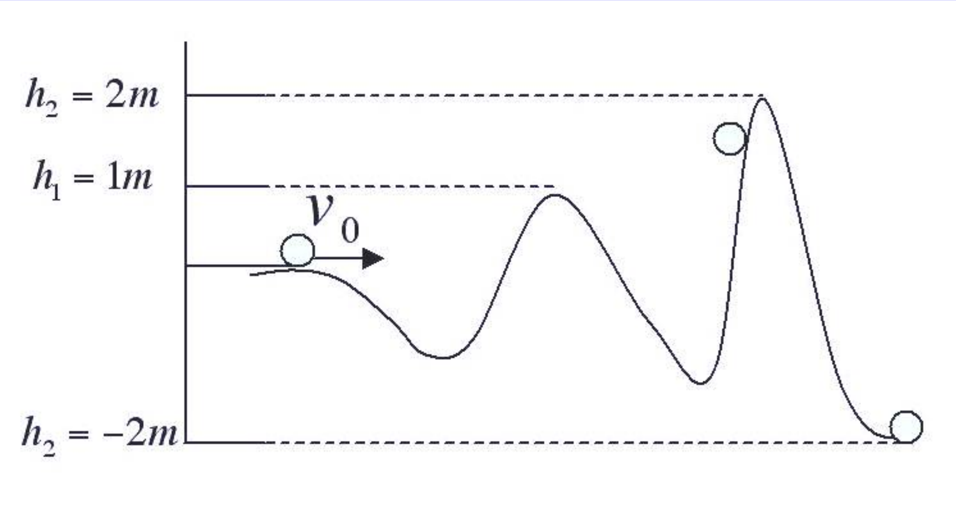
$$\Delta U + \Delta K = 0 \rightarrow -mgh + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = 0$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{5}mR^2\omega^2 = \frac{7}{10}mv^2$$

$$v = \sqrt{\frac{5}{7}2gh} = \sqrt{\frac{5}{7}}\sqrt{2gh} < \sqrt{2gh}$$

The ball has a lower velocity because some of the potential can been converted into rotational kinetic energy!

Flashback: Potential Energy Curves



A ball is rolled onto the path shown. If its initial velocity is ***v* = 6*m*/*s*** does the ball make it over the last hill?

When we first analyzed this problem, we ignored rotational kinetic energy and the ball did not quite make it over the last hill.

Now we ask the question:

$$E = \frac{1}{2}mv_i^2 + \frac{1}{2}I\omega_i^2 \overset{?}{>} mgh_2$$

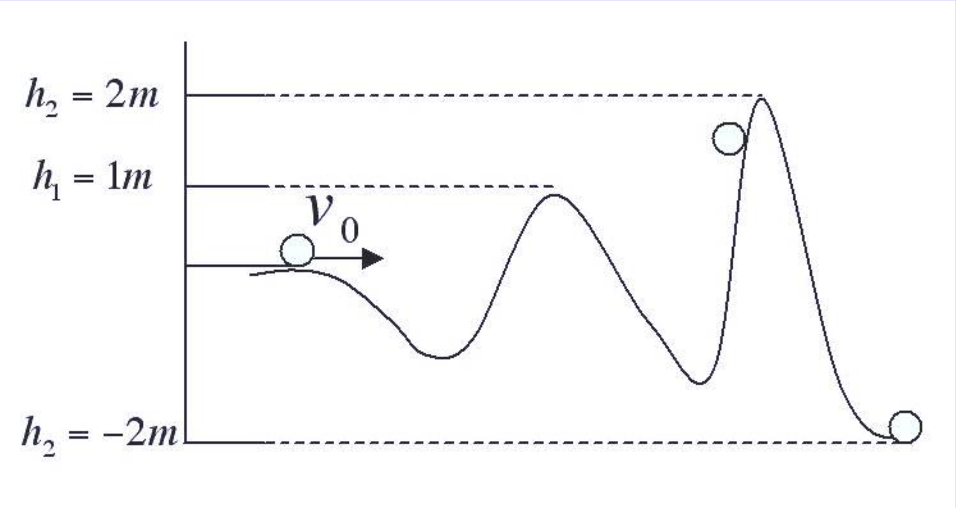
The answer is:

$$E = \frac{1}{2}mv_i^2 + \frac{1}{5}mv_i^2 = \frac{7}{10}mv_i^2 = 25.2m$$

$$E = 25.2m > 2(9.8)m = 19.6m$$

The additional rotational kinetic energy was enough to climb the last hill!

Flashback: Potential Energy Curves



A ball is rolled onto the path shown. If its initial velocity is ***v* = 6*m/s*** what is the final velocity of the ball?

