Chapter 13

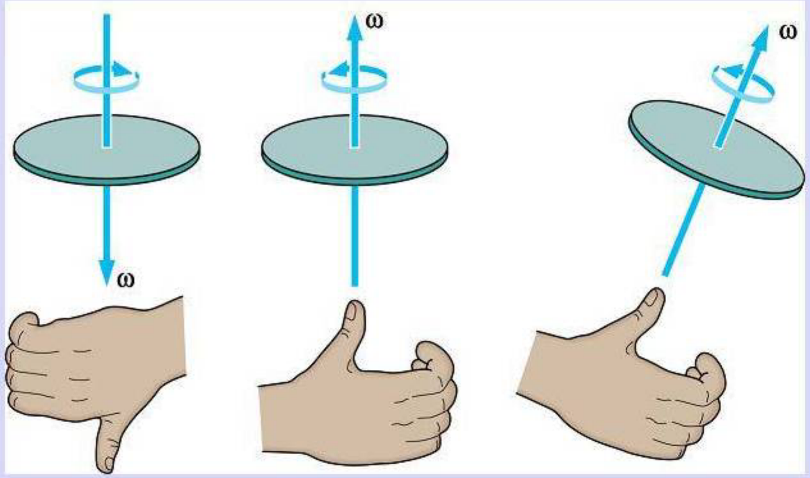
Angular Momentum
Rotational Dynamics

The "right hand rule" and the angular velocity vector

The "right hand rule" refers to the direction of a vector product result being in the **direction of your thumb** as you wrap your fingers around from the **first vector to the second**. This applies to **ALL vector products**.

The angular velocity vector actually comes from the vector product

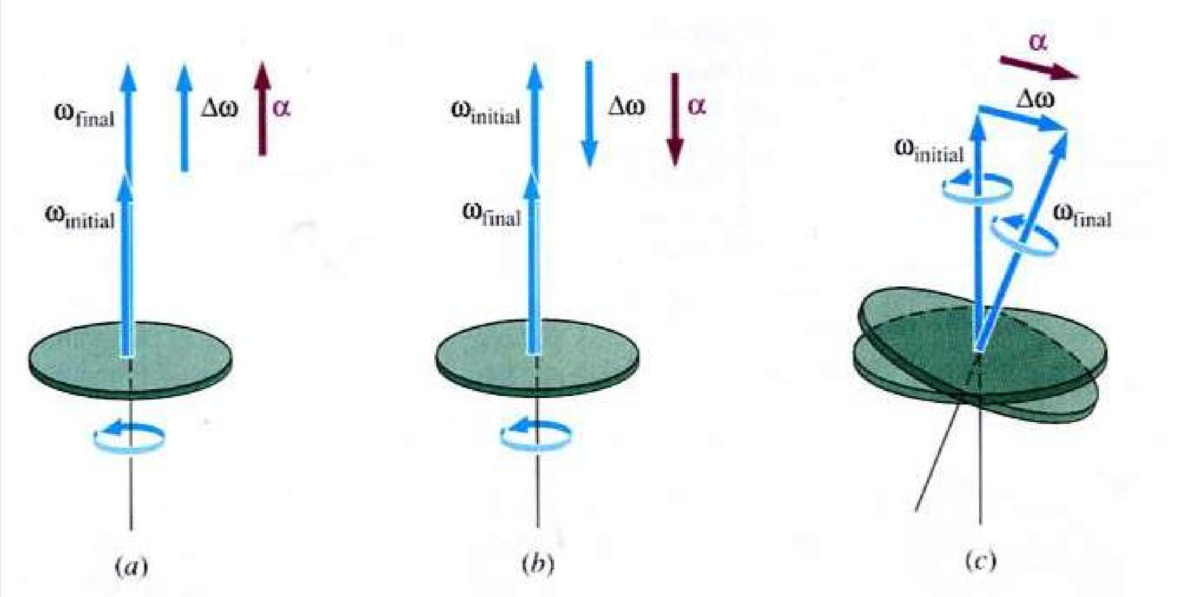
$$\boxed{\vec{\omega} = \frac{\vec{r} \times \vec{v}}{r^2}}$$



and points **along the axis of rotation** according to the right hand rule. Sign of ω obviously relates to direction of rotation.

