PHYSICS 2A

FINAL REVIEW

1-D Kinematics with Uniform Acceleration

Velocity as a function of ***t***:

$$v(t) - v_o = \int_0^t a\,dt = at \rightarrow v(t) = v_o + at$$

Position as a function of ***t***:

$$x(t) - x_o = \int_0^t v(t)\,dt = \int_0^t (v_o + at)\,dt$$

$$x(t) = x_o + v_o t + \frac{1}{2}at^2$$

Substituting for ***t***:

$$t = \frac{v(t) - v_o}{a}$$

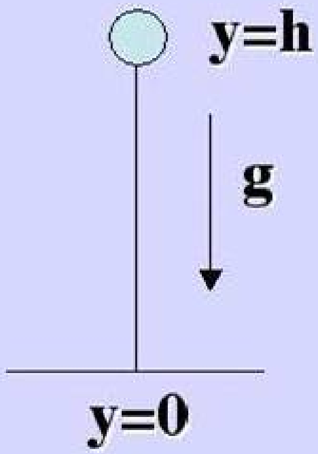
We can express ***v(x):***

$$\Delta x(t) = v_o \frac{v(t) - v_o}{a} + \frac{1}{2}a\left(\frac{v(t) - v_o}{a}\right)^2$$

$$2a\Delta x(t) = v^2(t) - v_o^2$$

Example: Gravity Accelerates Approximately Constantly

Earth's gravitational field accelerates all free bodies at approximately 9.8m/s^2 downwards



How long does it take for the body to reach the ground from a height h?

$$y_f = y_i + V_i t + \frac{1}{2} a t^2$$

$$0 = h + 0 + \frac{1}{2}(-g)t^2$$

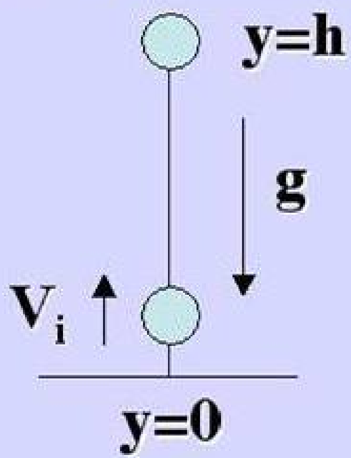
$$t = \sqrt{\frac{2h}{g}}$$

Thus a body of any mass falling from 2m would land in 0.64s

Example: The Velocity Needed to Throw a Stone to Height h

A stone is thrown directly upward, reaches a height h, and falls.

V=0 momentarily



Using equation (2) (without explicit time)

$$V_f^2 - V_i^2 = 2a(y_f - y_i)$$

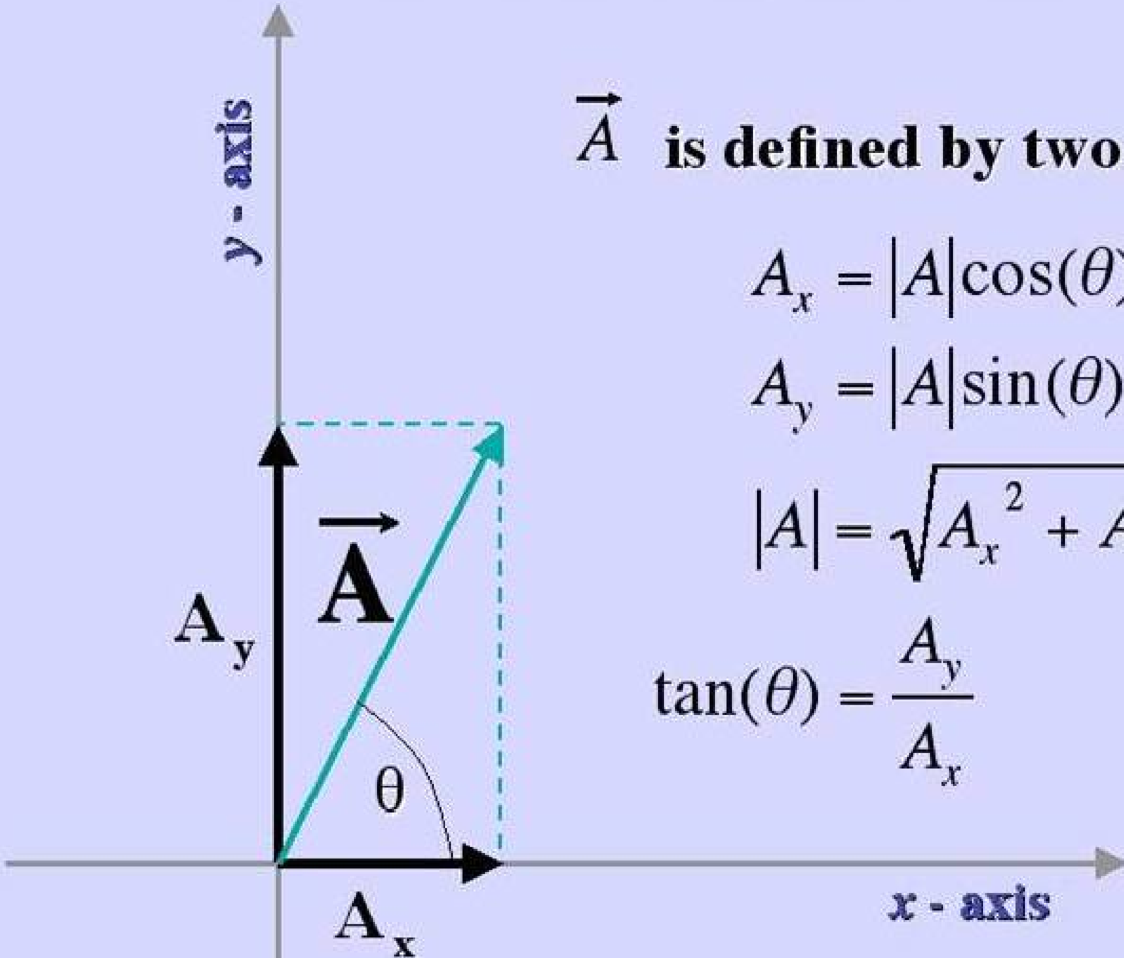
$$0 - V_i^2 = 2(-g)(h - 0)$$

$$V_i = \sqrt{2gh}$$

Example: For h=10m V_i=14m/s

The time to reach 10m: $V_f - V_i = (-g)t$ $\qquad t = \dfrac{V_i}{g} = 1.43s$

Vectors in 2 Dimensions



$\vec{A}$ **is defined by two components A_x, A_y**

$$A_x = |A|\cos(\theta)$$

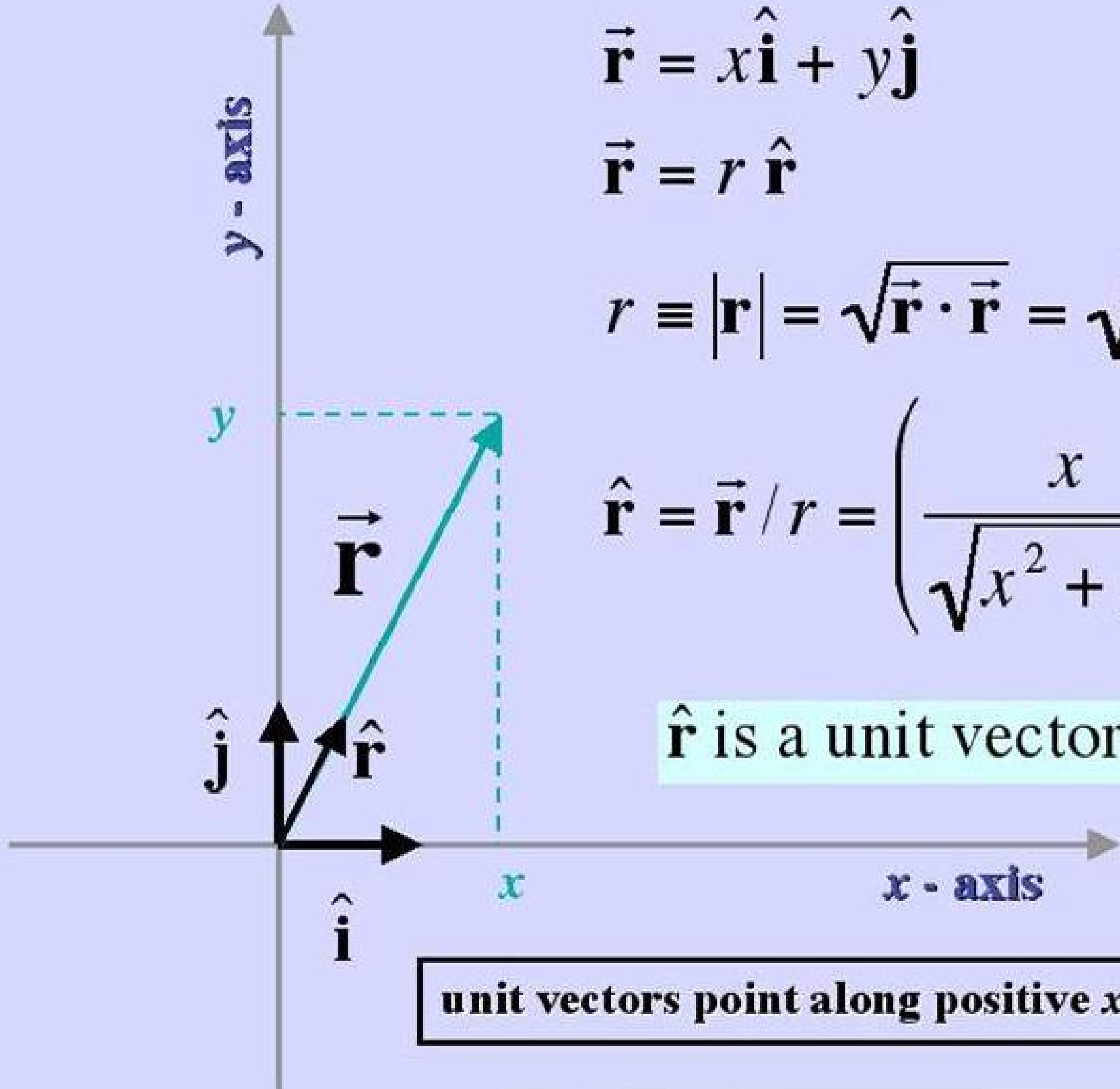
$$A_y = |A|\sin(\theta)$$

$$|A| = \sqrt{{A_x}^2 + {A_y}^2}$$

$$\tan(\theta) = \frac{A_y}{A_x}$$

We will begin the next lecture with vectors in two dimensions, where we will show that the concepts of single dimensional motion discussed above apply to each dimension individually.

Vectors in 2 Dimensions: Unit Vectors



$$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

$$\vec{\mathbf{r}} = r\,\hat{\mathbf{r}}$$

$$r \equiv |\mathbf{r}| = \sqrt{\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}} = \sqrt{x^2 + y^2}$$

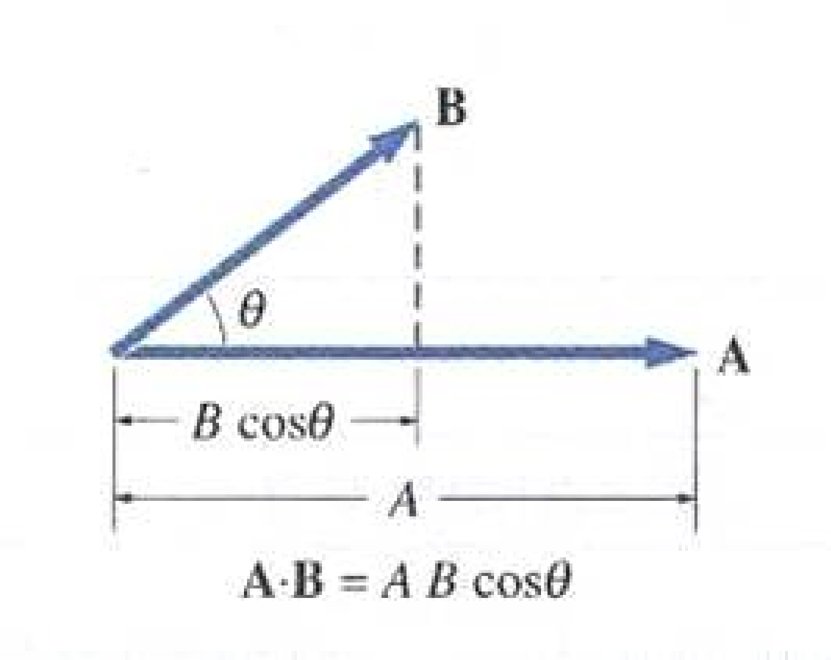
$$\hat{\mathbf{r}} = \vec{\mathbf{r}} / r = \left(\frac{x}{\sqrt{x^2 + y^2}}\right)\hat{\mathbf{i}} + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)\hat{\mathbf{j}}$$

$\hat{\mathbf{r}}$ is a unit vector along $\vec{\mathbf{r}}$, so $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1$

unit vectors point along positive *x* and *y* axes

$$\boxed{\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = 1 \quad \text{and} \quad \hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = 0}$$

Vectors in 2 Dimensions – Scalar Product



From our definitions up to this point we can define a vector $\vec{A}$ such that it points along the x axis.

By definition taking the scalar product of this vector with $\vec{B}$ yields the scalar

$$\vec{A} \bullet \vec{B} = AB\cos\theta$$

A and B are the magnitudes of their respective vectors and θ is the angle between them.

The axes can be rotated, but their magnitude and the angle between the two vectors remains unchanged. This means that this result is a general **invariant** result for any coordinate system where θ is the angle between the two vectors.

Velocity and Acceleration Vectors

Displacements, Velocities and Accelerations are all Vectors.

For velocity, what was previously discussed as a scalar differential equation is actually a vector differential equation:

$$\vec{\mathbf{V}}(\mathbf{t_a}) = \lim_{t_b \to t_a} \frac{\vec{x}_b - \vec{x}_a}{t_b - t_a} = \frac{d\vec{x}}{dt}$$

where in unit vector notation

$$\mathbf{v_x}\hat{\mathbf{i}} + \mathbf{v_y}\hat{\mathbf{j}} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}}$$

The acceleration is analogous

$$\vec{\mathbf{a}}(\mathbf{t_a}) = \lim_{t_b \to t_a} \frac{\vec{V}_b - \vec{V}_a}{t_b - t_a} = \frac{d\vec{V}}{dt} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} = \frac{dv_{\mathbf{x}}}{dt}\hat{\mathbf{i}} + \frac{dv_y}{dt}\hat{\mathbf{j}}$$

Example: Velocity/Acceleration 2D

An object's position is given by: $\vec{r} = (ct - bt^3)\hat{i} + dt^2\hat{j}$

$$c = 6.7m/s, \quad b = .81m/s^3, \quad d = 4.5m/s^2$$

a) What is its initial velocity?

$$\vec{v}(t=0) = \left.\frac{d\vec{r}}{dt}\right|_{t=0} = \left[(c - 3bt^2)\hat{i} + 2dt\hat{j}\right]_{t=0}$$

$$\vec{v}(t=0) = c\hat{i} = 6.7\hat{i}\,m/s$$

b) How long does it take for its velocity to rotate 90**º**?