When we first analyzed this problem, the ball did not quite make it over the last hill. Now the conservation of energy yields:

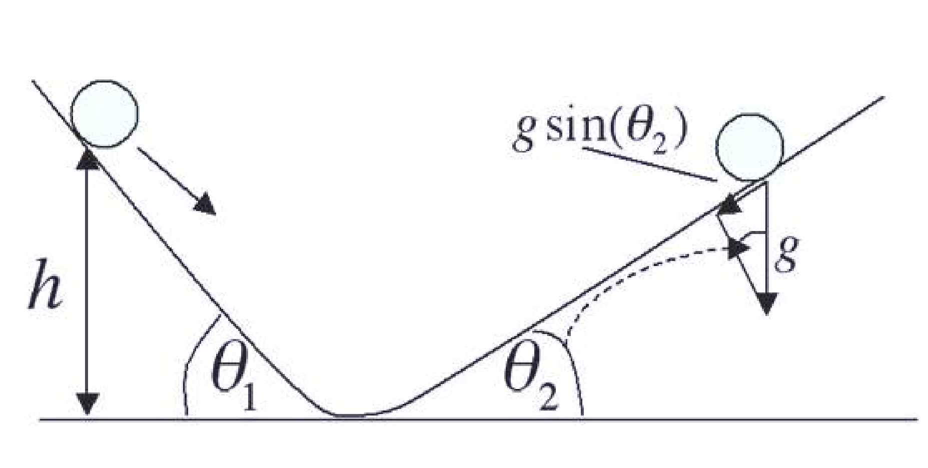
$$E = \frac{1}{2}mv_i^2 + \frac{1}{5}mv_i^2 = \frac{7}{10}mv_i^2 = \frac{7}{10}mv_f^2 - mgh_2$$

$$v_f^2 = v_i^2 + \frac{10}{7}gh_2 \rightarrow v_f = \sqrt{v_i^2 + 10gh_2/7}$$

$$v_f = \sqrt{36 + 10(19.6)/7} = 8.0m/s$$

The conservation of energy including rotational kinetic energy was enough to climb the last hill but it also resulted in a lower velocity than that for a particle with $\boldsymbol{v_i}$ **= 6 *m/s*** that falls **2*m*!**

Balls Rolling on an Inclined Track (again)



Before we found the time that a sliding ball (starting from rest) spent on each side of the track. How will the rotational energy change that ratio?

Since both balls start from the same height, hence they both have the same velocity at the bottom:

$$v_f = \sqrt{\frac{10}{7}gh}$$

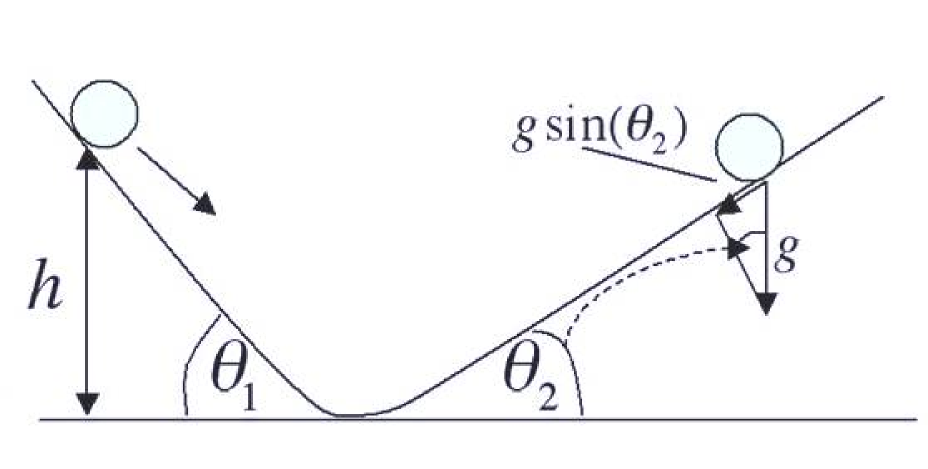
From kinematics we know that ***v = at***.
Hence:

$$v_f = a_1 t_1 = a_2 t_2 \rightarrow \frac{t_1}{t_2} = \frac{a_2}{a_1} = \frac{v_f^2}{2d_2}\frac{2d_1}{v_f^2}$$

$$\frac{t_1}{t_2} = \frac{d_1}{d_2} = \frac{h}{\sin\theta_1}\frac{\sin\theta_2}{h} = \frac{\sin\theta_2}{\sin\theta_1}$$

This is the same ratio that we obtained without rotation.
Both accelerations are reduced by the same factor.

Balls Rolling on an Inclined Track (again)



How long will it take to reach the bottom sliding versus rolling?

Balls accelerate over the same distance. However they obtain different velocities at the bottom. Hence:

From kinematics the ratio of the final velocities leads to:

$$\frac{v_{slid}}{v_{roll}} = \frac{\sqrt{2a_{slid}d}}{\sqrt{2a_{roll}d}} = \sqrt{\frac{a_{slid}}{a_{roll}}} = \sqrt{\frac{7}{5}}$$

$$\frac{t_{slid}}{t_{roll}} = \frac{v_{slid}}{a_{slid}}\frac{a_{roll}}{v_{roll}} = \sqrt{\frac{7}{5}}\frac{5}{7} = \sqrt{\frac{5}{7}}$$

The time on the incline for a sliding mass is less (larger accelerations and velocities) than a rolling ball.