Angular Velocity and Acceleration as Vectors

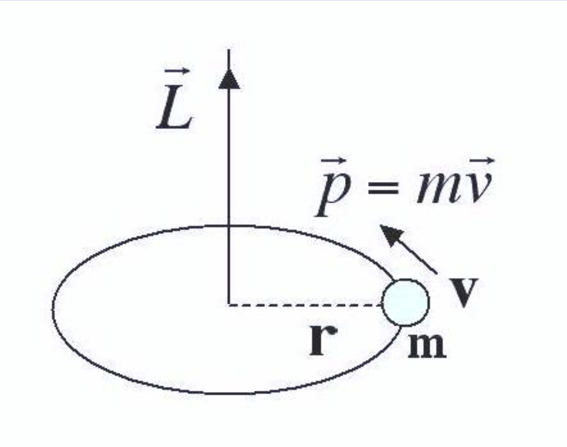
(a) An increase in angular speed alone means that $\boldsymbol{\Delta\omega}$ is parallel to ω. (b) A decrease in angular speed alone means that $\boldsymbol{\Delta\omega}$ is antiparallel to w. (c) If ω only changes direction, then for small $\boldsymbol{\Delta\omega}$ the angular acceleration is perpendicular to ω.



A change in the direction of the angular velocity without a change in angular speed is analogous to the radial acceleration for circular motion. Remember angular velocity and angular acceleration are vectors!

$$\boxed{\vec{\alpha} = \frac{d\vec{\omega}}{dt}}$$

Angular Momentum and Torque



Angular momentum is defined as

$$\boxed{\vec{L} = \vec{r} \times \vec{p}}$$

By analogy with Newton's 2nd for linear motion we stated that:

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

Now consider a system of particles for which the total angular momentum is:

$$\vec{L} = \sum_i \vec{L}_i = \sum_i (\vec{r}_i \times \vec{p}_i)$$

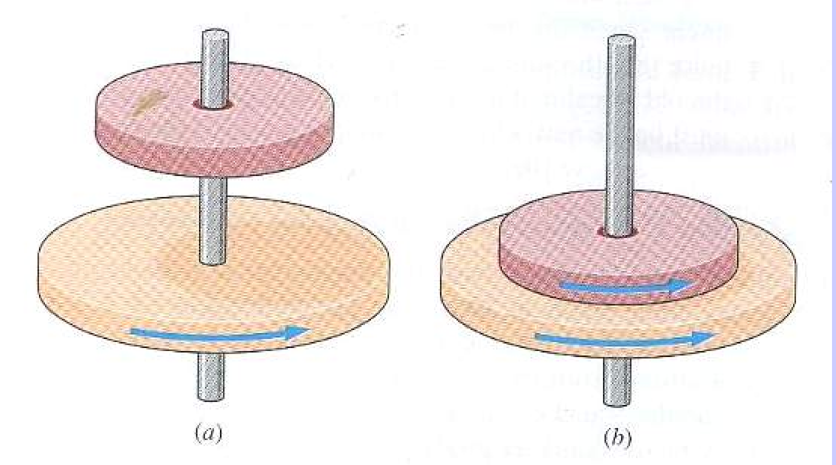
Taking the time derivative of the angular momentum yields:

$$\frac{d\vec{L}}{dt} = \sum_i \left(\frac{d\vec{r}_i}{dt} \times \vec{p}_i + \vec{r}_i \times \frac{d\vec{p}_i}{dt} \right) = \sum_i \vec{r}_i \times \vec{F}_i$$

$$\frac{d\vec{L}}{dt} = \sum_i \vec{\tau}_i = \vec{\tau}_{net}$$

The analogy is complete! Note when there is no external torque, angular momentum is conserved!