For this to happen the ***x*** component of the velocity must vanish, or:

$$c - 3bt^2 = 0 \rightarrow t = \sqrt{c/3b} = 1.66\,\mathrm{sec}$$

c) How much does its speed change during this time?

$$\vec{v}\left(t = \sqrt{c/3b}\right) = 2d\sqrt{c/3b}\,\hat{j} = 14.94\hat{j}$$

$$\Delta v = 14.94 - 6.7 = 8.24m/s$$

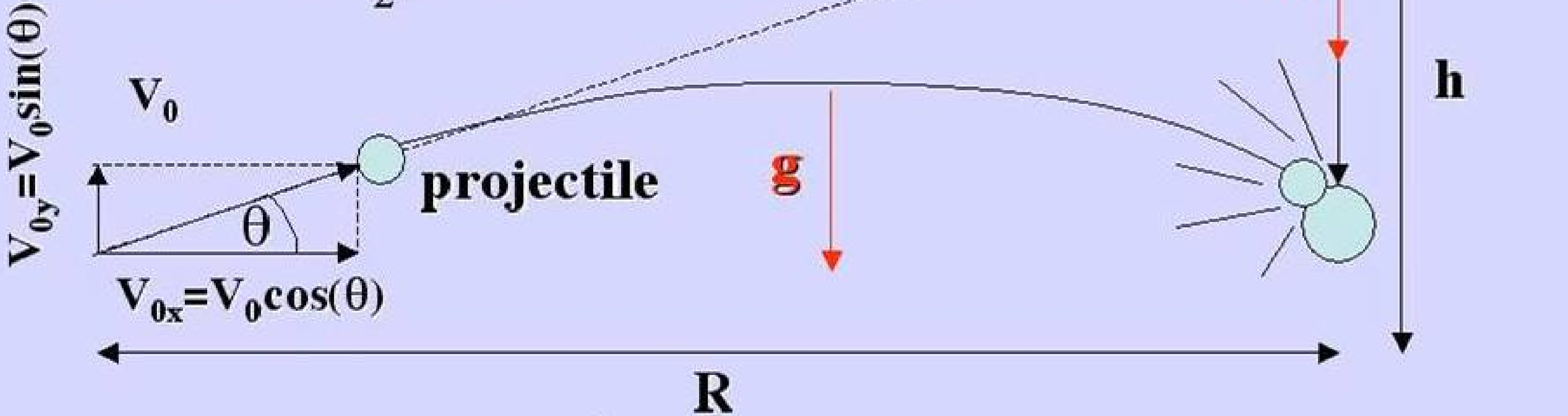
Example: Shoot the Monkey!

We first construct the position vectors

where $h = R\tan(\theta)$

$$\vec{r}_p = (V_0\cos(\theta)t)\hat{i} + (V_0\sin(\theta)t - \frac{1}{2}gt^2)\hat{j}$$

$$\vec{r}_t = (R)\hat{i} + (h - \frac{1}{2}gt^2)\hat{j}$$



When projectile reaches R,

$$R = V_0\cos(\theta)t$$

$$t = \boxed{\frac{R}{V_0\cos(\theta)}}$$

At what time are the y positions equal?

$$h - \frac{1}{2}gt^2 = V_{0y}t - \frac{1}{2}gt^2$$

$$t = \frac{h}{V_{0y}} = \frac{R\tan(\theta)}{V_0\sin(\theta)} = \boxed{\frac{R}{V_0\cos(\theta)}}$$

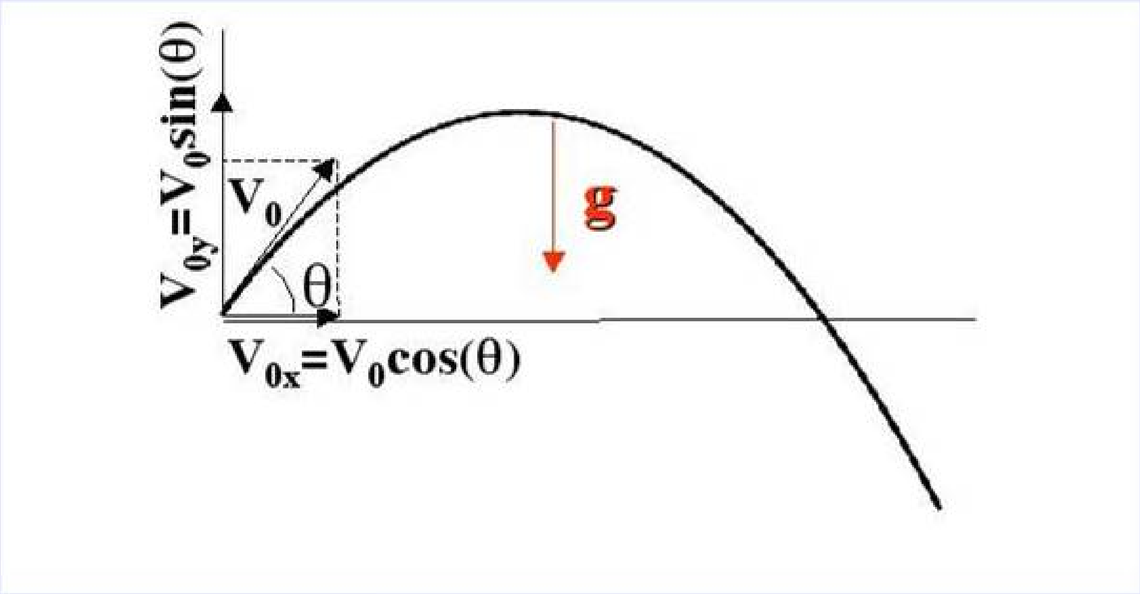
The Same Time! They Collide!

Projectile Trajectory is Parabolic

We can show this by eliminating time from the problem and deriving the function y(x). First, solve for the time from the position equation in x

$$x = V_0 \cos(\theta) t$$

$$t = \frac{x}{V_0 \cos(\theta)}$$



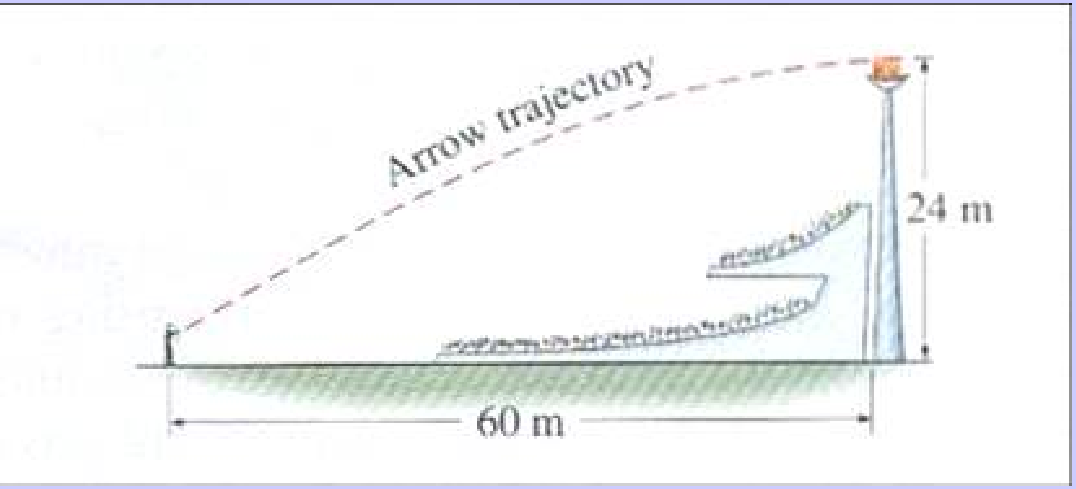
Then substitute in for t in position equation for y

$$y = V_0 \sin(\theta)\left(\frac{x}{V_0 \cos(\theta)}\right) - \frac{1}{2} g \left(\frac{x}{V_0 \cos(\theta)}\right)^2$$

$$\boxed{y = \tan(\theta) x - \frac{g}{2V_0^2 \cos^2(\theta)} x^2}$$

Trajectory is always parabolic in x

Example: Olympic Flame



In the '92 Olympics the Olympic Flame was lit by a flaming arrow. Given the geometry shown, find the initial velocity to reach flame at the peak of the trajectory.

The peak of the trajectory equation occurs when *dy/dx* = 0, or:

$$y = \tan(\theta)x - \frac{1}{2}\frac{g}{V_0^2\cos^2\theta}x^2$$

$$\left.\frac{dy}{dx}\right|_{x=\ell} = \tan\theta - \frac{g\ell}{V_0^2\cos^2\theta} = 0$$

$$V_0^2\cos^2\theta = \frac{g\ell}{\tan\theta} \rightarrow V_0^2 = \frac{g\ell}{\sin\theta\cos\theta}$$

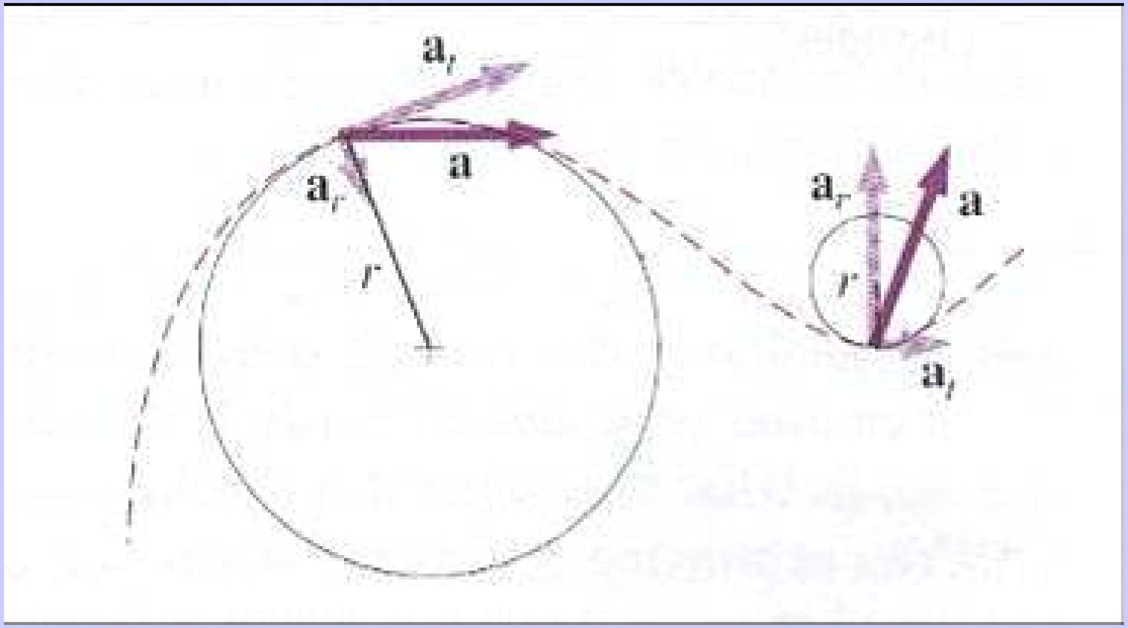
Finding the height when *dy/dx* = 0:

$$y = \tan(\theta)x - \frac{1}{2}\frac{g}{V_0^2\cos^2\theta}x^2$$

$$y(\ell) = h = \tan(\theta)\ell - \frac{1}{2}\frac{g\ell^2}{g\ell/\tan\theta}$$

$$h = \frac{1}{2}\tan(\theta)\ell$$

Nonuniform Circular Motion



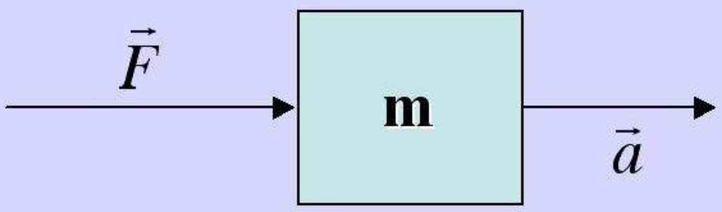
Each point along the path can be characterized by a radius of curvature, *r*. An object with speed *v* has a radial acceleration of v^2/r and a tangential acceleration of magnitude dv/dt . In general both *v* and *r* change as the object moves.

$$\vec{a} = a_t\hat{\phi} - a_r\hat{r} \quad \text{with} \quad a_t = \frac{dv}{dt} \quad \text{and} \quad a_r = \frac{v^2}{r} = \omega^2 r$$

Newton's Second Law of Motion

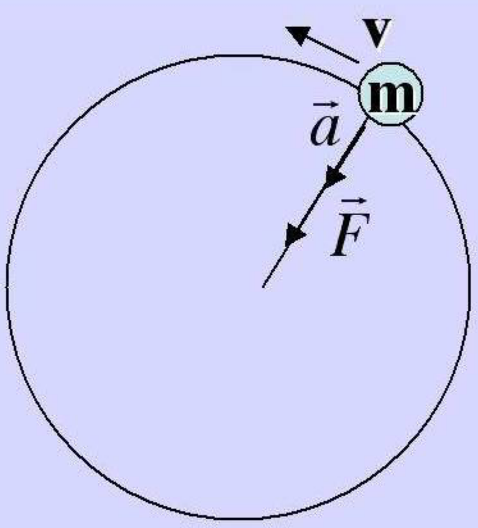
The Law of Force, Mass and Acceleration: The force on a body is equal to its change in momentum with time. For constant mass, this is the mass times the acceleration.

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt}$$ **or** $$\vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a}$$ **For m constant**



$$\vec{F} = m\vec{a}$$

For a body in linear motion this accelerates the body.



In uniform circular motion there is a radial force inward.

More About Newton's Second Law

$\sum \vec{F}$ is the net force

This is the vector sum of all the forces acting on the object

Newton's Second Law can be expressed in terms of components:

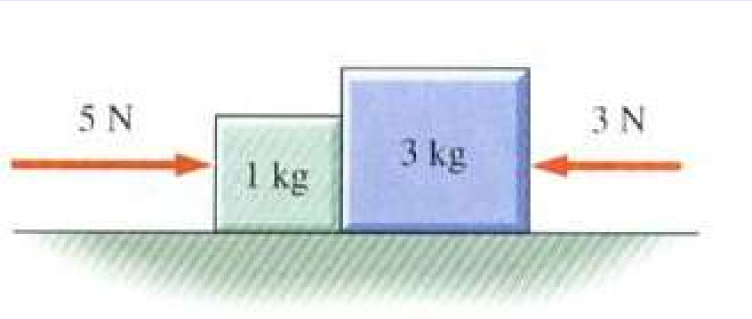
$\Sigma F_x = m\, a_x$

$\Sigma F_y = m\, a_y$

$\Sigma F_z = m\, a_z$

Most philosopher's of science consider Newton's 2nd to be the definition of a force.

Example: Newton's 3rd - Pushing Those Blocks



On a surface with negligible friction, there are two opposing forces, $\boldsymbol{F_1 = 5N}$ and $\boldsymbol{F_2 = -3N}$, acting on two blocks of mass $\boldsymbol{m_1 = 1kg}$ and $\boldsymbol{m_2 = 3kg}$. (a) What is the force, $\boldsymbol{F_{21}}$, of the second block acting on the first?

