Chapter 14

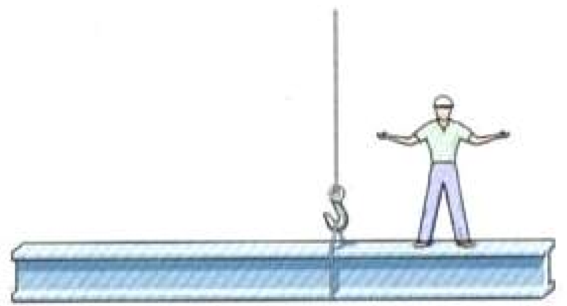
Static Equilibrium II

Chapter 15

Oscillatory Motion

Conditions for Equilibrium

A body is in **static** equilibrium when it is stationary and when both "the net external force and the net external torque are zero!"



(a)

$$\sum_i \vec{F}_i = 0$$

and

$$\sum_i \vec{\tau}_i = 0$$

If the tension in the support cable does not act at the center of "gravity" for the beam and the worker then the beam will rotate! Not good!

For static equilibrium we also have

$$\sum_i \vec{p}_i = 0$$

Center of Gravity

The center of gravity is the point at which the gravitational force acts in balance.

Consider the calculation of the torque acting on a group of point masses due to gravitational force.

$$\vec{\tau}_{net} = \sum_i \vec{\tau}_i$$

For each point mass if the acceleration of gravity is g (uniform) the force is $m_i g$

$$\vec{\tau}_{net} = \sum_i \vec{r}_i \times m_i \vec{g}$$

The mass is scalar and can be moved, and we can multiply and divide by the total mass M without changing τ.

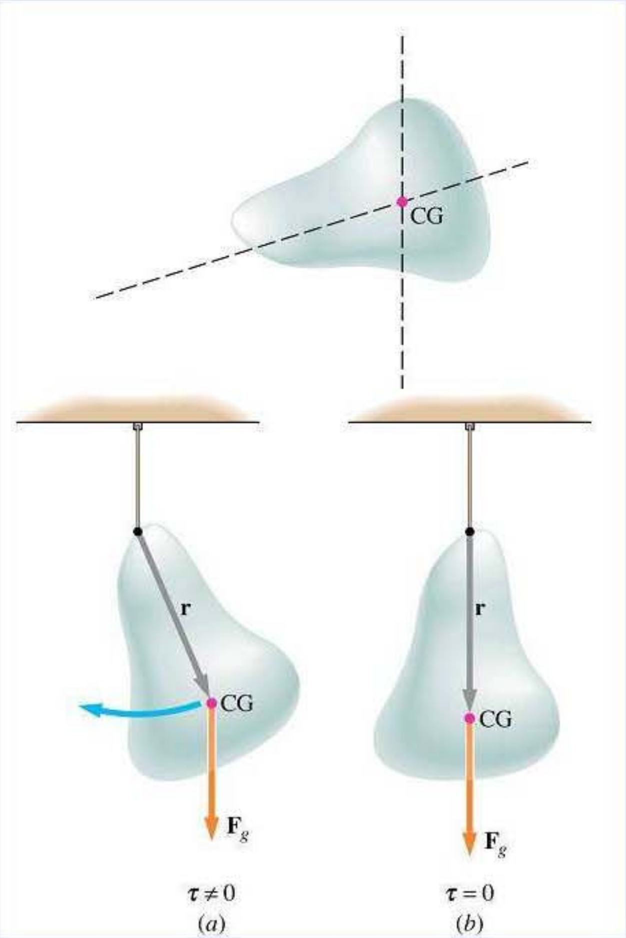
$$\vec{\tau}_{net} = \sum_i \frac{m_i \vec{r}_i}{M} \times M\vec{g}$$

The first term is the center of mass. The second is the net gravitational force on all the objects.

$$\boxed{\vec{\tau}_{net} = \vec{R}_{cm} \times M\vec{g}}$$

Thus if g is uniform, R_{cm} is the center of gravity.

Example: Hanging Equilibrium



For a hanging solid object the center of gravity will lie directly under the hanging point when in equilibrium.