Example: Conservation of Angular Momentum

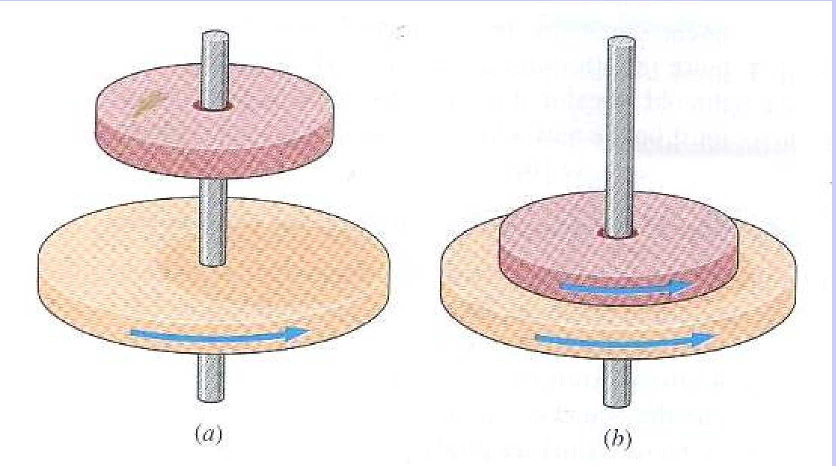


The lower disk has a mass M_1 and a radius R_1, while the upper disk has a mass M_2 and a radius R_2. The lower disk as a initial angular frequency of ω_i. The upper disk drops freely down and frictional forces act to bring the two disks to the same angular speed. (a) What is the final angular speed, ω_f?

In the absence of any external torque, angular momentum is conserved.
For this one dimensional problem this is expressed as:

$$
\begin{aligned}
L_i &= L_f \\
I_1\omega_i &= (I_1 + I_2)\omega_f \\
\omega_f &= \frac{I_1}{I_1 + I_2}\omega_i = \frac{M_1R_1^2}{M_1R_1^2 + M_2R_2^2}\omega_i
\end{aligned}
$$

Example: Conservation of Angular Momentum



The lower disk has a mass $\boldsymbol{M_1}$ and a radius $\boldsymbol{R_1}$, while the upper disk has a mass $\boldsymbol{M_2}$ and a radius $\boldsymbol{R_2}$. The lower disk as a initial angular frequency of $\boldsymbol{\omega_i}$. The upper disk drops freely down and frictional forces act to bring the two disks to the same angular speed. (b) how much energy was lost to friction?

The initial kinetic energy is $\boldsymbol{E_i = \frac{1}{2} I_1 \omega_i^2}$, while the final kinetic energy is:

$$E_f = \frac{1}{2}(I_1 + I_2)\omega_f^2 = \frac{1}{2}(I_1 + I_2)\left(\frac{I_1}{I_1 + I_2}\right)^2 \omega_i^2 = \frac{1}{2}\frac{I_1^2}{I_1 + I_2}\omega_i^2$$

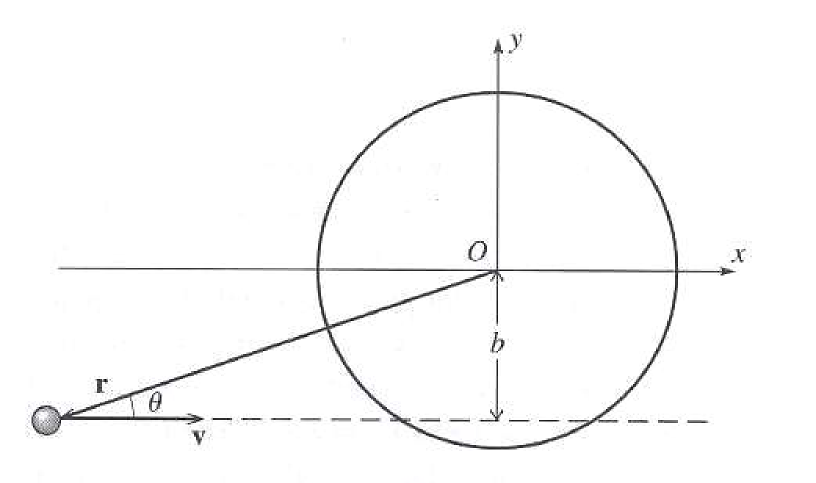
The change in kinetic energy is:

$$\Delta E = E_f - E_i = \frac{1}{2}\frac{I_1^2}{I_1 + I_2}\omega_i^2 - \frac{1}{2}I_1\omega_i^2$$

$$\Delta E = \frac{1}{2}\left(\frac{I_1}{I_1 + I_2} - 1\right)I_1\omega_i^2 = -\frac{1}{2}\left(\frac{I_2}{I_1 + I_2}\right)I_1\omega_i^2$$