From Newton's 2nd the acceleration of both blocks is:

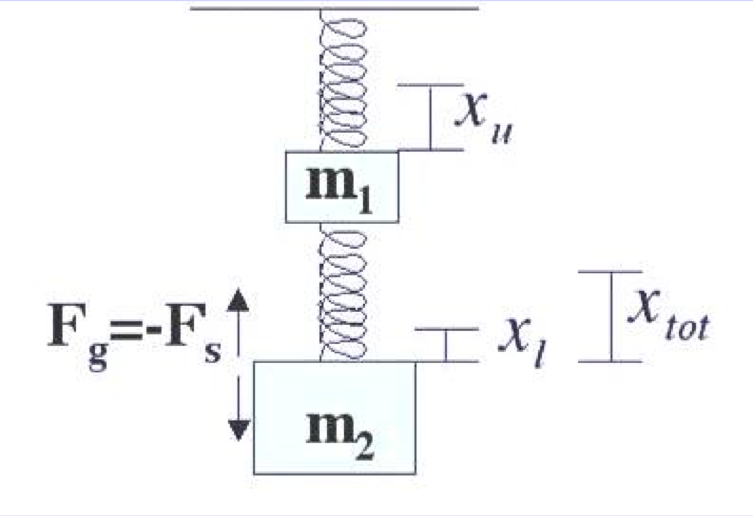
$$a = \frac{F_{net}}{m_{tot}} = \frac{F_1 + F_2}{m_1 + m_2} = \frac{5-3}{1+3} = \frac{1}{2} m/s^2$$

The net force acting on $\boldsymbol{m_2 = 3kg}$, is the force of the first block acting on the second, $\boldsymbol{F_{12}}$, and $\boldsymbol{F_2 = -3N}$. From Newton's 2nd:

$$F_{12} + F_2 = m_2 a \rightarrow F_{12} = m_2 a - F_2 = \frac{3}{2} + 3 = 4\frac{1}{2} N$$

From Newton's 3rd: $$F_{21} = -F_{12} = -4\frac{1}{2} N$$

Example: Springs in Series with an Additional Mass



Two springs each with a spring constant of ***k* = 20N/m** support two mass, ***m*$_1$ = .2kg** and ***m*$_2$ = .4kg** as shown. Find the displacement from equilibrium of each spring.

For this configuration the lower spring is supporting ***m*$_2$**. From Hook's law its displacement is:

$$x_l = \frac{m_2 g}{k} = \frac{.4 \times 9.8}{20} = .196m = 19.6cm$$

From Newton's 3rd, the upper spring is supporting both masses. From Hook's law:

$$x_u = \frac{(m_2 + m_1)g}{k} = \frac{.6 \times 9.8}{20} = .294m = 29.4cm$$

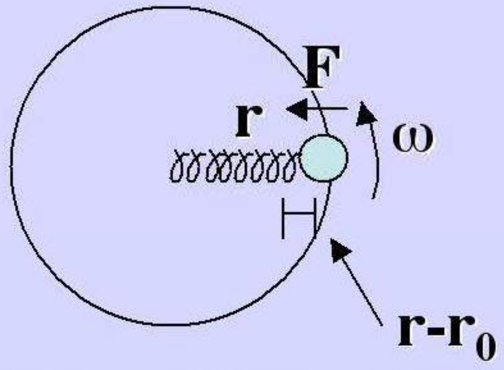
The total displacement is:

$$x_{tot} = x_u + x_l = 29.4 + 19.6 = 49cm$$

Example: Hook's Law

A mass m is in uniform circular motion at angular frequency ω on a spring, which displaces a distance $r-r_0$. What is the constant k of the spring?

The displacement is relative to the "unstretched or compressed" length of the spring. Thus the force:



$$F = -k(r - r_o)$$

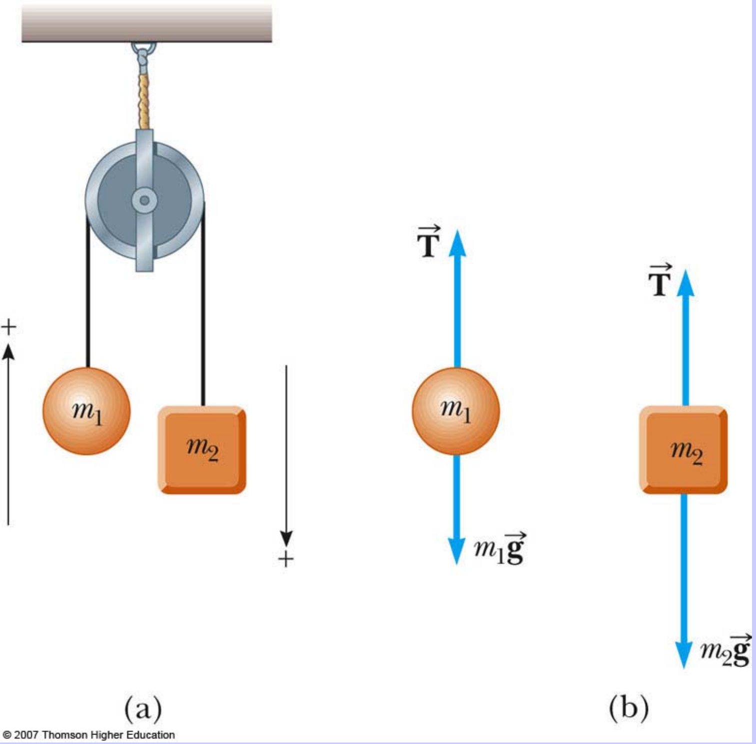
But this force is equal and opposite to the centripetal force for uniform circular motion which points outward:

$$F = m\frac{v^2}{r} \quad \text{where} \quad v = r\omega$$

Equating them: $$\boxed{k = \frac{r}{r - r_0} m\omega^2}$$

Remember that the change in a springs length is NOT the springs length!

Example: Atwood's Machine



Applying Newton's 2nd for each object (often called the equations of motion or EOM) yields (note signs in each equation):

$$m_2 g - T = m_2 a \quad \text{and} \quad T - m_1 g = m_1 a$$

To solve this system of equations we start by summing them to eliminate ***T***,

$$(m_2 - m_1)g = (m_2 + m_1)a \rightarrow a = \frac{m_2 - m_1}{m_2 + m_1} g$$

Does this result make physical sense?

Incline With Friction

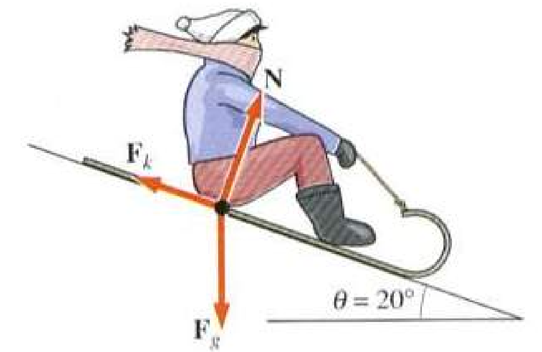
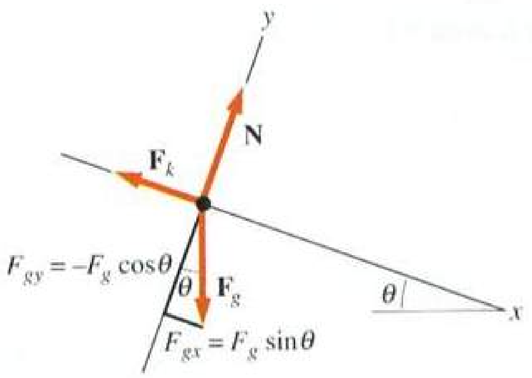


FIGURE 6-33 Forces on the child and sled.



A child slides down a **20º** slope with a coefficient of kinetic friction $\boldsymbol{\mu_k}$ **= .085**.
(a) What is the child's acceleration?

From the Free-Body diagram, the vector equation for the sled is:

$$\vec{F}_g + \vec{F}_k + \vec{N} = m\vec{a}$$

From the Free-Body diagram we see that the component equations are:

$$F_g \sin\theta - F_k = mg\sin\theta - \mu_k N = ma$$
$$N - mg\cos\theta = 0$$

Substituting for the normal force in the *x* component equation followed by dividing by ***m*** yields:

$$a = g(\sin\theta - \mu_k \cos\theta) = 9.8(\sin 20^\circ - .085\cos 20^\circ) = 2.57 m/s^2$$

Incline With Friction

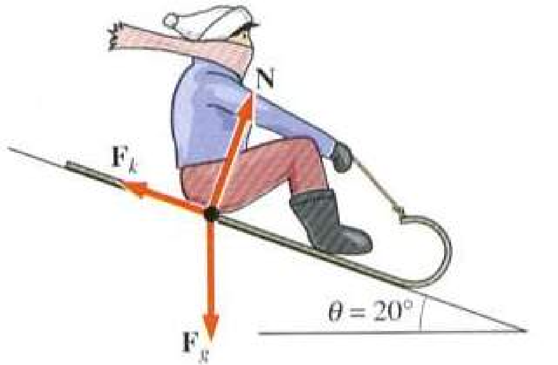
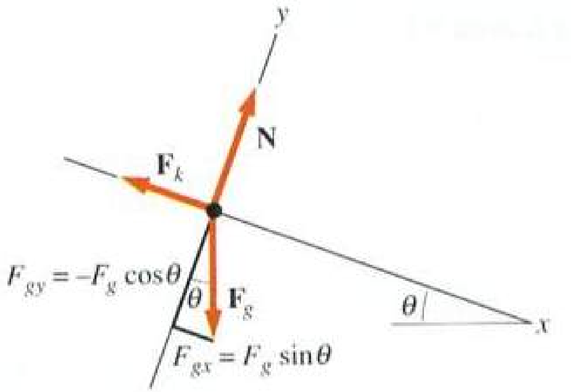


FIGURE 6-33 Forces on the child and sled.



A child slides down a **20º** slope with a coefficient of kinetic friction $\boldsymbol{\mu_k}$ **= .085**.
(b) At what angle will the child velocity remain constant?

From the results for a general angle we saw

$$a = g(\sin\theta - \mu_k \cos\theta)$$

The acceleration vanishes when

$$\sin\theta = \mu_k \cos\theta \rightarrow \mu_k = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

What happens when the angle of the incline, $\boldsymbol{\theta}$, approaches $\boldsymbol{\pi/2}$?

What happens when the angle of the incline, $\boldsymbol{\theta}$, is less than $\mathbf{tan^{-1}}(\boldsymbol{\mu_k})$?

Two Blocks & Incline Plane

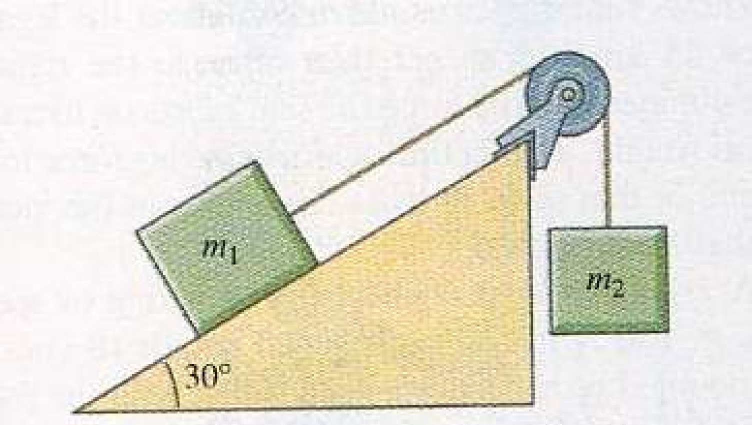


FIGURE 6-76 Problems 67, 68.

Assuming frictionless surfaces, find the acceleration of $\boldsymbol{m_2}$.

From a free-body diagram the vector EOM for $\boldsymbol{m_1}$ is:

$$\vec{T} + \vec{N} + \vec{F}_g = m\vec{a}$$

Choosing a coordinate system in which the *x* axis is parallel to the incline the component equations are:

$$T - m_1 g \sin\theta = m_1 a \quad \text{and} \quad N - m_1 g \cos\theta = 0$$

The EOM for $\boldsymbol{m_2}$ is particularly simple:

$$m_2 g - T = m_2 a$$

It is important to note that we have assumed that $\boldsymbol{m_2}$ is accelerating down and $\boldsymbol{m_1}$ is accelerating up the incline. We could have done the reverse, but we must be consistent. That is, $\boldsymbol{m_1}$ and $\boldsymbol{m_2}$ cannot both accelerate up (or down).

Two Blocks & Incline Plane

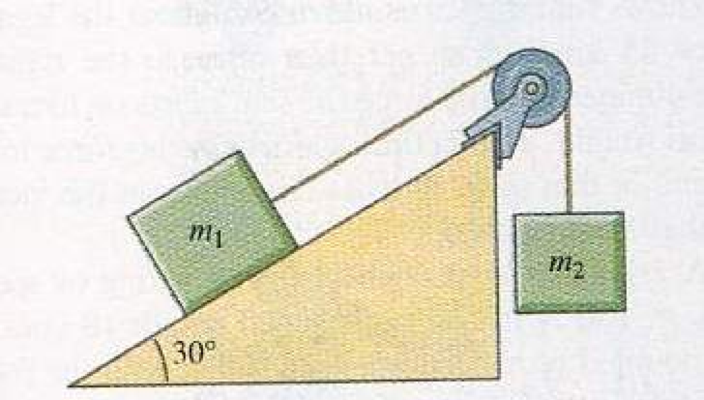


FIGURE 6-76 Problems 67, 68.

Assuming frictionless surfaces, find the acceleration of $\boldsymbol{m_2}$.

The two relevant equations are:

$$T - m_1 g \sin\theta = m_1 a \quad \text{and} \quad m_2 g - T = m_2 a$$

Taking into account that ***sin(θ) = ½,*** the acceleration is:

$$a = \frac{m_2 - m_1/2}{m_1 + m_2} g$$

As long as $\mathbf{m_2} > \frac{1}{2}\, \mathbf{m_1}$ then the acceleration is consistent with $\boldsymbol{m_1}$ moving up the incline. If $\boldsymbol{m_2} < \frac{1}{2}\, \boldsymbol{m_1}$ then $\boldsymbol{m_1}$ will accelerate down the incline.

What if there is friction on the surface of the incline?

$$T - m_1 g \sin\theta \mp \mu_k m_1 g \cos\theta = m_1 a \quad \text{and} \quad m_2 g - T = m_2 a$$

Work-Kinetic Energy Theorem

The kinetic energy for a single particle of mass ***m*** traveling at speed ***v*** is defined as:

$$\boxed{K = \frac{1}{2}mv^2 = \frac{1}{2}m\vec{v}\cdot\vec{v}}$$

The time derivative of this expression is easily evaluated as

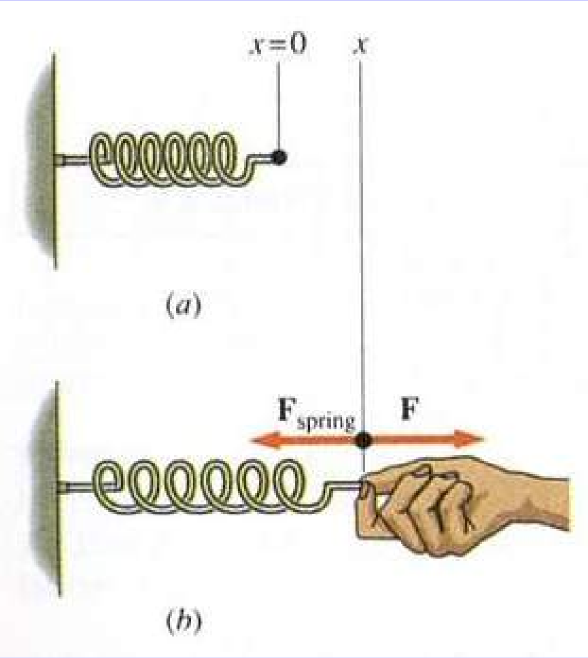
$$\frac{dK}{dt} = m\vec{v}\cdot\frac{d\vec{v}}{dt} = m\frac{d\vec{v}}{dt}\cdot\vec{v} = \vec{F}_{net}\cdot\frac{d\vec{r}}{dt}$$

Now we can multiply this expression by *dt* to find $dK = \vec{F}_{net}\cdot d\vec{r}$
Integrating this expression along the path of the particle we find

$$\boxed{\Delta K = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_{net}\cdot d\vec{r} = W}$$

This is a statement of the **Work-KE theorem**,
"The change in a particle's kinetic energy between two points is equal to the work done by the net force along the path between the two points."