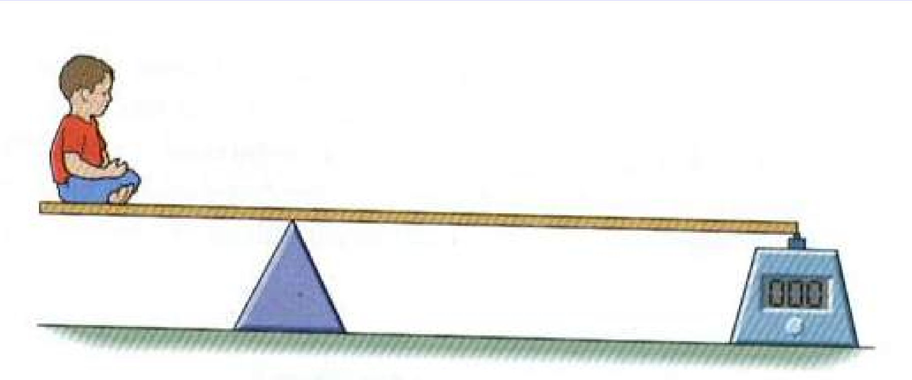
If the object is hung from two pivot points, the vertical lines beneath them will intersect at the center of gravity.

The center of gravity will behave like a point mass on a pendulum for a hanging body. When the center of gravity is not beneath the pivot, a torque acts on the body, and it is not in equilibrium

Later we will define the equilibrium condition mathematically as
For the pendulum, x is the arclength along the swing.
The derivative of gravitational potential at the bottom is zero.

$$\frac{dU}{dx} = 0$$

Example: Teeter-Totter



Consider a teeter-totter as shown in the figure. If a child of mass ***m*** is a distance ***x*** from the fulcrum and the weight of mass ***M*** is a distance ***y***, what is the ratio of ***x/y*** for the teeter-totter to be in equilibrium?

As usual we must first choose an origin from which to find the torques. First we consider the fulcrum as the origin:

$$mgx = Mgy \rightarrow x/y = M/m$$

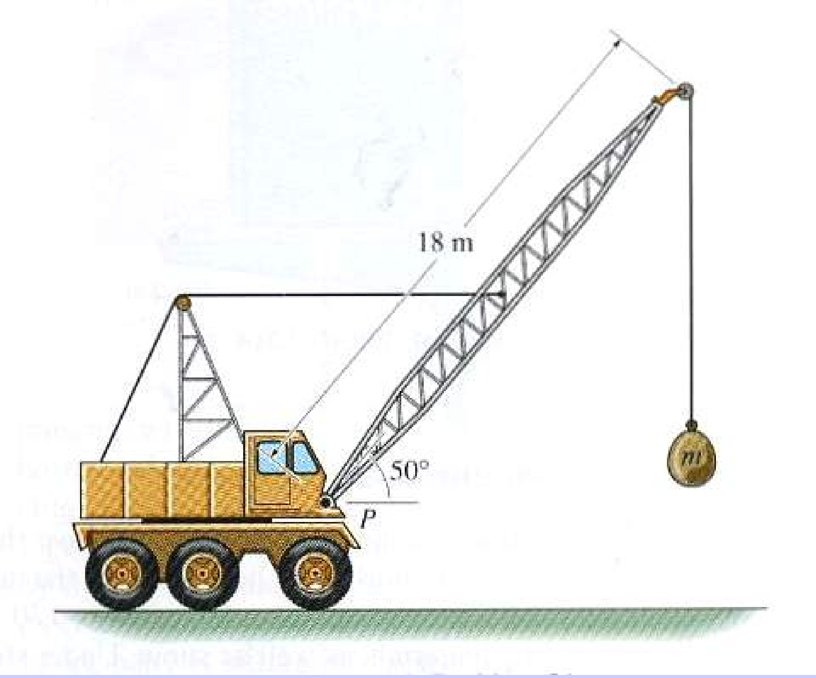
The larger the mass ***M***, the further the child must sit from the fulcrum.

Now choose the location of the weight as the origin. The normal force at the fulcrum must equal the weight of the child and the mass ***M***. Hence:

$$mg(x+y) = (M+m)gy \rightarrow mx = My$$

This leads to the same result. What if the board is at an angle $\boldsymbol{\theta}$ wrt the horizontal?

Example: Crane Equilibrium



Consider the crane as shown in the figure. The boom is supported by a cable attached to its center point. Find the tension in the cable when angle of the boom is **50º** above the horizontal and the mass of the boom is ***1700kg*.** The hanging mass is ***2200kg***.