Example: Conservation of Angular Momentum

A uniform turntable of mass ***M*** and radius ***R*** is a rest on a frictionless axle. A lump of putty, mass ***m***, approaches the turntable with a velocity ***v*** along a line that passes a distance ***b*** from the center of the turntable and sticks to its edge. Find the resulting angular frequency, ***ω***.



Again angular momentum is conserved. The initial angular momentum is that of the putty:

$$L_i = mvr\sin\theta = mvb$$

The final angular momentum is:

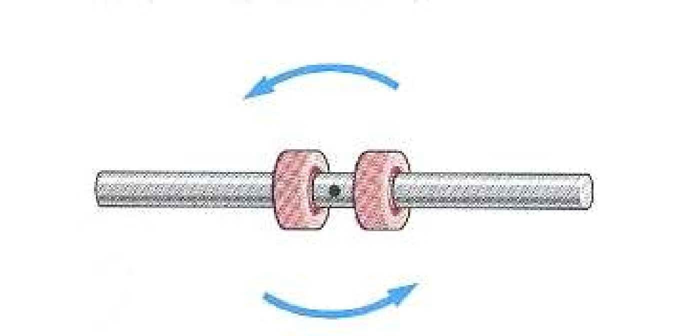
$$L_f = I_{tt}\omega + mR^2\omega = \left(\frac{1}{2}M + m\right)R^2\omega$$

Setting them equal and solving for ***ω***:

$$\omega = \frac{mvb}{m + M/2}\,\frac{vb}{R^2}$$

Example: Conservation of Angular Momentum

Two small beads of mass ***m*** are free to slide on a frictionless rod of mass ***M*** and length ***l*** as shown. Initially the beads are held together as the rods center and the rod is spinning at angular velocity $\boldsymbol{\omega_o}$. Find the expression for the angular velocity, $\boldsymbol{\omega}$, of the rod as a function of the location of the beads.



Again angular momentum is conserved. The initial angular momentum is that of the rod:

$$L_i = I_{rod}\omega_i = \frac{1}{12}Ml^2\omega_i$$

The angular momentum ***L*(*r*)**:

$$L(r) = I_{rod}\omega + 2mr^2\omega = \left(\frac{1}{12}Ml^2 + 2mr^2\right)\omega$$

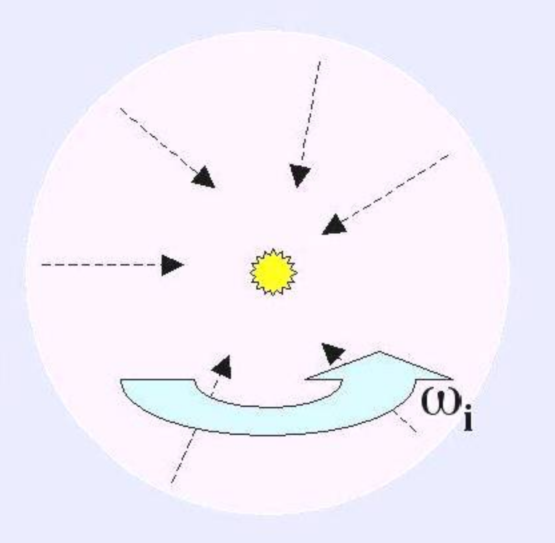
Setting them equal and solving for $\boldsymbol{\omega}$:

$$\omega(r) = \frac{Ml^2}{Ml^2 + 24mr^2}\omega_i$$

How does the angular frequency change when the beads leave the rod?