Example: Work to Stretch a Spring

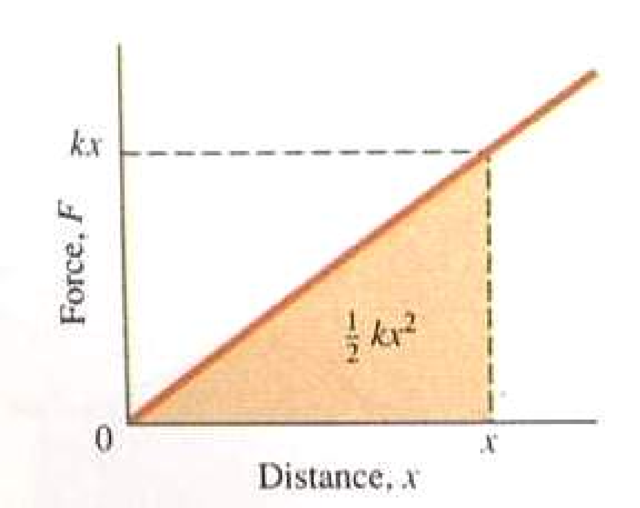


From Hook's Law a spring exerts a force proportional to its displacement from equilibrium:

$$F = -kx$$

This is the force by the spring on the hand stretching it. From Newton's 3rd, the force exerted by the hand is ***kx***. The work done by the hand is the integral:

$$W = \int_0^x kx'\,dx' = \frac{1}{2}kx^2$$



What would the work be if the hand compressed the spring?

Example: Kinetic Energy and Springs

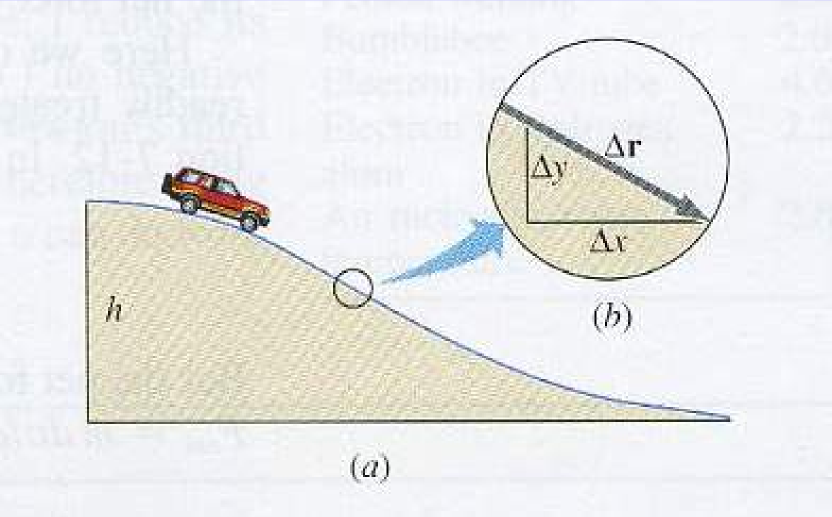
A spring with spring constant $\boldsymbol{k}$ is compressed a distance $\boldsymbol{A}$ and while being attached to an object of mass $\boldsymbol{m}$. The spring is then released. What is the speed of the object when the spring returns to its original equilibrium position?

The work done by the spring on the object is $\boldsymbol{W = \frac{1}{2} k A^2}$. From the Work-Kinetic Energy Theorem:

$$W = \frac{1}{2}kA^2 = \frac{1}{2}mv^2 \rightarrow v = \sqrt{\frac{k}{m}}A$$

The details of using the force of the spring to find the acceleration and then using kinematics to find the velocity are not required. The Work-Kinetic Energy Theorem solves the problem with minimal effort.

Example: Work and the Gravitational Force



How much work does the force of gravity do on a car as it drives from the top of the hill, (***y*=*h***) to the bottom (***y*=*0***)?

The force of gravity is $\vec{F}_g = -mg\hat{j}$.

The path integral for the work done by gravity,

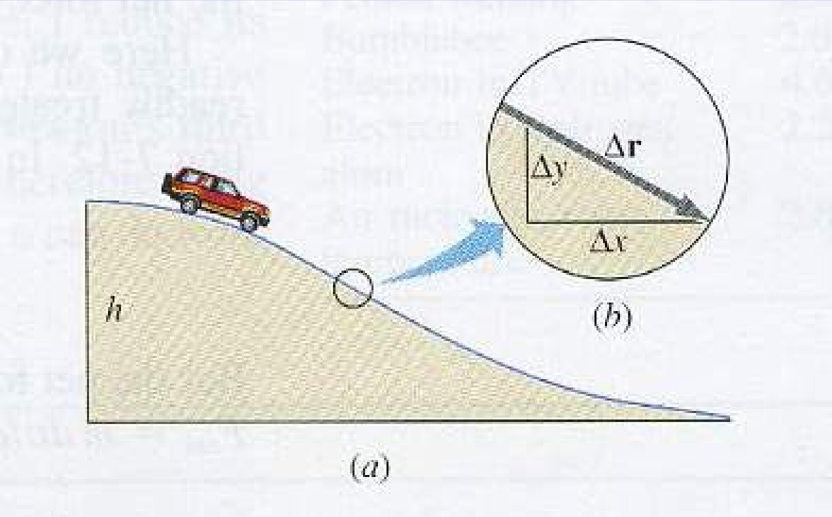
$$W = \int_1^2 \vec{F}_g \cdot d\vec{r} = -\int_1^2 mg\hat{j} \cdot d\vec{r} = -\int_h^0 mgdy$$

$$W = -mg(0 - h) = mgh$$

Note that the details of the path didn't matter for this problem, only the change in height, ***h***, was relevant.

Does the sign make sense for this result?

Example: Work and the Gravitational Force



Assuming that the car started from rest, how fast is the car traveling when it reaches the bottom of the hill (ignoring friction)?

The work done by gravity as the rolls down the hill was found to be:

$$W = mgh$$

From the work-energy theorem:

$$W = mgh = \frac{1}{2}mv^2 \rightarrow v = \sqrt{2gh}$$

Again the work-energy theorem solves this problem with minimal effort!

Conservation of Energy

From the Work-Energy Theorem the work done on an object is equal to the change in its kinetic energy:

$$W = m\int \frac{d\vec{v}}{dt} \cdot d\vec{r} = \int \vec{F}_{net} \cdot d\vec{r} = \Delta K = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

If we consider separately the work done by conservative forces, W_c, and non-conservative forces, $\boldsymbol{W_{nc}}$:

$$\Delta K = W_c + W_{nc}$$

Potential energy was defined as the negative of the work done by conservative forces: $\boldsymbol{\Delta U = - W_c}$. Hence:

$$\Delta K + \Delta U = W_{nc}$$

In the absence of non-conservative forces, the total mechanical energy is conserved!

$$\boxed{\Delta K + \Delta U = 0 \rightarrow \frac{1}{2}mv_i^2 + U_i = \frac{1}{2}mv_f^2 + U_f = E}$$

Force and Potential Energy

Consider a force pushing a body along the x axis. The work being done by the force is:

$$W = -\Delta U$$

We also know that

$$W = F_x \Delta x$$

Combining these two we can write:

$$F_x = -\frac{\Delta U}{\Delta x}$$

This applies to 3D motion in general:

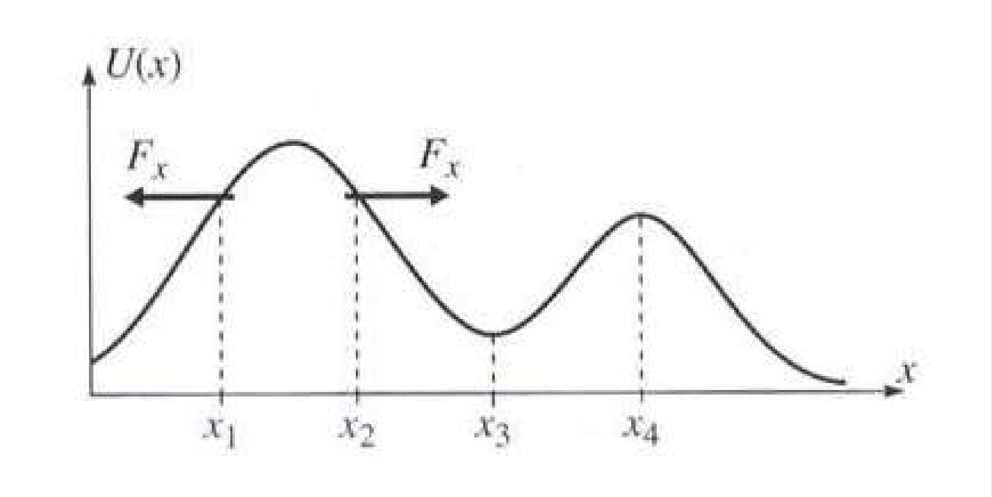
$$\vec{F} = -\left(\frac{dU}{dx}\hat{i} + \frac{dU}{dy}\hat{j} + \frac{dU}{dz}\hat{k}\right)$$

Or, in gradient notation:

$$\vec{F} = -\vec{\nabla} U$$

Here, $\vec{\nabla}$ is a vector differential operator.

Force and Potential Energy



Again think of the potential energy plot as a picture of a roller coaster. The force

$$F_x = -\frac{dU}{dx}$$

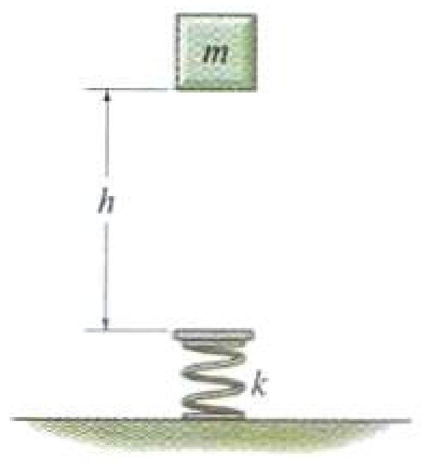
tends to push the object downhill as shown in the plot at $\boldsymbol{x=x_1}$ and $\boldsymbol{x=x_2}$.

Note that at the points $\boldsymbol{x_3}$ and $\boldsymbol{x_4}$ where $\boldsymbol{dU/dx = 0}$, $\boldsymbol{U}$ is a minimum or a maximum. The object is in equilibrium as the net force vanishes at those points. However $\boldsymbol{x_3}$ is a point of stable equilibrium (why?) and $\boldsymbol{x_4}$ is a point of unstable equilibrium (why?).

For example consider the potential energy for a spring:

$$U(x) = \frac{1}{2}kx^2 \rightarrow F = -\frac{dU}{dx} = -kx$$

Vertical Spring



A mass ***m*** is dropped from a height ***h*** above the top of a spring with spring constant ***k***. What is the maximum compression of the spring?

If zero for the gravitational potential energy is chosen to be the height at the top of the spring, then the conservation of energy for this problem is:

$$E = mgh = -mgx + \frac{1}{2}kx^2$$

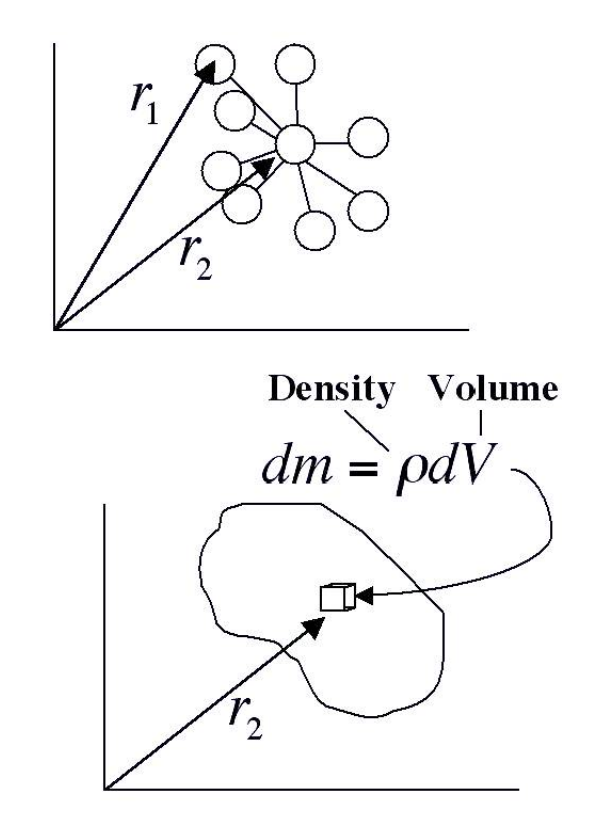
Note that it is important to note that the mass does not come to rest until the spring obtains maximum compression!

Now it a simple problem of solving the quadratic equation:

$$\frac{1}{2}kx^2 - mgx - mgh = 0 \rightarrow x = \frac{mg}{k}\left(1 + \sqrt{1 + 2kh/mg}\right)$$

What is the physical significance of the other root?

Center of Mass



Mathematically we define the center of mass as the average of the mass weighted vector displacement of the individual particles. Defining the total mass as ***M***.

$$M = m_1 + m_2 + m_3 + \cdots = \sum_{i=1}^{N} m_i$$

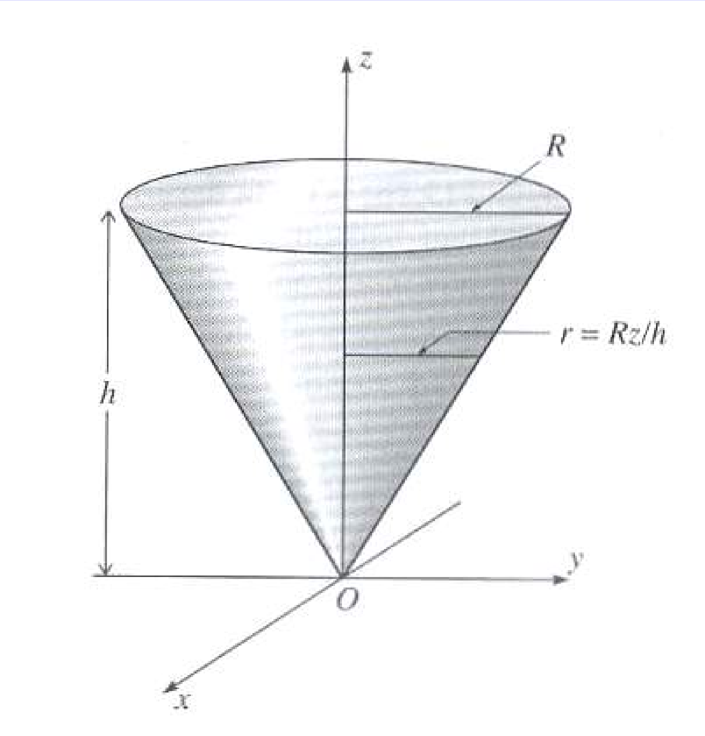
This allows us to define the center of mass as:

$$\vec{R}_{cm} = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots) = \frac{1}{M}\sum_{i=1}^{N} m_i\vec{r}_i$$

For continuous media both sums become integrals:

$$M = \int dm, \quad \text{and} \quad \vec{R}_{cm} = \frac{1}{M}\int \vec{r}\,dm$$

Example: Center of Mass – Uniform Solid Cone



From symmetry considerations the center of mass must lie on the *z* axis. All that is left is to perform the integral to determine Z_{cm}.

$$Z_{cm} = \frac{1}{M}\int z\,dm = \frac{1}{M}\int z\rho\,dV = \frac{1}{V}\int z\,dV$$

At a height *z* (radius *r*) the volume element is:

$$dV = A(z)dz = \pi r^2(z)dz$$

Since *r(z)* satisfies the relationship $r = Rz/h$ the integral for Z_{cm} becomes:

$$Z_{cm} = \frac{1}{V}\int z\,dV = \frac{1}{V}\int z\pi r^2(z)dz$$

$$Z_{cm} = \frac{1}{V}\int z\pi\frac{R^2}{h^2}z^2dz = \frac{\pi R^2}{h^2(\pi R^2 h/3)}\int_0^h z^3dz$$

$$Z_{cm} = \frac{3}{h^3}\frac{h^4}{4} = \frac{3}{4}h \quad \text{(from the vertex)}$$

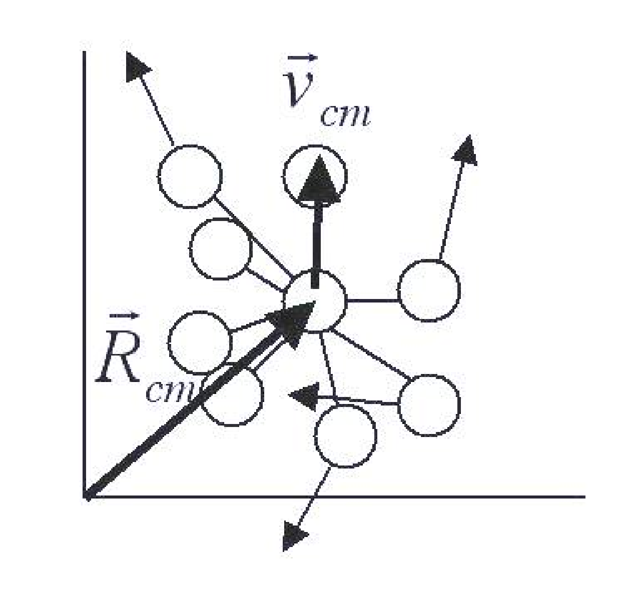
Motion of the Center of Mass

The total momentum of a system of particles is equal to the momentum of the center of mass. In the absence of any **net external force** this momentum is **conserved**.