Summing torques about the pivot point ***P***:

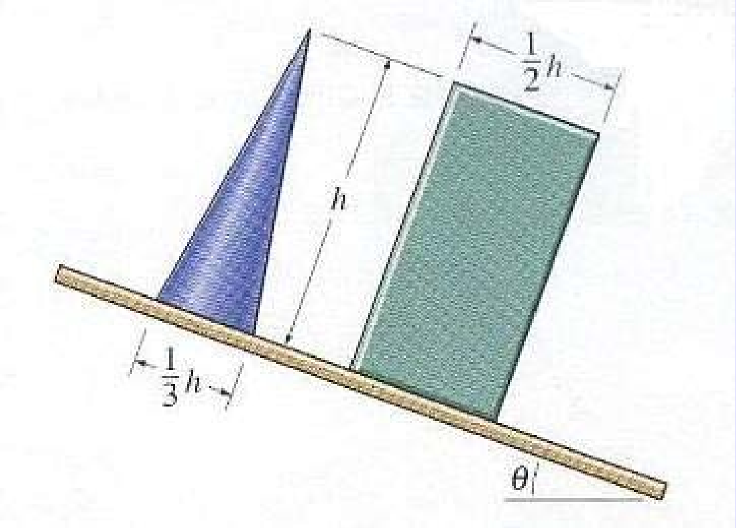
$$T(9\sin 50^\circ) - 1700g(9\cos 50^\circ) - 2200g(18\cos 50^\circ) = 0$$

Solving for ***T***:

$$T = \frac{4400 + 1700}{\sin 50^\circ}(9.8\cos 50^\circ) = 50kN$$

Again the normal forces at the pivot balance all of the forces.

Example: Tip or Slide, that is the Question.



A rectangular block, twice as high as it is wide, is resting on an incline with an angle $\boldsymbol{\theta}$ and a coefficient of static friction $\boldsymbol{\mu}$. What condition on $\boldsymbol{\mu}$ will cause the block to slide before it tips?

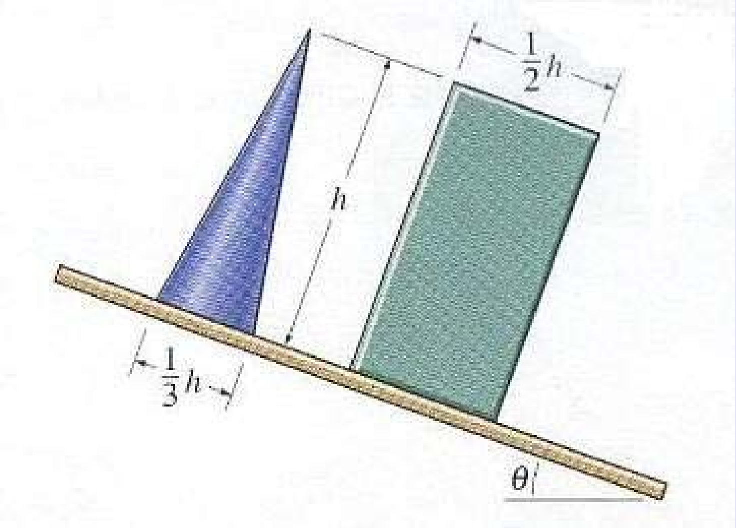
The maximum torque resulting from the normal force occurs when the normal force acts at the lowest edge of the block. If the torque from the gravitational force is greater than this restoring torque, the block will tip!

The torque about the lowest edge must vanish. This means that at the steepest angle prior to tipping the center of mass must lie directly above this point. This angle is given by **tan** $\boldsymbol{\theta}$ **= ½**. For the block not to slide:

$$\mu mg\cos\theta - mg\sin\theta \geq 0 \rightarrow \mu \geq \tan\theta = 1/2$$

Hence if $\boldsymbol{\mu}$ is less than ½ the block will slide before it tips!

Example: Tip or Slide, that is the Question.



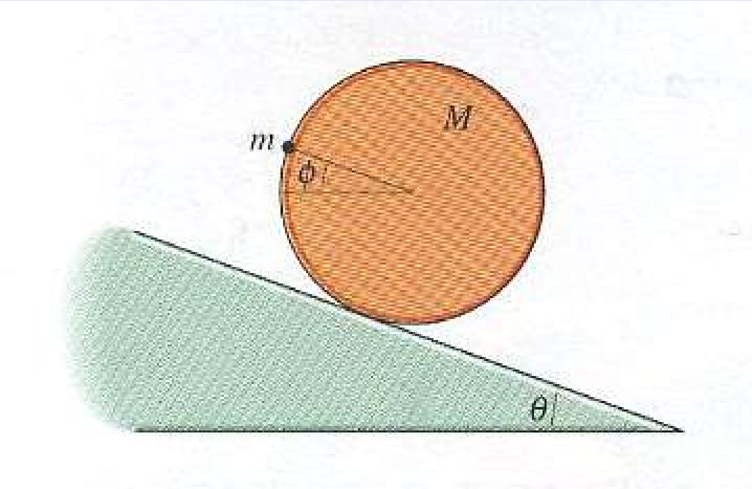
A cone, three times as high as it is wide, is resting on an incline with an angle $\boldsymbol{\theta}$ and a coefficient of static friction $\boldsymbol{\mu}$. What condition on $\boldsymbol{\mu}$ will cause the cone to slide before it tips?