Example: Conservation of Angular Momentum

A uniform spherical cloud of interstellar gas has a total mass of $\boldsymbol{m = 2x10^{30}kg}$ and a radius of $\boldsymbol{R = 10^{13}m}$. It is rotating with a period of $\boldsymbol{1.4x10^{6}\ yrs}$. If the cloud collapses to form a star of radius $\boldsymbol{r = 7x10^{8}m}$, what is its rotational period.



Again angular momentum is conserved, this time for two uniform spherical objects.

$$I_i\omega_i = I_f\omega_f \rightarrow I_iT_f = I_fT_i$$

$$T_f = \frac{I_f}{I_i}T_i = \frac{r^2}{R^2}T_i = \frac{49\times10^{16}}{10^{26}}1.4\times10^{6} = .00686 yrs$$

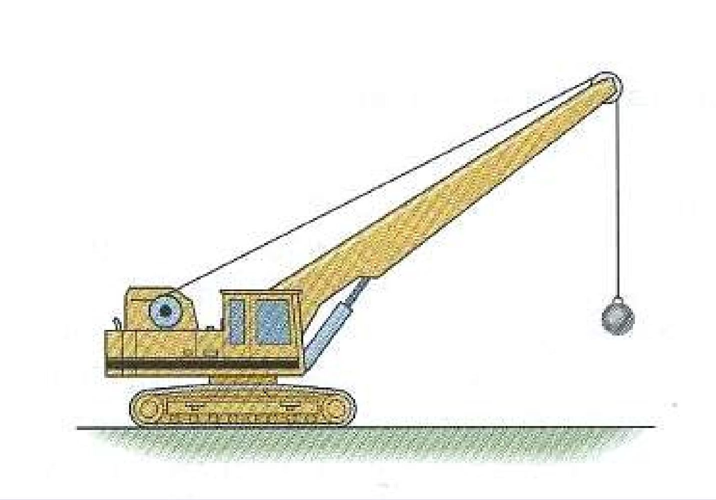
$$T_f = 2.5 days$$

These are the approximate values for our Sun.

The problem is never actually this simple.

Rotational Dynamics

The crane shown in the figure consists of a hollow drum of mass $\boldsymbol{m_d}$ and radius $\boldsymbol{r_d}$ that is driven by an engine to wind up the cable. The cable passes over a solid cylindrical pulley of mass $\boldsymbol{m_p}$ and radius $\boldsymbol{r_p}$. How much torque must the engine apply to the drum to lift the weight, mass $\boldsymbol{m_w}$, with an acceleration $\boldsymbol{a}$?



There are three free-body diagrams with corresponding EOM's:

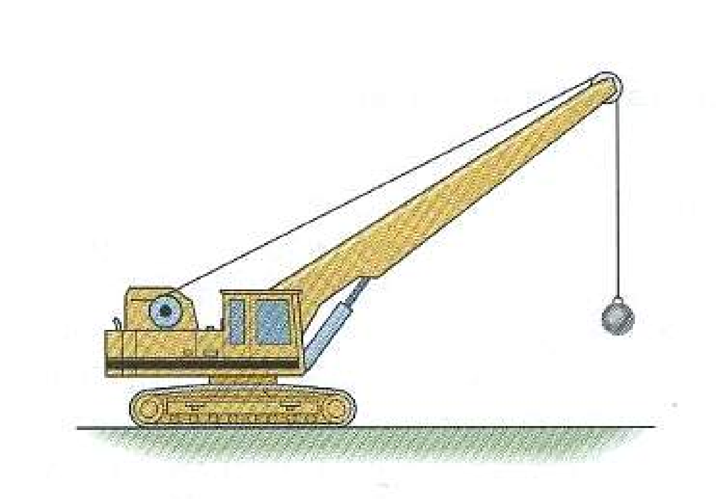
$$
\begin{aligned}
T_1 - m_w g &= m_w a \\
(T_2 - T_1) r_p &= I_p \alpha = I_p \frac{a}{r_p} \\
\tau - T_2 r_d &= I_d \alpha = I_d \frac{a}{r_d}
\end{aligned}
$$