To see this consider the time derivative of the center of mass:

$$\frac{d}{dt}\vec{R}_{cm} = \vec{v}_{cm} = \frac{1}{M}\sum_{i=1}^{N} m_i \frac{d}{dt}\vec{r}_i = \frac{1}{M}\sum_{i=1}^{N} m_i \vec{v}_i$$

$$\vec{P}_{cm} = \sum_{i=1}^{N} m_i \frac{d}{dt}\vec{r}_i = M\vec{v}_{cm} = \sum_{i=1}^{N} m_i \vec{v}_i$$



Even though individual particles may be moving relative to the center of mass, the center of mass maintains a uniform velocity.

External Forces and the Center of Mass

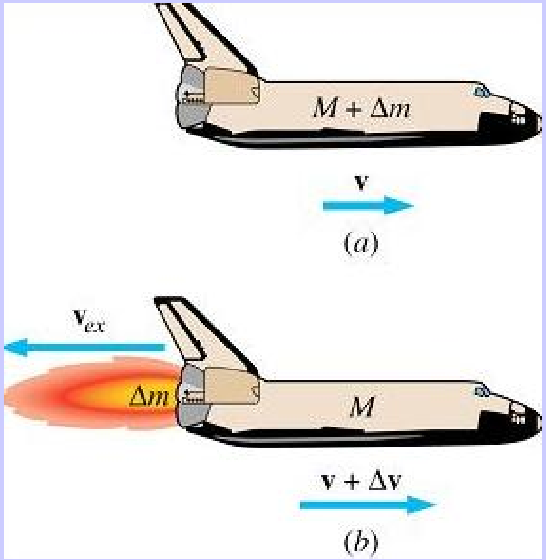
The sum of all the net forces on each of the particles determines the acceleration of the center of mass.

$$\frac{d}{dt}\vec{P}_{cm} = \sum_{i=1}^{N} m_i \frac{d}{dt}\vec{v}_i = \sum_{i=1}^{N} m_i \vec{a}_i = \sum_{i=1}^{N} \vec{F}_{i-net}$$

However, we need to consider the sum of the forces on each of the particles. Some of the forces on the ***i*** th particle are due to external forces (e.g. external gravitational field). There are also forces between the particles themselves (at least a gravitational attraction). This could make the problem virtually intractable, but Newton's 3rd comes to the rescue. It is the basis for recognizing that **the sum of the internal forces over all of the particles cancel!** It is only the sum of all the external forces that induce an acceleration of the center of mass.

$$\boxed{\frac{d}{dt}\vec{P}_{cm} = \sum_{i=1}^{N} \vec{F}_{i-ext}}$$

Rockets



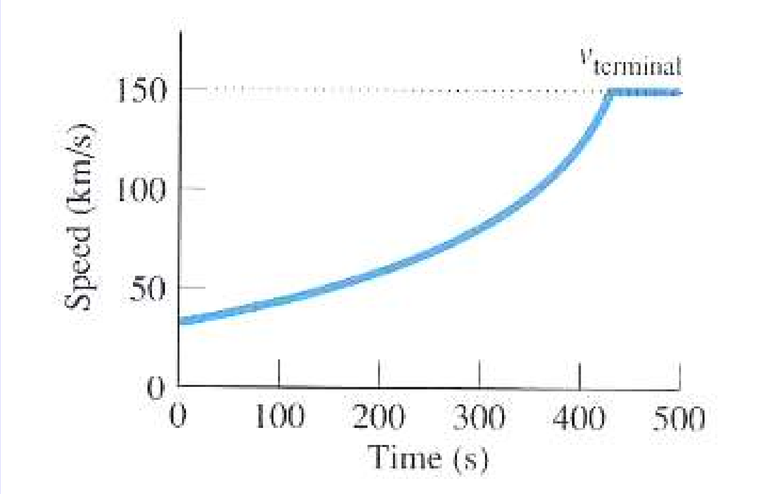
To solve for the velocity as a function of time we multiply our EOM,

$$F_T = M\frac{dv}{dt} = -v_{ex}\frac{dM}{dt}$$

by ***dt***, separate and integrate.

$$\int_i^f dv = v_f - v_i = -v_{ex}\int_i^f \frac{dM}{M} = v_{ex}\ln\frac{M_i}{M_f}$$

The velocity as a function of time is shown in the plot:



Inelastic vs. Elastic Collisions

Inelastic collisions: the **momentum** is conserved, **not** the **energy**

Elastic collisions: the **momentum and energy** are conserved

For a **totally inelastic** collision between two objects, the objects stick together and become one mass. Only momentum conserved:

$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1 + m_2)\vec{v}_f$$

For a **totally elastic** collision between two objects, the objects are unchanged in form, and rebound perfectly while conserving kinetic energy as well as momentum:

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

$$\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$$

Inelastic collisions are rarely totally inelastic, elastic collisions are rarely totally elastic. Reality is somewhere between and these are the limits.

Elastic Collisions in One-Dimension

For an elastic collisions in one-dimension both energy and momentum are conserved:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

Note that for elastic collisions in one dimension there are two equations for two unknowns.

Rearranging and simplifying:

$$m_1 (v_{1i} - v_{1f}) = m_2 (v_{2f} - v_{2i})$$

$$m_1 \left(v_{1i}^2 - v_{1f}^2\right) = m_2 \left(v_{2f}^2 - v_{2i}^2\right)$$

Now we can divide the second equation by the first and then rearrange:

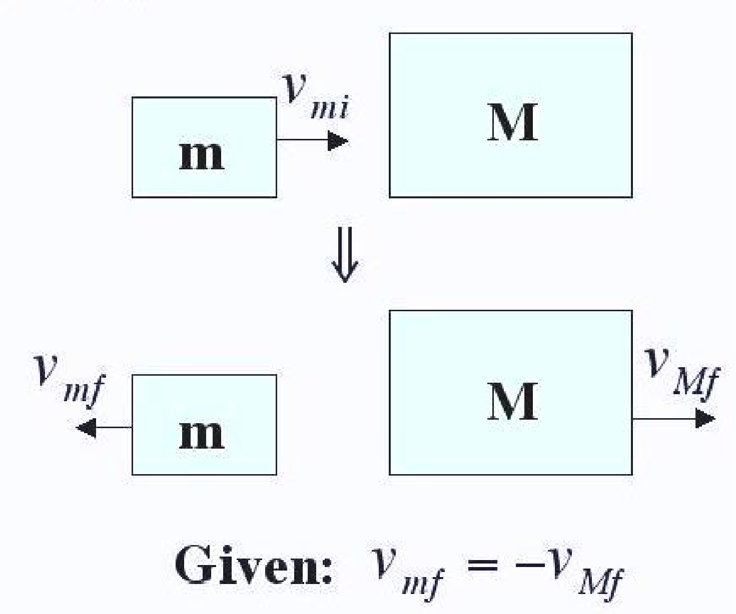
$$m_1 v_{1i} + m_2 v_{2i} = m_2 v_{2f} + m_1 v_{1f}$$

$$v_{1i} - v_{2i} = v_{2f} - v_{1f}$$

This pair of equations (linear) is much easier to work with than the original. Also note that the relative velocities become reversed!

Elastic Collisions in One-Dimension

Example:



In a one dimensional elastic collision ***M*** is initially at rest. If both masses end up with the same speed, $\boldsymbol{v_{mf} = - v_{Mf}}$, how are ***m*** and ***M*** related?

With $\boldsymbol{v_{Mi} = 0}$ and $\boldsymbol{v_{mf} = - v_{Mf}}$, our equations become:

$$mv_{mi} = mv_{mf} + Mv_{Mf} \rightarrow mv_{mi} = mv_{mf} - Mv_{mf}$$

$$v_{mi} = v_{Mf} - v_{mf} \rightarrow v_{mi} = -2v_{mf}$$

Substituting for $\boldsymbol{v_{mi}}$ we find:

$$-2mv_{mf} = mv_{mf} - Mv_{mf}$$

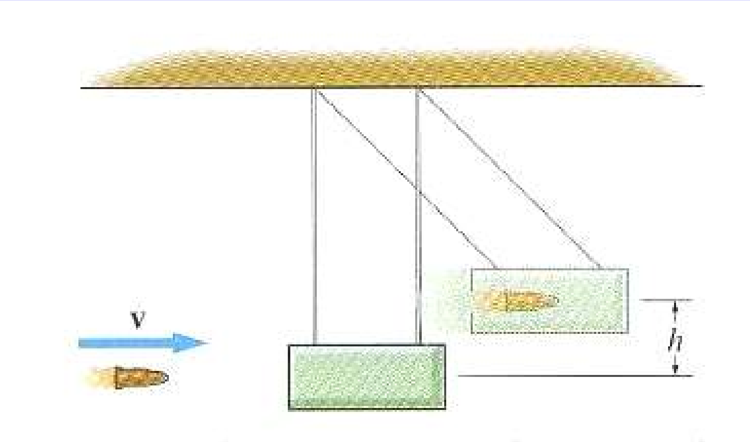
with the result

$$M = 3m$$

As we already noted, this pair of equations is much easier to work with than the original conservation of energy and conservation of momentum!

Inelastic Collisions in Two-Dimensions Example: Ballistic Pendulum

If a bullet with mass ***m*** strikes the wooden block with mass ***M***, find the initial velocity of the bullet if the block rises to a height ***h*** after impact.



Initially momentum is conserved so that:

$$mv = (M + m)V$$

Conserving energy as the block (and bullet) rise in the gravitational field we have:

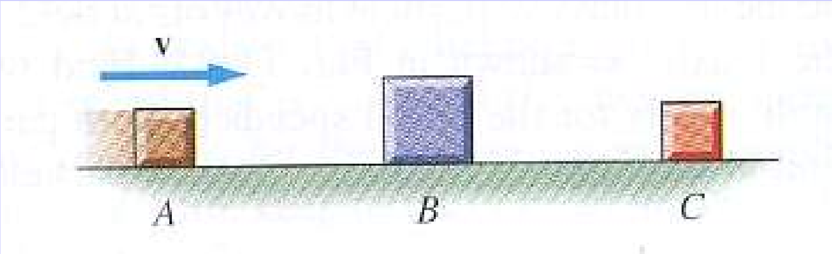
$$\frac{1}{2}(M + m)V^2 = (M + m)gh \rightarrow V = \sqrt{2gh}$$

Solving for ***v*** :

$$v = \frac{M + m}{m}\sqrt{2gh}$$

Elastic Collisions in One-Dimension
Three Blocks

Blocks B and C have masses ***2m*** and ***m*** respectively and are at rest on a frictionless surface. Block A also of mass ***m*** is heading at block B with a velocity ***v***. Assuming that all collisions are elastic what is the final velocity of each block



The conservation equations for each collision are:

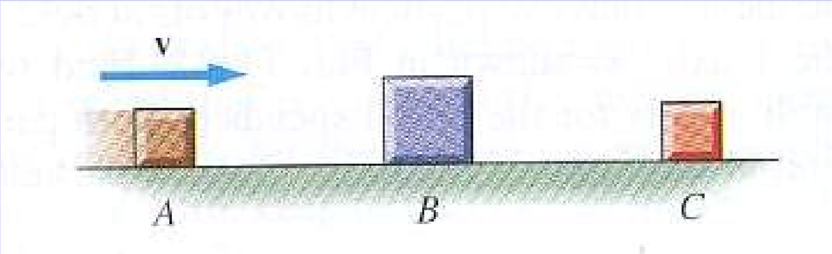
$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \text{ and } v_{1i} - v_{2i} = v_{2f} - v_{1f}$$

For the first collision these become: $v = v_{Af} + 2v_{Bf}$ and $v = v_{Bf} - v_{Af}$

With solutions: $v_{Bf} = 2v/3$ and $\boxed{v_{Af} = -v/3}$

Elastic Collisions in One-Dimension
Three Blocks

Blocks B and C have masses ***2m*** and ***m*** respectively and are at rest on a frictionless surface. Block A also of mass ***m*** is heading at block B with a velocity ***v***. Assuming that all collisions are elastic what is the final velocity of each block



The conservation equations for each collision are:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \text{ and } v_{1i} - v_{2i} = v_{2f} - v_{1f}$$

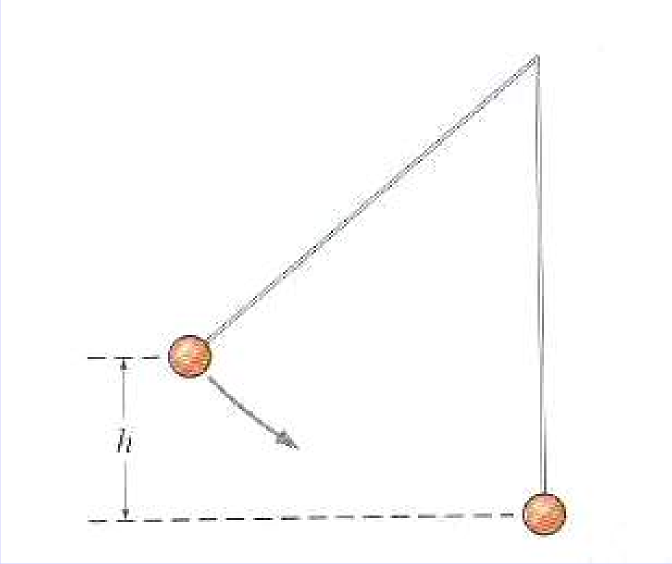
For the second collision these become: $4v/3 = 2v_{Bf} + v_{Cf}$ and $2v/3 = v_{Cf} - v_{Bf}$

With solutions: $\boxed{v_{Cf} = 8v/9}$ and $\boxed{v_{Bf} = 2v/9}$

Note that the total final momentum is equal to *mv*.

One Final Example

Two identical pendulum bobs are suspended from strings of equal length. One is released from a height ***h***. When it hits the second bob they stick together. What is the maximum height that the pair rise to on the opposite side?