Again the torque about the lowest edge must vanish. For a cone the center of mass is ***h*/4** above the base. At steepest angle prior to tipping the center of mass must lie directly above the lowest edge. This angle is given by **tan $\boldsymbol{\theta}$ = 2/3**. For the block not to slide:

$$\mu mg\cos\theta - mg\sin\theta \geq 0 \rightarrow \mu \geq \tan\theta = 2/3$$

Hence if $\boldsymbol{\mu}$ is less than **2/3** the block will slide before it tips!

Example: Weighted Disk on Incline



Consider a uniform disk of mass ***M*** and radius ***R*** with a weight of mass ***m*** on the rim. The disk is on an incline of angle $\boldsymbol{\theta}$ with friction (rolls and not slides). Find the angle $\boldsymbol{\phi}$ as shown such that the disk is in static equilibrium.

In equilibrium the forces are balanced by the normal force and the frictional force.

To balance torques it is convenient to choose the point of contact between the incline and the disk as the origin. The moment arm to the center of the disk is:

$$\vec{R} = R\sin\theta\,\hat{i} + R\cos\theta\,\hat{j}$$

The moment arm for the weight on the rim is:

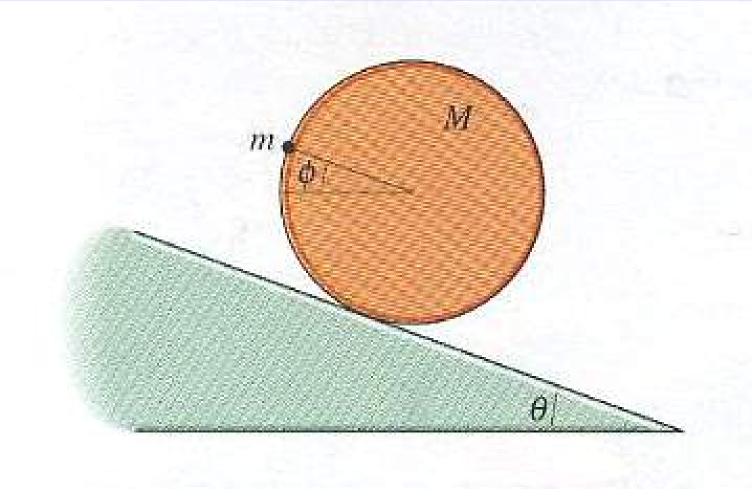
$$\vec{r} = \vec{R} - R\cos\phi\,\hat{i} + R\sin\phi\,\hat{j}$$

The expression for the total torque:

$$\vec{\tau} = \vec{R} \times \left(-Mg\hat{j}\right) + \vec{r} \times \left(-mg\hat{j}\right) = 0$$

$$\vec{\tau} = -MgR\sin\theta\,\hat{k} - mgR(\sin\theta - \cos\phi)\hat{k} = 0$$

Example: Weighted Disk on Incline



Consider a uniform disk of mass ***M*** and radius ***R*** with a weight of mass ***m*** on the rim. The disk is on an incline of angle $\boldsymbol{\theta}$ with friction (rolls and not slides). Find the angle $\boldsymbol{\phi}$ as shown such that the disk is in static equilibrium.

The expression for the net torque:

$$\vec{\tau} = -MgR\sin\theta\hat{k} - mgR(\sin\theta - \cos\phi)\hat{k} = 0$$

Solving for $\boldsymbol{\phi}$:

$$m\cos\phi = m\sin\theta + M\sin\theta \rightarrow \cos\phi = (1 + M/m)\sin\theta$$

$$\phi = \cos^{-1}((1 + M/m)\sin\theta)$$

The normal force is what is required for equilibrium normal to the incline. The friction force results in equilibrium parallel to the incline. It is convenient to choose the origin for the moment arms at the point of contact between the incline and the disk so that it is not required to know these forces.