To solve these equations we divide equations 2 and 3 by $\boldsymbol{r_p}$ and $\boldsymbol{r_d}$ respectively and add:

$$\tau / r_d - m_w g = m_w a + I_p \frac{a}{r_p^2} + I_d \frac{a}{r_d^2}$$

Rotational Dynamics

The crane shown in the figure consists of a hollow drum of mass $\boldsymbol{m_d}$ and radius $\boldsymbol{r_d}$ that is driven by an engine to wind up the cable. The cable passes over a solid cylindrical pulley of mass $\boldsymbol{m_p}$ and radius $\boldsymbol{r_p}$. How much torque must the engine apply to the drum to lift the weight, mass $\boldsymbol{m_w}$, with an acceleration $\boldsymbol{a}$?



Substituting in the values for the moment of inertias:

$$I_p = \frac{1}{2} m_p r_p^2 \quad \text{and} \quad I_d = m_d r_d^2$$

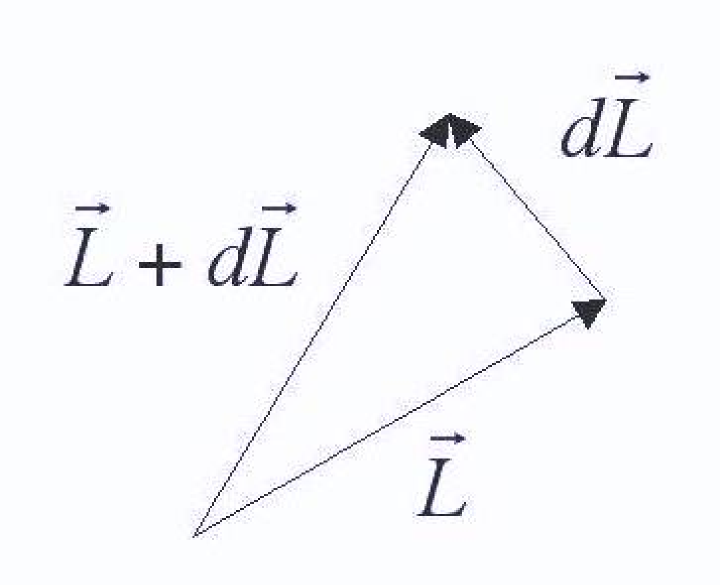
Yields:

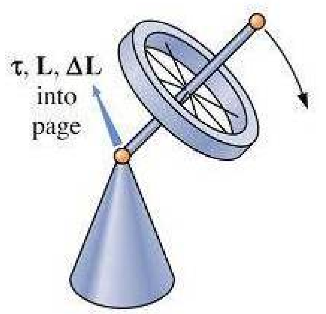
$$\tau = m_w r_d (g + a) + \left(\frac{1}{2} m_p r_p + m_d r_d \right) a$$

The first term on the RHS is what you would obtain for massless pulley and drum. Finite mass contributes the additional terms due to rotational inertia.

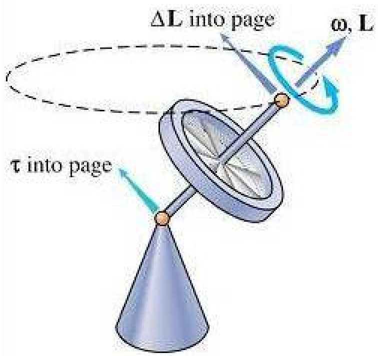
3D Angular Momentum: Precession

If an applied **torque** imparts angular momentum **perpendicular** to a spinning object's **angular momentum**, and the rotational axis is fixed at a point, the magnitude of the angular momentum won't change, only the direction. This is called **precession**.



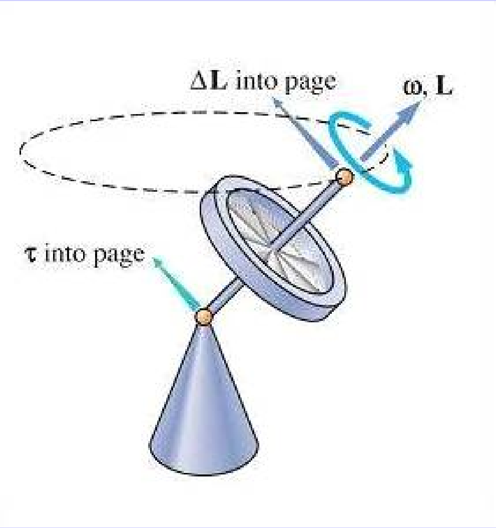


Without the spin, the gyroscope falls over and the torque is perpendicular to the rotation of the fall.