Clearly the answer is ***h/4***, WHY?

Calculating the Moment of Inertia

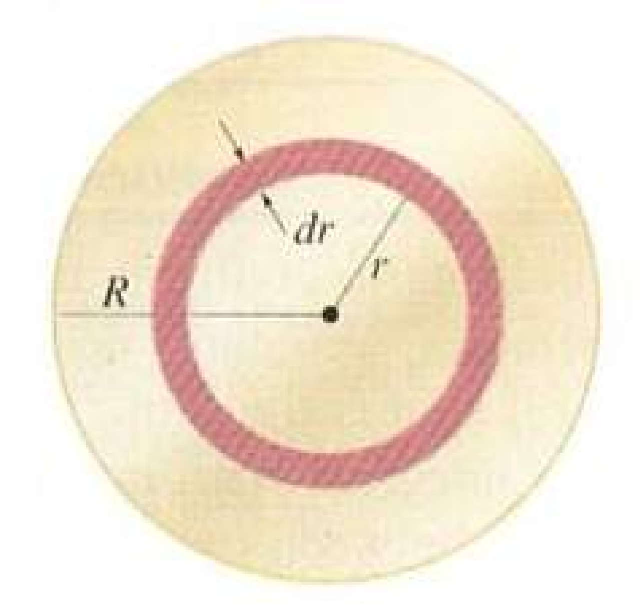
For point masses and continuous media the moment of inertia is calculated in a similar way as the center of mass, but with r^2 instead of r (and we do not divide by the total mass).

For a collection of point masses	$I = \sum_i r_i^2 m_i$
For a continuous medium of mass	$I = \int r^2 dm$

Similarly to the center of mass, the moment of inertia depends on the choice of origin through r.

Note that *r* is the distance to the axis of rotation, and *I* is not equal to $M\,r^2_{cm}$.

Example: Moment of Inertia of a Thin Disk about its Center



Due to circular symmetry we only have to integrate along the radial direction. The differential mass element is:

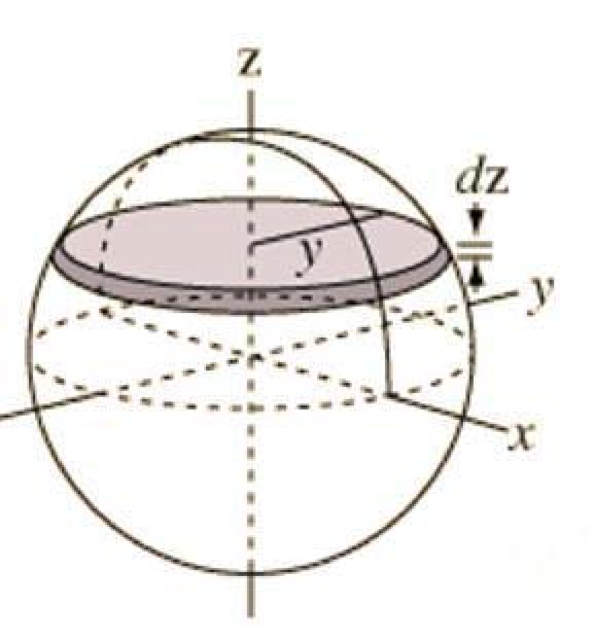
$$dm = \frac{M}{A} 2\pi r dr$$

Performing the moment of inertia integral:

$$I = \int r^2 dm = \int_0^R \frac{M}{A} 2\pi r^3 dr = \frac{2\pi M}{\pi R^2}\frac{R^4}{4} = \frac{1}{2}MR^2$$

This result can be used to sum thin disks for any objects with circular symmetry such as a sphere or disk etc.

Example: Thin Spherical Shell



For a thin spherical shell the mass element is:

$$dm = \frac{M}{A}(2\pi y)Rd\theta = 2\pi\frac{M}{A}R^2 \sin\theta d\theta$$

Finding the mass element was the hard part.
The integral for the moment of inertia is:

$$I = \int r^2 dm = \int y^2 dm = \frac{2\pi M}{A}R^4 \int_0^\pi \sin^3\theta d\theta$$

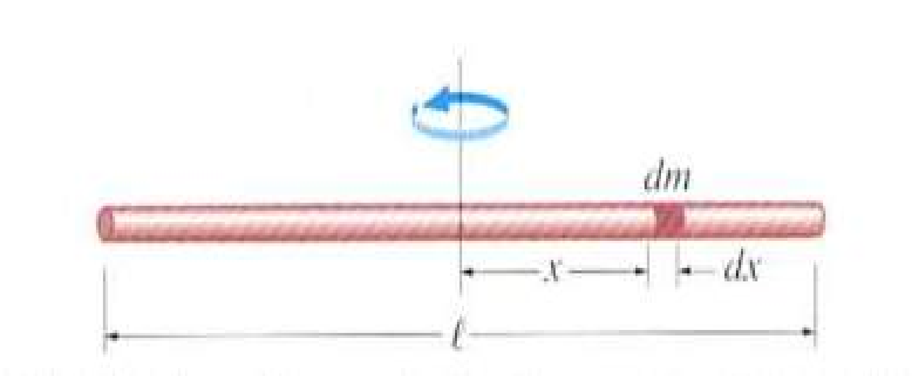
$$I = \frac{2\pi M}{4\pi R^2}R^4 \int_0^\pi (1 - \cos^2\theta)\sin\theta d\theta = \frac{1}{2}MR^2\left(2 - \frac{2}{3}\right)$$

$$I = \frac{2}{3}MR^2$$

Could we have used the moment of inertia for a solid sphere of radius R and subtracted the moment of inertia for a solid sphere (same density) of slightly smaller radius, $\boldsymbol{R\text{-}\delta R}$?

Absolutely, give it a try!

Example: Moment of Inertia of a Rod



Find the moment of inertia for a rod of mass *m* and length ℓ about an axis perpendicular to the rod through its (a) center and its (b) end.

(a) The integral for *I* with an axis through its center is straightforward

$$I = \int_{-\ell/2}^{\ell/2} x^2 dm = \int_{-\ell/2}^{\ell/2} x^2 \mu dx = \frac{m}{\ell}\int_{-\ell/2}^{\ell/2} x^2 dx$$

with the result:

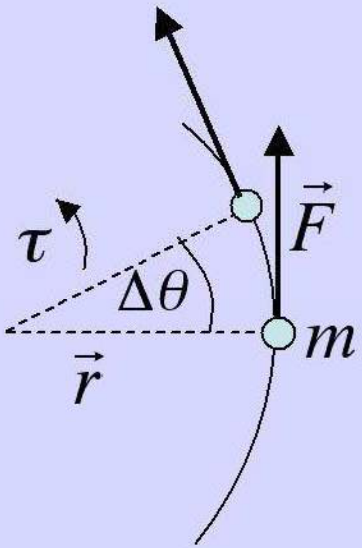
$$I = \frac{m}{\ell}\,\frac{2(\ell/2)^3}{3} = \frac{1}{12}m\ell^2$$

What is particularly interesting about this result is to note what happens when we consider its momentum of inertia about an axis through its end.

Rotational Energy

In the simplest sense we learned $\Delta W = F\Delta x$ **in linear motion.**

What are the work and energy in rotational motion?



If we integrate the torque we find the work done in the angular acceleration, generating rotational kinetic energy:

$$W = \int \tau d\theta = \int_{\theta_i}^{\theta_f} I\alpha d\theta = \int_{\theta_i}^{\theta_f} I\frac{d\omega}{dt} d\theta$$

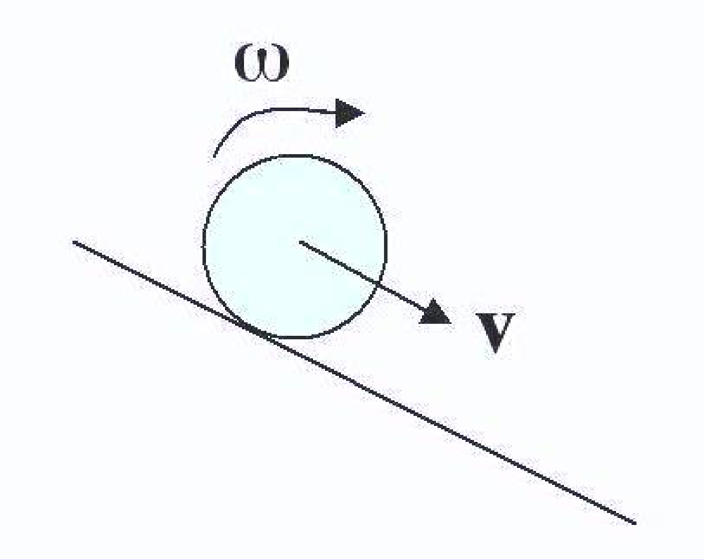
$$W = I\int_{\omega_i}^{\omega_f} \frac{d\theta}{dt} d\omega = I\int_{\omega_i}^{\omega_f} \omega d\omega = \frac{1}{2} I(\omega_f^2 - \omega_i^2)$$

The K_{rot} is as we expect, with I and ω analogs of linear motion.

$$\boxed{K_{rot} = \frac{1}{2} I\omega^2}$$

How does this effect rolling motion?

Kinetic Energy for a Rolling Object



For a rolling object the total kinetic energy is the sum of the linear and rotational kinetic energies.

$$K_{roll} = K_{lin} + K_{rot} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

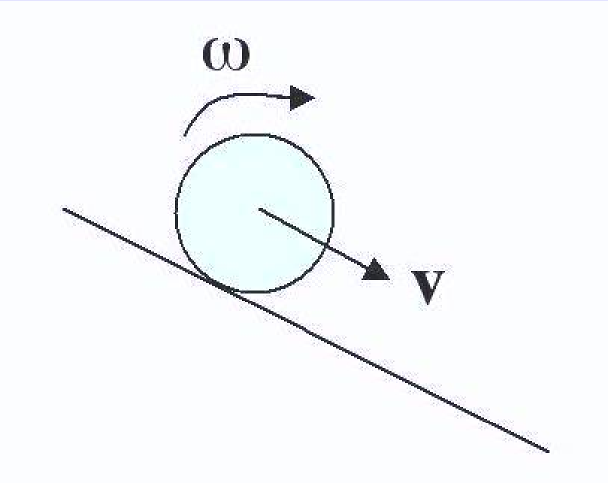
If a sphere rolls down an incline (without slipping), what fraction of its kinetic energy is rotational?

The non-slip condition is $\boldsymbol{v = R\omega}$. Hence the ratio of the KE's is:

$$\frac{K_{rot}}{K_{lin} + K_{rot}} = \frac{I\omega^2}{mv^2 + I\omega^2} = \frac{(2/5)mR^2\omega^2}{mv^2 + (2/5)mR^2\omega^2}$$

$$\frac{K_{rot}}{K_{lin} + K_{rot}} = \frac{2v^2}{5v^2 + 2v^2} = \frac{2}{7}$$

Example: A Ball Rolling Down an Incline



After rolling down a height ***h***, what is the speed of the ball after starting from rest?

Even in the presence of friction, if the ball rolls and does not slip, the total mechanical energy is conserved. (Without friction the ball will only slide.)

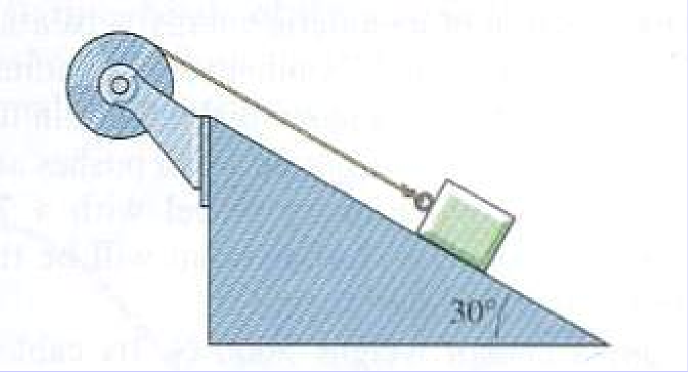
$$\Delta U + \Delta K = 0 \rightarrow -mgh + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = 0$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{5}mR^2\omega^2 = \frac{7}{10}mv^2$$

$$v = \sqrt{\frac{5}{7}2gh} = \sqrt{\frac{5}{7}}\sqrt{2gh} < \sqrt{2gh}$$

The ball has a lower velocity because some of the potential can been converted into rotational kinetic energy!

Example: Incline Plane w Pulley & Friction



A block of mass ***m*** is attached by a massless string to a solid cylinder pulley of mass ***M*** and radius ***R***. When the mass is released it accelerates down the plane with an acceleration ***a***. Find the coefficient of kinetic friction, $\boldsymbol{\mu_k}$.

Newton's 2nd for the mass ***m*** yields:

$$mg\sin\theta - T - \mu_k mg\cos\theta = ma$$

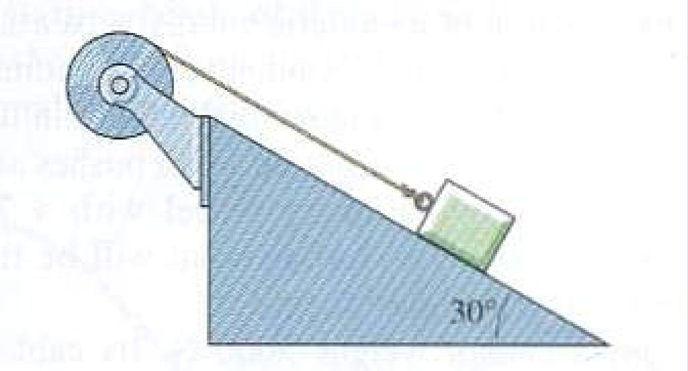
Again (the rope does not slip) the tangential acceleration of the pulley is equal to this acceleration:

$$a = \alpha R = \frac{\tau}{I} R = \frac{R^2}{I} T$$

For a solid cylinder $\boldsymbol{I = \frac{1}{2} MR^2}$ and after combining the equations:

$$(m + M/2)a = mg\sin\theta - \mu_k mg\cos\theta$$

Example: Incline Plane w Pulley & Friction



A block of mass ***m*** is attached by a massless string to a solid cylinder pulley of mass ***M*** and radius ***R***. When the mass is released it accelerates down the plane with an acceleration ***a***. Find the coefficient of kinetic friction, $\boldsymbol{\mu_k}$.

$$(m + M/2)a = mg\sin\theta - \mu_k mg\cos\theta$$

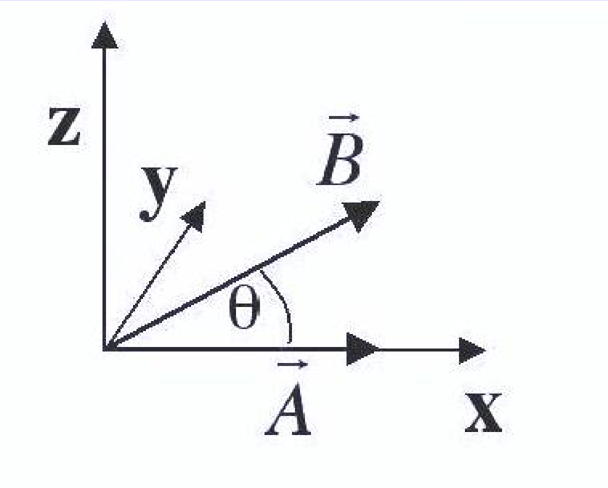
Solving for $\boldsymbol{\mu_k}$:

$$\mu_k = \frac{mg\sin\theta - (m + M/2)a}{mg\cos\theta} = \tan\theta - \frac{(m + M/2)}{m\cos\theta}\frac{a}{g}$$

If $\boldsymbol{\mu_k = \tan\theta}$ then there is no acceleration and the moment of inertia of the pulley is irrelevant. However for steeper angles the moment of inertia again reduces the acceleration of the mass sliding down the incline.