Stability of Equilibria

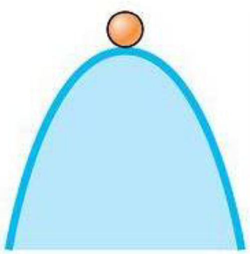
An unperturbed body in equilibrium will have no translational or rotational acceleration. A perturbation adds a small amount displacement to the system, and the response of the system to that displacement defines the stability.

(*a*)

$$\frac{d^2U}{dx^2} > 0$$

stable

A perturbed body in a stable equilibrium will return to that equilibrium state as the energy of the perturbation dissipates or damps.



(*b*)

$$\frac{d^2U}{dx^2} < 0$$

unstable

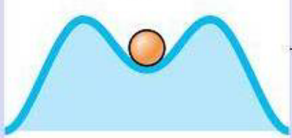
A perturbed body in unstable equilibrium will cause the perturbation to grow.

(*c*)

$$\frac{d^2U}{dx^2} = 0$$

neutral

Neutral equilibrium will remain as perturbed.

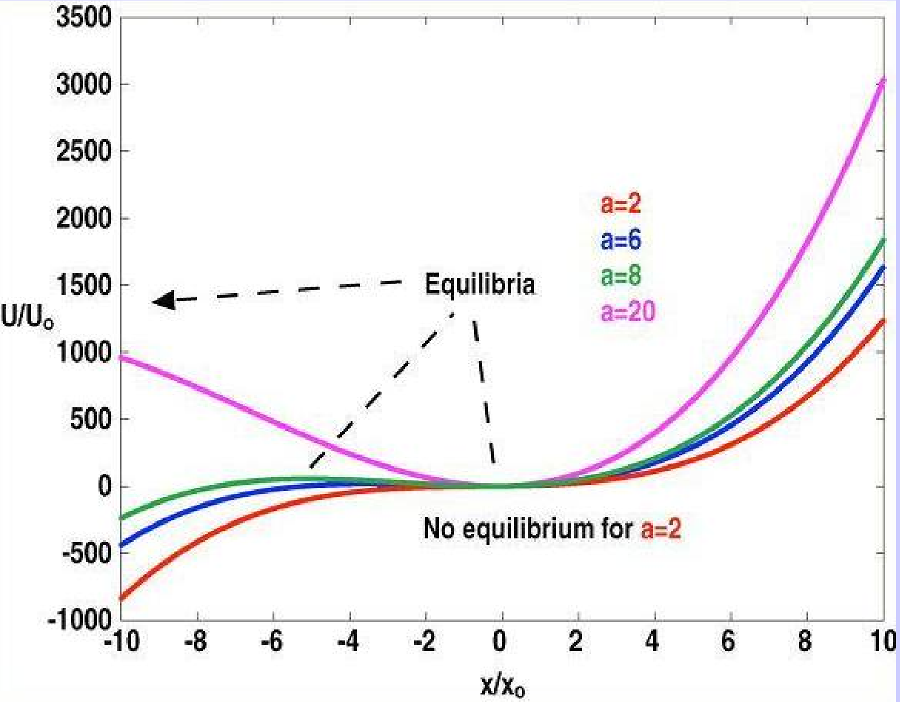


(*d*)

metastable

Metastable equilibrium has a threshold for instability. (locally stable)

Example: Equilibrium Conditions



The potential energy as a function of ***x*** is:

$$U(x) = U_o\left(\frac{x^3}{x_o^3} + a\frac{x^2}{x_o^2} + 4\frac{x}{x_o}\right)$$

For what values of a will there be two static equilibria? Comment on the stability of these equilibria.

Equilibrium is determined by ***dU/dx* = 0**. Taking the derivative of ***U***:

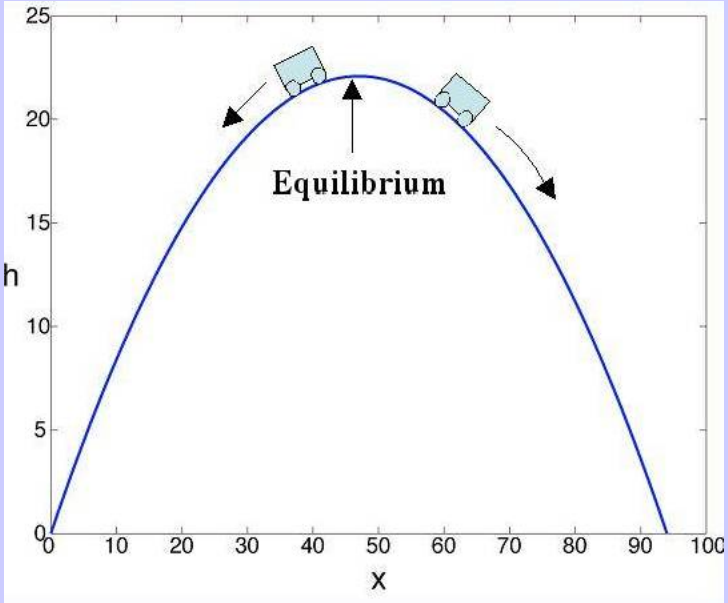
$$\frac{dU}{dx} = \frac{U_o}{x_o}\left(3\frac{x^2}{x_o^2} + 2a\frac{x}{x_o} + 4\right) = 0 \rightarrow \frac{x_{1,2}}{x_o} = \frac{1}{3}\left(-a \pm \sqrt{a^2 - 12}\right)$$