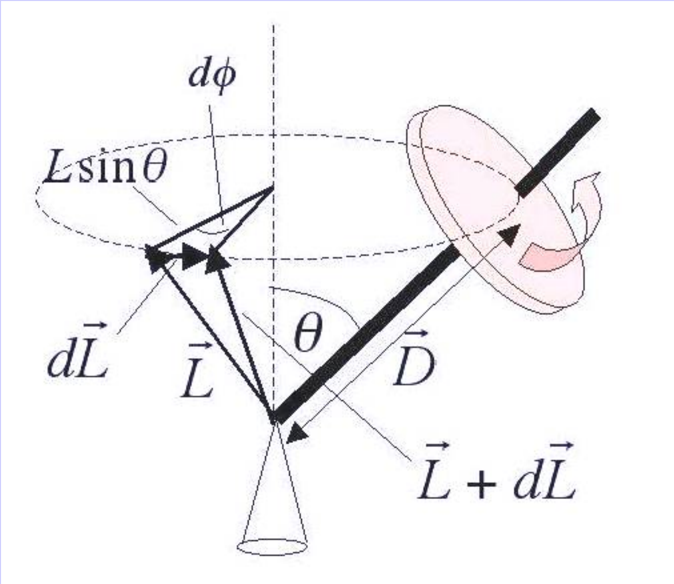


With the spin, the torque from the pull of gravity is perpendicular to the angular momentum of the spin. This causes L to change direction around the pivot point.

3D Angular Momentum: Precession



What is the rate of precession for the gyro? First we determine the torques and the angular momentum, then apply $\boldsymbol{\tau} = \boldsymbol{dL/dt}$. Consider the figure below:



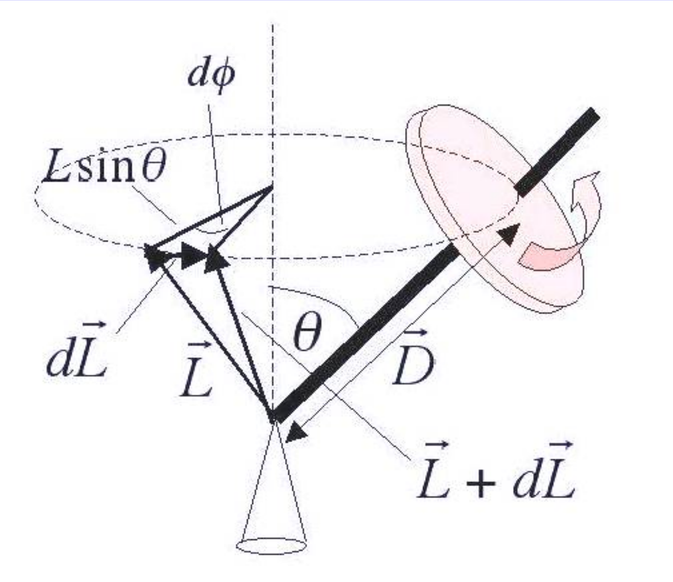
Assume that the rotational axis is massless. Then the torque due to gravity is:

$$\vec{\tau} = \vec{D} \times m\vec{g} \rightarrow \tau = mgD\sin\theta$$

A small change in the angular momentum, ***dL***, is:

$$dL = L\sin\theta d\phi$$

3D Angular Momentum: Precession



What is the rate of precession for the gyro? First we determine the torques and the angular momentum, then apply $\boldsymbol{\tau = dL/dt}$.