Could we have solved this problem with the work-energy theorem? You betcha! Now $\boldsymbol{\Delta K}$ includes rotational kinetic energy as well. Give it a try!

The Vector Product (Cross Product)



The vector product can be formed by expanding the vectors in their basis and applying the sinθ between the individual basis vectors.

$$\hat{i}\times\hat{i}=0,\quad \hat{j}\times\hat{j}=0,\quad \hat{k}\times\hat{k}=0$$
$$\hat{i}\times\hat{j}=\hat{k},\quad \hat{j}\times\hat{k}=\hat{i},\quad \hat{k}\times\hat{i}=\hat{j}$$
$$\hat{j}\times\hat{i}=-\hat{k},\quad \hat{k}\times\hat{j}=-\hat{i},\quad \hat{i}\times\hat{k}=-\hat{j}$$

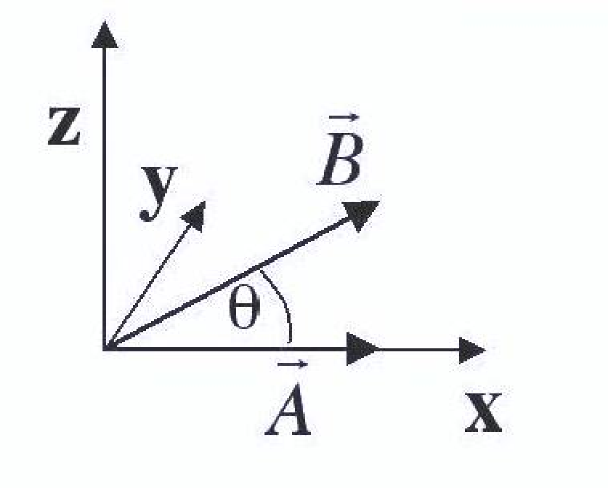
The net result is:

$$\vec{A}\times\vec{B}=\left(A_x\hat{i}+A_y\hat{j}+A_z\hat{k}\right)\times\left(B_x\hat{i}+B_y\hat{j}+B_z\hat{k}\right)$$
$$\vec{A}\times\vec{B}=(A_yB_z-A_zB_y)\hat{i}+(A_zB_x-A_xB_z)\hat{j}+(A_xB_y-A_yB_x)\hat{k}$$

For **A** and **B** in the x – y plane as shown:

$$\vec{A}\times\vec{B}=AB\sin\theta\,\hat{k}$$

The Vector Product (Cross Product)



The net result of this is we can express the vector product of **A** and **B** as the determinant:

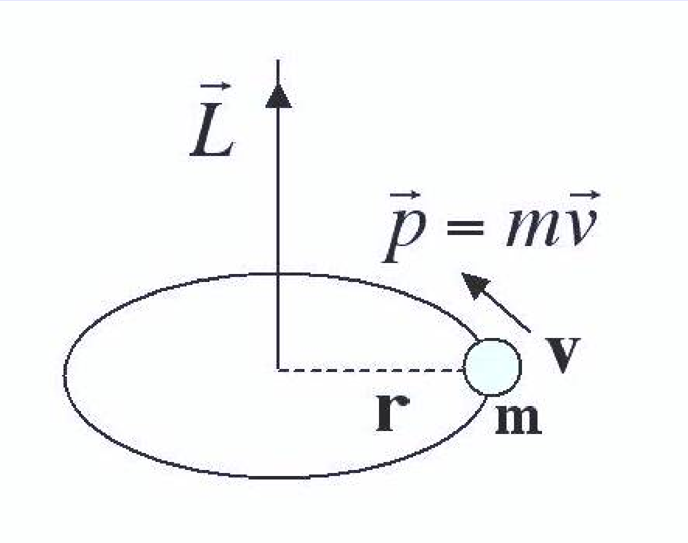
$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} \hat{i} & A_x & B_x \\ \hat{j} & A_y & B_y \\ \hat{k} & A_z & B_z \end{vmatrix}$$

The magnitude of the vector product of **A** and **B**:

$$\left|\vec{A} \times \vec{B}\right| = AB\sin\theta$$

The vector **A** x **B** is normal to the plane formed by the two vectors **A** and **B** with a direction determined by the "**right hand rule**".

Angular Momentum



Angular momentum for a particle is defined as

$$\boxed{\vec{L} = \vec{r} \times \vec{p}}$$

From our new definition of angular velocity:

$$\vec{\omega} = \frac{\vec{r} \times \vec{v}}{r^2}$$

we have the result:

$$\vec{L} = m(\vec{r} \times \vec{v}) = mr^2\vec{\omega}$$

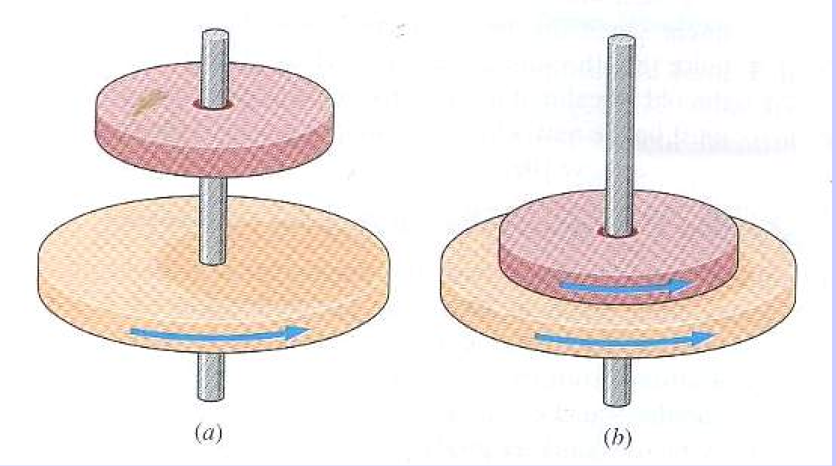
Summing (integrating) over multiple particles leads to:

$$\boxed{\vec{L} = I\vec{\omega}}$$

The torque is the rate of change of angular momentum.

$$\boxed{\vec{\tau} = I\vec{\alpha} = \frac{d\vec{L}}{dt}}$$

Example: Conservation of Angular Momentum



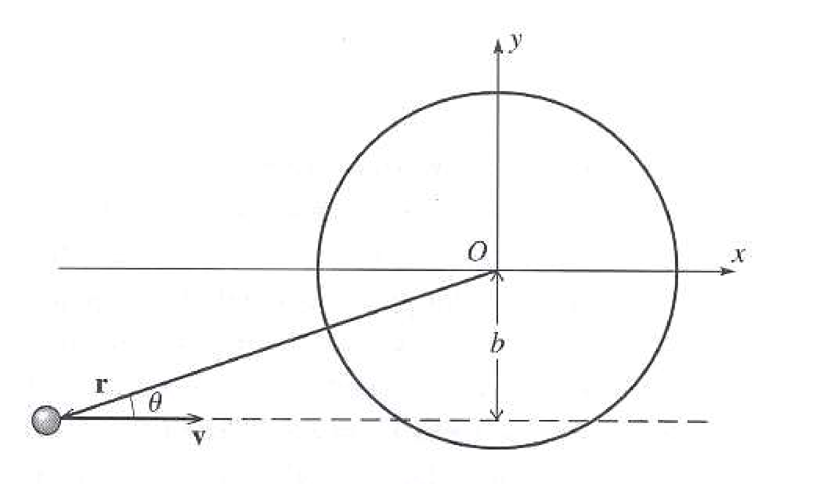
The lower disk has a mass $\boldsymbol{M_1}$ and a radius $\boldsymbol{R_1}$, while the upper disk has a mass $\boldsymbol{M_2}$ and a radius $\boldsymbol{R_2}$. The lower disk as a initial angular frequency of $\boldsymbol{\omega_i}$. The upper disk drops freely down and frictional forces act to bring the two disks to the same angular speed. (a) What is the final angular speed, $\boldsymbol{\omega_f}$?

In the absence of any external torque, angular momentum is conserved.
For this one dimensional problem this is expressed as:

$$
\begin{aligned}
L_i &= L_f \\
I_1\omega_i &= (I_1 + I_2)\omega_f \\
\omega_f &= \frac{I_1}{I_1 + I_2}\omega_i = \frac{M_1R_1^2}{M_1R_1^2 + M_2R_2^2}\omega_i
\end{aligned}
$$

Example: Conservation of Angular Momentum

A uniform turntable of mass ***M*** and radius ***R*** is a rest on a frictionless axle. A lump of putty, mass ***m***, approaches the turntable with a velocity ***v*** along a line that passes a distance ***b*** from the center of the turntable and sticks to its edge. Find the resulting angular frequency, ***ω***.



Again angular momentum is conserved. The initial angular momentum is that of the putty:

$$L_i = mvr\sin\theta = mvb$$

The final angular momentum is:

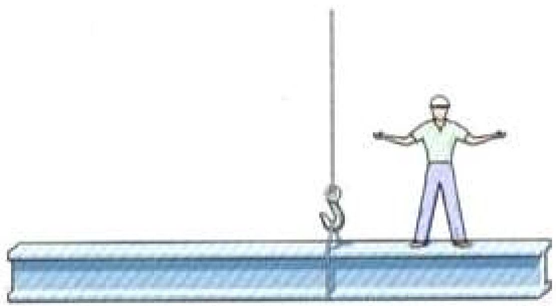
$$L_f = I_{tt}\omega + mR^2\omega = \left(\frac{1}{2}M + m\right)R^2\omega$$

Setting them equal and solving for ***ω***:

$$\omega = \frac{m}{m + M/2}\frac{vb}{R^2}$$

Conditions for Equilibrium

A body is in **static** equilibrium when it is stationary and when both "the net external force and the net external torque are zero!"


(a)

$$\sum_i \vec{F}_i = 0$$

and

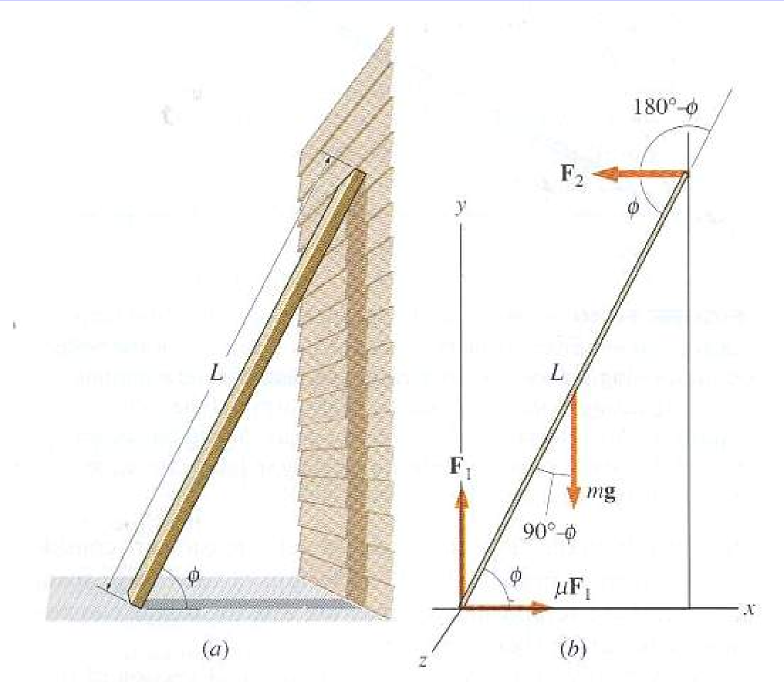
$$\sum_i \vec{\tau}_i = 0$$

If the tension in the support cable does not act at the center of "gravity" for the beam and the worker then the beam will rotate! Not good!

For static equilibrium we also have

$$\sum_i \vec{p}_i = 0$$

Example: Leaning Board Against a Wall



A board of mass m and length L is leaning against a wall. The wall is frictionless and the coefficient of static friction between the floor and the board is $\boldsymbol{\mu}$. Find the minimum angle $\boldsymbol{\phi}$ at which the board can be leaned without slipping.

The component force equations yield:

$$F_1 = mg \quad \text{and} \quad \mu F_1 = F_2$$

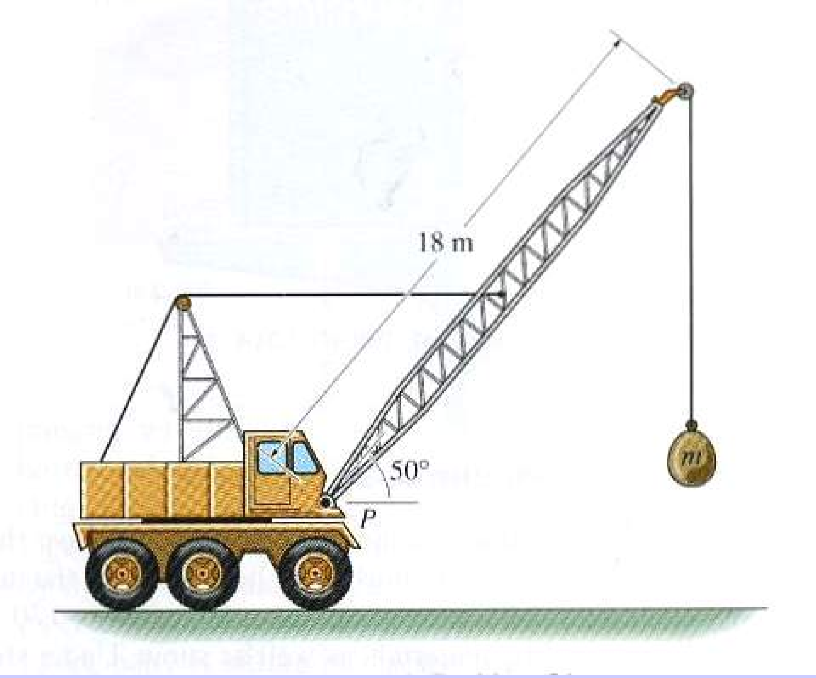
The most convenient origin for the torque equation is the bottom of the board. Balancing torques about that point:

$$mg\frac{L}{2}\sin(\pi/2-\phi) = F_2 L\sin\phi = \mu mgL\sin\phi$$

$$2\mu = \sin(\pi/2-\phi)/2\sin\phi = \cot\phi$$

Solving for $\boldsymbol{\phi}$: $\tan\phi = 1/2\mu \rightarrow \phi = \tan^{-1}(1/2\mu)$ Does this make sense?

Example: Crane Equilibrium



Consider the crane as shown in the figure. The boom is supported by a cable attached to its center point. Find the tension in the cable when angle of the boom is **50º** above the horizontal and the mass of the boom is ***1700kg.*** The hanging mass is ***2200kg***.

Summing torques about the pivot point ***P***:

$$T(9\sin 50^\circ) - 1700g(9\cos 50^\circ) - 2200g(18\cos 50^\circ) = 0$$

Solving for ***T***:

$$T = \frac{4400 + 1700}{\sin 50^\circ}(9.8\cos 50^\circ) = 50kN$$

Again the normal forces at the pivot balance all of the forces.

Stability of Equilibria

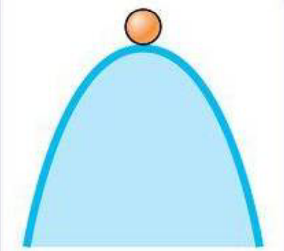
An unperturbed body in equilibrium will have no translational or rotational acceleration. A perturbation adds a small amount displacement to the system, and the response of the system to that displacement defines the stability.

(*a*)

$$\frac{d^2U}{dx^2} > 0$$

stable

A perturbed body in a stable equilibrium will return to that equilibrium state as the energy of the perturbation dissipates or damps.