For two real roots ***a*²** **> 12**, ***a*** **~ 3.5**. Taking the second derivative yields:

$$\frac{d^2U(x_{1,2})}{dx^2} = \frac{U_o}{x_o^2}\left(6\frac{x_{1,2}}{x_o} + 2a\right) = \pm 2\frac{U_o}{x_o^2}\sqrt{a^2 - 12}$$

Right point is metastable.
Left point is unstable.

Example: Unstable Equilibrium



The top of a roller coaster track is described by:

$$h = .94x - .01x^2$$

Here ***h*** and ***x*** are measured in meters. (a) Find the point on the track where the cars are in static equilibrium. (b) Is the equilibrium stable or unstable? (c) What is the height of the track at the equilibrium point?

(a) The potential energy is given by ***U(x) = mgh***. For equilibrium ***dU*/*dx* = 0**.

$$\frac{dU}{dx} = mg(.94 - .02x) = 0 \rightarrow x = 47m$$

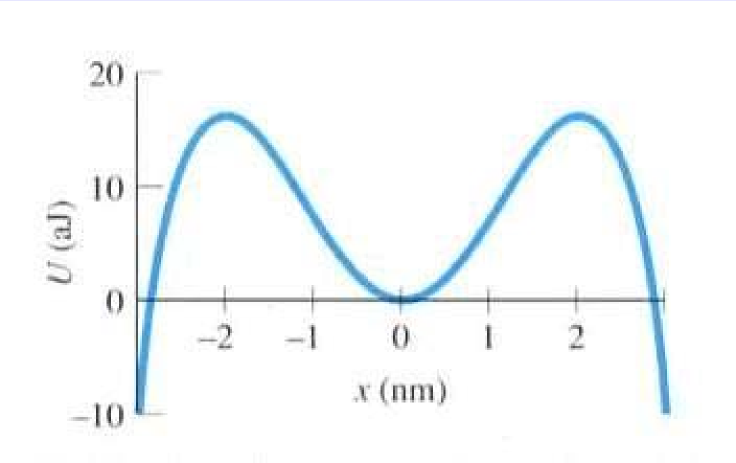
(b) The second derivative determines stability:

$$\frac{d^2U}{dx^2} = -.02mg < 0 \rightarrow \text{unstable}$$

(c) The height at ***x* = 47** is:

$$h = .94(47) - .01(47)^2 = 22m$$

Example: Multiple Equilibrium Points



In a certain semiconductor the local potential is:

$$U(x) = ax^2 - bx^4$$

Here ***x*** is the position of the electron in nm, and ***U*** is its potential energy in **a*J* = 10^{-18}*J***. The constants are ***a = 8 aJ/nm^2*** and ***b = 1 aJ/nm^3***. Locate the equilibrium positions for the electron and describe their stability.

For equilibrium ***dU*/*dx* = 0**:

$$\frac{dU}{dx} = 2ax - 4bx^3 = 0$$

$$x = 0, \ \pm\sqrt{a/2b} = 0, \ \pm 2nm$$

The second derivative determines stability:

$$\frac{d^2U(x)}{dx^2} = 2a - 12bx^2$$

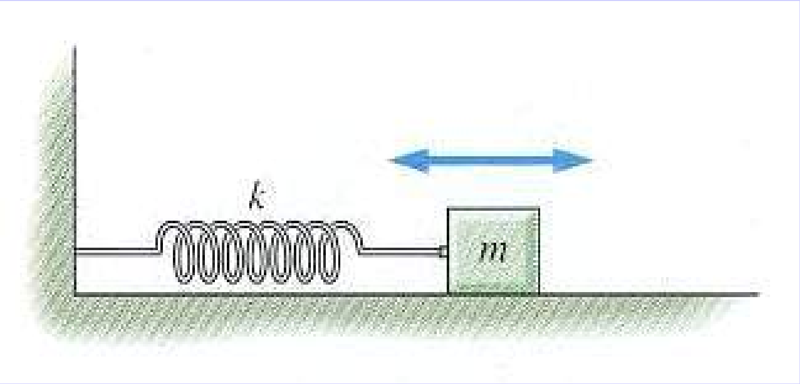
$$\frac{d^2U(x=0)}{dx^2} = 2a = 16 > 0 \rightarrow \text{stable}$$