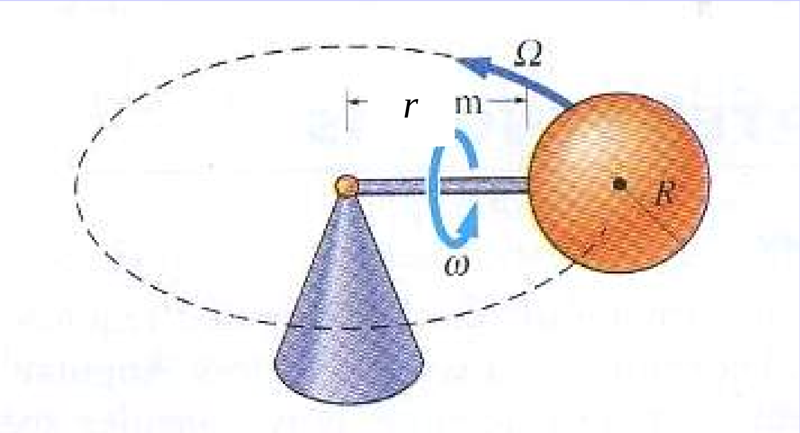
$$dL = \tau dt = mgD\sin\theta dt = L\sin\theta d\phi$$

Solving for the precession rate, $\boldsymbol{\Omega = d\phi/dt}$:

$$\boxed{\Omega = \frac{d\phi}{dt} = \frac{mgD}{L}}$$

The precession rate is independent of $\boldsymbol{\theta}$!

Example: Precession



A uniform solid sphere is mounted on a shaft of negligible mass and length r. The shaft rests on a frictionless pivot. The sphere is spinning at an angular rate $\boldsymbol{\omega}$ while precessing at in a horizontal circle at a rate $\boldsymbol{\Omega}$. Find the radius, ***R***, of the sphere.

The precession rate is given by:

$$\Omega = \frac{d\phi}{dt} = \frac{mgD}{L}$$

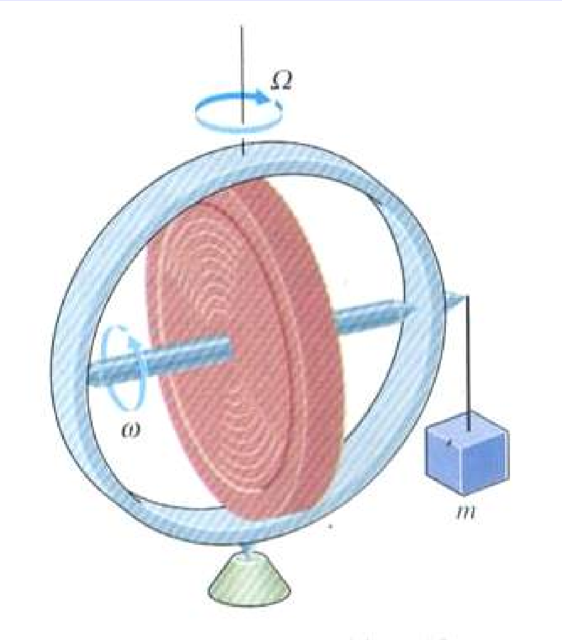
For this problem $\boldsymbol{D = r + R}$ and $\boldsymbol{L = I\omega = 2/5\, mR^2\, \omega}$. Using these values in the expression for the rate of precession:

$$\Omega = \frac{g(r+R)}{(2/5)R^2\omega} \rightarrow gr + gR = \frac{2}{5}\omega\Omega R^2$$

The solution for this quadratic:

$$R = \frac{5g}{4\omega\Omega}\left(1 + \sqrt{1 + 8\omega\Omega r/5g}\right)$$

Example: Precession



Initially a gyroscope is spinning with angular speed ω and is perfectly balanced so that it is not precessing. When a mass m is hung from the frame the gyro precesses about the vertical axis at a rate $\mathbf{\Omega}$. Find the rotational inertia of the gyro.

The torque due to gravity is: $\tau = mgR$

The torque is the rate of change of the angular momentum. Hence:

$$\tau = mgR = \frac{dL}{dt} = L\Omega = I\omega\Omega$$

Solving for ***I***: $I = \dfrac{mgR}{\omega\Omega}$