$$\frac{d^2U(x=\pm 2)}{dx^2} = 16 - 12(4) < 0 \rightarrow \text{unstable}$$

Chapter 15

Oscillatory Motion

Simple Harmonic Motion



Simple harmonic motion results when an object is subject to a linear restoring force and is called **simple harmonic motion**, SHO.

Mathematically such a force is described as:

$$F = -kx$$

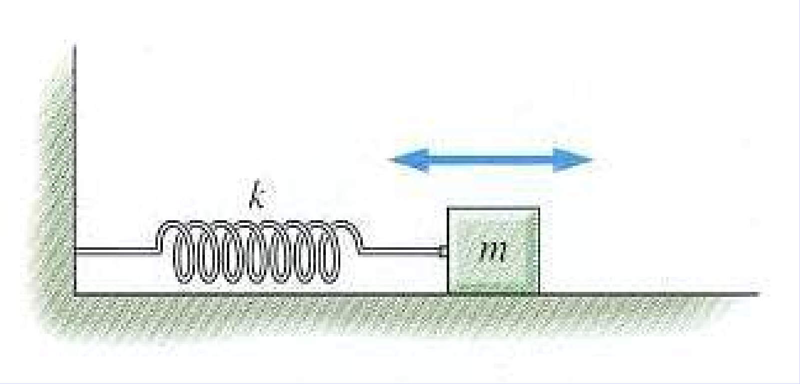
The is the force exerted by an ideal spring of spring constant ***k***.
From Newton's 2nd we can write:

$$F = m\frac{d^2x}{dt^2} = -kx$$

An object experiencing such a force means that when it is displaced from equilibrium there is a force proportional to the distance from equilibrium that accelerates the object back towards its equilibrium position.

How do we describe such motion?

Simple Harmonic Motion



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From Newton's 2nd:

$$F = m\frac{d^2x}{dt^2} = -kx$$

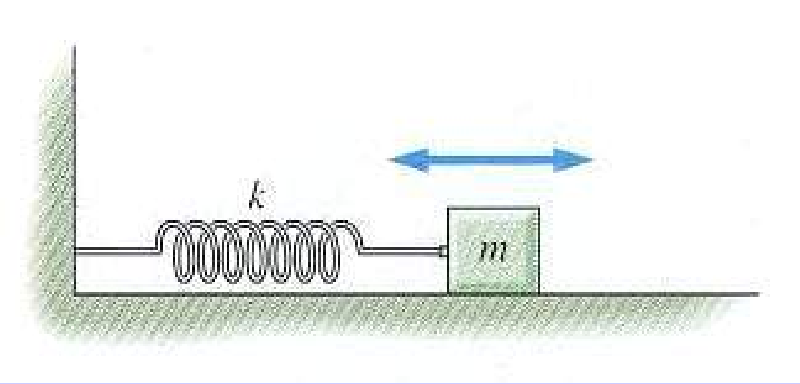
The solution to this equation is of the form ***x = A cos(ωt) + B sin(ωt)***. To see this we simply substitute this function into the "differential equation" represented by Newton's 2nd. Taking the derivatives:

$$\frac{dx}{dt} = -\omega A\sin\omega t + \omega B\cos\omega t$$

$$\frac{d^2x}{dt^2} = -\omega^2 A\cos\omega t - \omega^2 B\sin\omega t$$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

Simple Harmonic Motion



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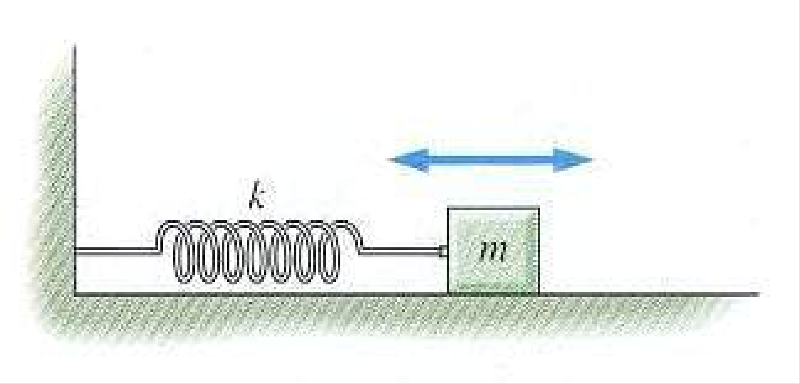
Substituting this result into Newton's 2nd:

$$F = m\frac{d^2x}{dt^2} = -m\omega^2 x = -kx$$

This equation is satisfied if $\boldsymbol{\omega^2 = k/m}$ **!** But what is this ω ? We know from the form of $\boldsymbol{x}$ that when $\boldsymbol{\omega T = 2\pi}$ the sine or cosine function returns to its value it had when $\boldsymbol{t = 0}$. So $\boldsymbol{T}$ is the period of the sine or cosine function. Hence $\boldsymbol{\omega = 2\pi / T = 2\pi f}$, where $\boldsymbol{f}$ is the frequency of oscillation.

Just as with angular velocity, ω is measured in radians per second. When $\boldsymbol{\omega T = 2\pi}$ radians the trignometric functions recycle. The motion is **oscillatory** with frequency $\boldsymbol{f = \omega/2\pi}$ where $\omega = \sqrt{k/m}$.

Simple Harmonic Motion



Simple harmonic motion results when an object is subject to a linear restoring force and is called **simple harmonic motion**, SHO. The general solution is then:

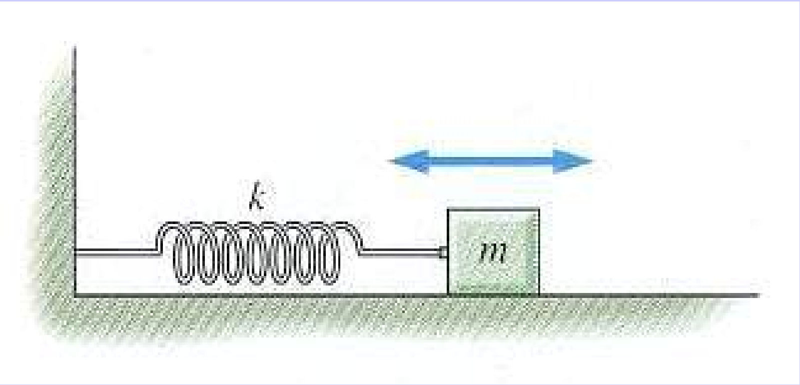
$$x = A\cos\omega t + B\sin\omega t \quad \text{with} \quad \omega = \sqrt{k/m}$$

What about the unknown constants ***A*** and ***B***? They are determined by the initial conditions. For example if at $\boldsymbol{t = 0}$ the system satisfies $\boldsymbol{x = x_o}$ and $\boldsymbol{v = 0}$, then:

$$x(t = 0) = A = x_o \quad \text{and} \quad v = dx/dt = \omega B = 0$$

The solution for this initial condition becomes: $x(t) = x_o \cos\omega t$

Simple Harmonic Motion



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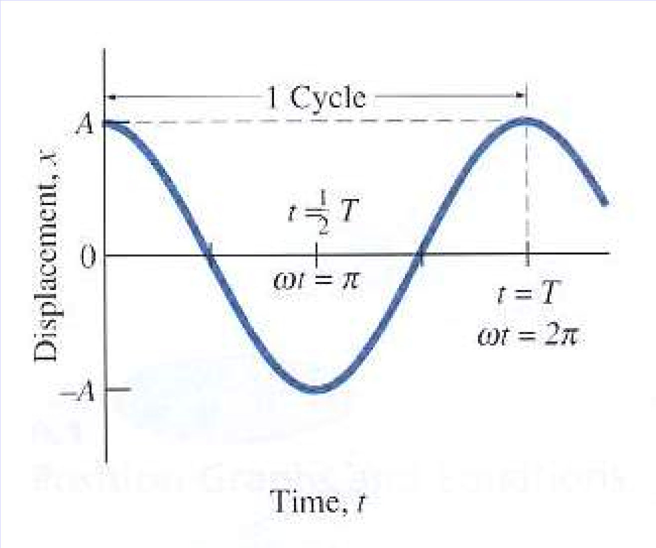
Graphically this solution is given by:



Does the expression for angular frequency

$$\omega = \sqrt{k/m}$$

make sense? What happens when the spring constant ***k*** increases, what about the dependence on ***m***?