(*b*)

$$\frac{d^2U}{dx^2} < 0$$

unstable

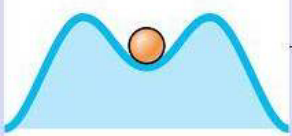
A perturbed body in unstable equilibrium will cause the perturbation to grow.

(*c*)

$$\frac{d^2U}{dx^2} = 0$$

neutral

Neutral equilibrium will remain as perturbed.

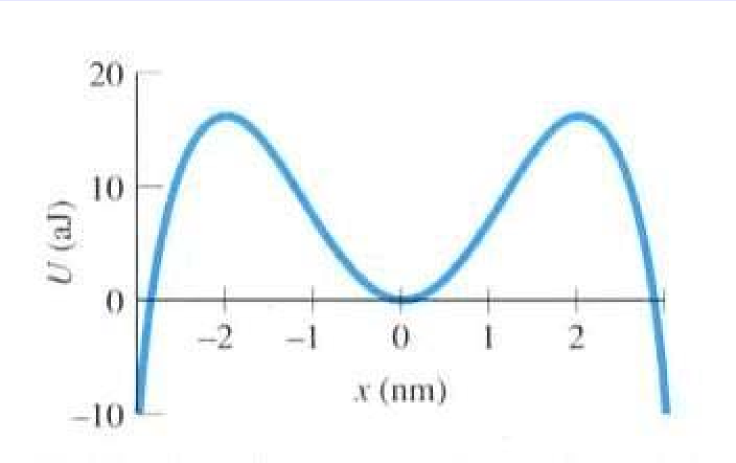


(*d*)

metastable

Metastable equilibrium has a threshold for instability. (locally stable)

Example: Multiple Equilibrium Points



In a certain semiconductor the local potential is:

$$U(x) = ax^2 - bx^4$$

Here ***x*** is the position of the electron in nm, and ***U*** is its potential energy in **a*J* = 10^{-18}*J***. The constants are ***a = 8 aJ/nm^2*** and ***b = 1 aJ/nm^3***. Locate the equilibrium positions for the electron and describe their stability.

For equilibrium ***dU/dx* = 0**:

$$\frac{dU}{dx} = 2ax - 4bx^3 = 0$$

$$x = 0,\ \pm\sqrt{a/2b} = 0,\ \pm 2nm$$

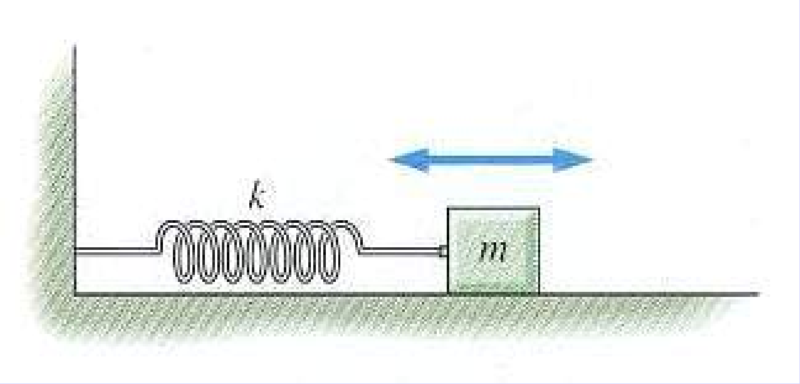
The second derivative determines stability:

$$\frac{d^2U(x)}{dx^2} = 2a - 12bx^2$$

$$\frac{d^2U(x=0)}{dx^2} = 2a = 16 > 0 \rightarrow \text{stable}$$

$$\frac{d^2U(x=\pm 2)}{dx^2} = 16 - 12(4) < 0 \rightarrow \text{unstable}$$

Simple Harmonic Motion



Simple harmonic motion results when an object is subject to a linear restoring force and is called **simple harmonic motion**, SHO.

Mathematically such a force is described as:

$$F = -kx$$

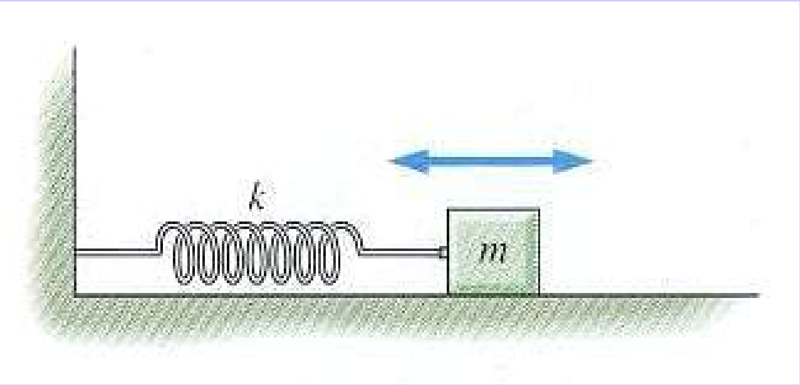
The is the force exerted by an ideal spring of spring constant ***k***.
From Newton's 2nd we can write:

$$F = m\frac{d^2x}{dt^2} = -kx$$

An object experiencing such a force means that when it is displaced from equilibrium there is a force proportional to the distance from equilibrium that accelerates the object back towards its equilibrium position.

How do we describe such motion?

Simple Harmonic Motion



Simple harmonic motion results when an object is subject to a linear restoring force and is called **simple harmonic motion**, SHO. The general solution is then:

$$x = A\cos\omega t + B\sin\omega t \quad \text{with} \quad \omega = \sqrt{k/m}$$

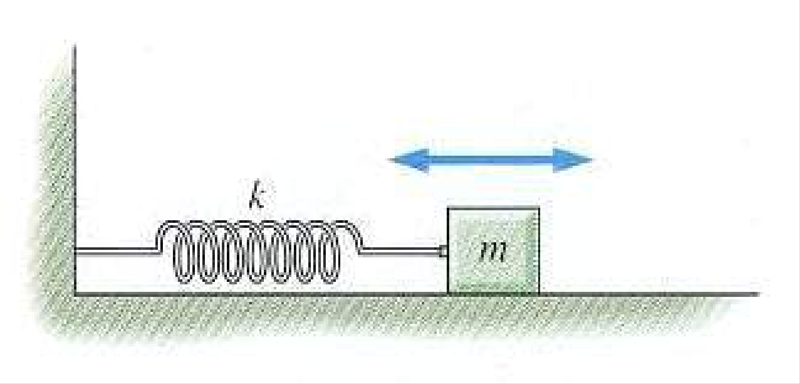
What about the unknown constants ***A*** and ***B***? They are determined by the initial conditions. For example if at $\boldsymbol{t = 0}$ the system satisfies $\boldsymbol{x = x_o}$ and $\boldsymbol{v = 0}$, then:

$$x(t = 0) = A = x_o \quad \text{and} \quad v = dx/dt = \omega B = 0$$

The solution for this initial condition becomes:

$$x(t) = x_o \cos\omega t$$

Simple Harmonic Motion



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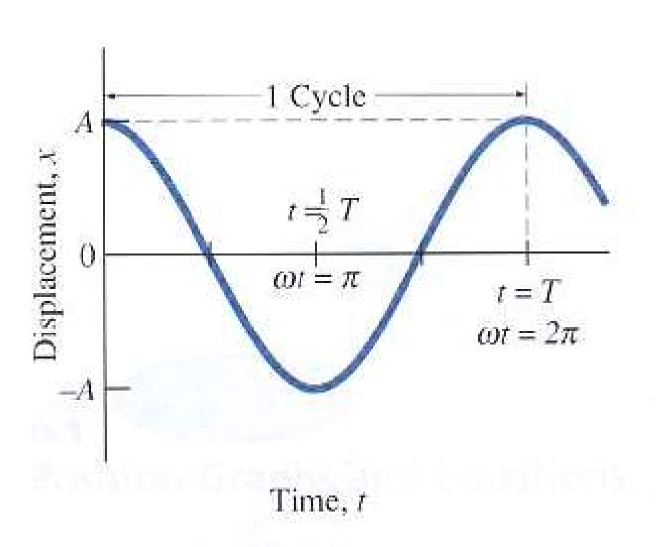
Graphically this solution is given by:



Does the expression for angular frequency

$$\omega = \sqrt{k/m}$$

make sense? What happens when the spring constant ***k*** increases, what about the dependence on ***m***?

S H O - Velocity and Acceleration

It is useful to consider the velocity and acceleration as it relates to the displacement. For this we will use the solution that includes the phase.

$$x(t) = A\cos(\omega t + \phi)$$

The velocity is the first derivative:

$$v(t) = \frac{dx}{dt} = -A\omega\sin(\omega t + \phi)$$

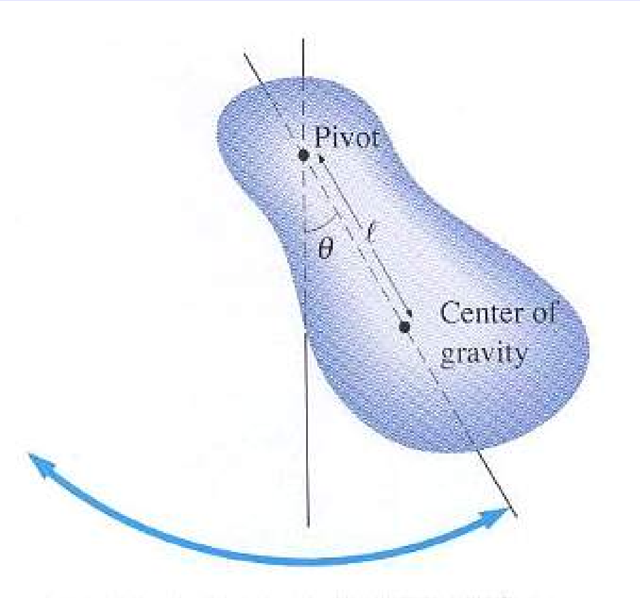
From this we see that the velocity is out of phase with the displacement. When the displacement is maximum, the velocity is zero. Similarly when the velocity is maximum the displacement is zero.

The acceleration is the second derivative:

$$a(t) = \frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t + \phi)$$

The acceleration **always** has the opposite sign of the displacement, i.e. the object is under the influence of a restoring force!

Example: Physical Pendulum



Consider the physical pendulum as shown in the figure. The center of gravity is a distance ***l*** from the pivot point. The equation of motion obtained from finding the torques about the pivot point is:

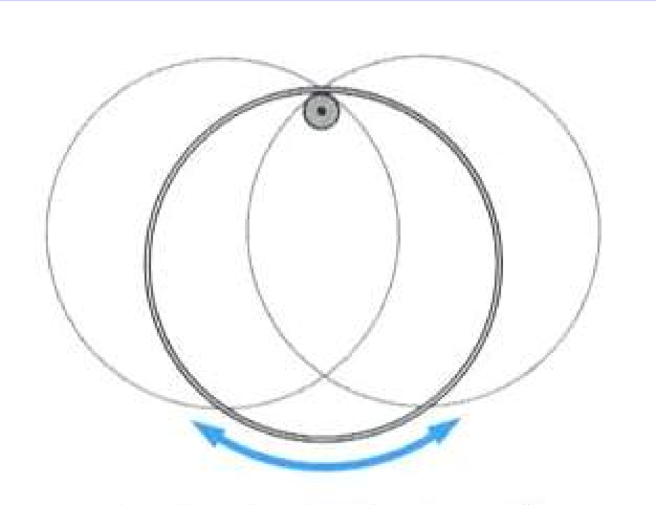
$$\frac{d^2\theta}{dt^2} + \frac{mg\ell}{I}\sin\theta = 0$$

This is the same as we obtained earlier, only now the momentum of inertia does not necessarily have a simple form. In the limit $\boldsymbol{\theta \ll 1}$ the solution is:

$$\theta(t) = A\cos(\omega t + \phi) \quad \text{with} \quad \omega^2 = mg\ell/I$$

Remember in this expression ℓ is the distance from the pivot to the CG.

Example: Physical Pendulum - Hoop



Consider a hoop of mass ***m*** and radius ***R***. It is oscillating about a thin horizontal rod as shown. Find its period of oscillations.

The moment of inertia of a hoop about an axis through its center is $\boldsymbol{I_{cm} = m R^2}$. From the parallel axis theorem if it is rotating about an axis on its circumference, $\boldsymbol{I = I_{cm} + mR^2 = 2mR^2}$.

The angular frequency for small amplitude oscillations is:

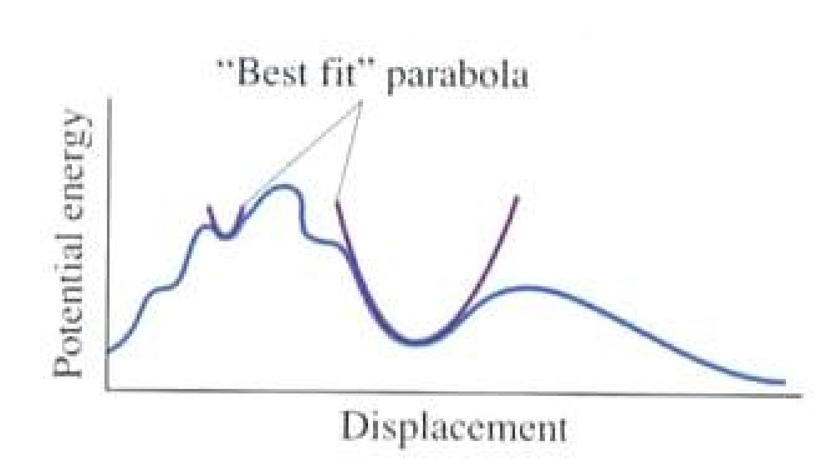
$$\omega = \sqrt{mg\ell/I} = \sqrt{mgR/2mR^2} = \sqrt{g/2R}$$

The period is:

$$T = 2\pi/\omega = 2\pi\sqrt{2R/g}$$

Oscillatory Motion and Potential Energy Functions

For small displacements from a position of stable equilibrium, $\boldsymbol{x_o}$, the potential can usually be approximated by an upright parabola. Consider a Taylor series about $\boldsymbol{x_o}$:



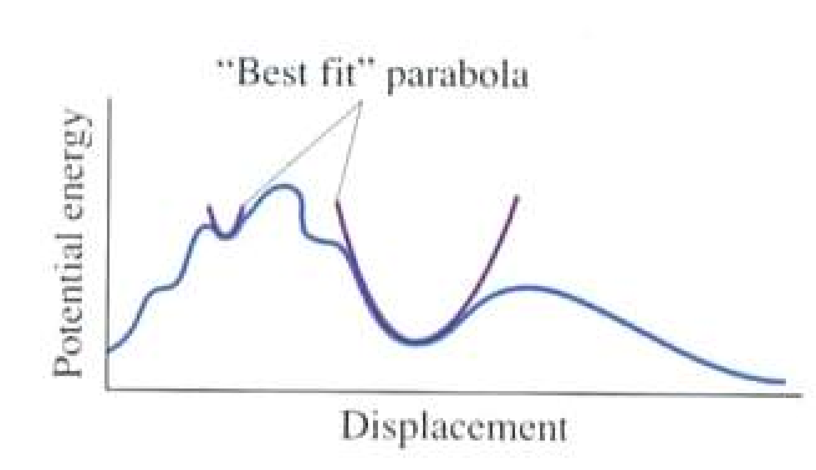
$$U(x) \simeq U(x_o) + \frac{dU(x_o)}{dx}(x - x_o) + \frac{1}{2}\frac{d^2U(x_o)}{dx^2}(x - x_o)^2$$

At the minimum the first derivative of the potential vanishes. Any potential energy has an arbitrary constant as it is the change in potential energy that is important. As long as the displacement from equilibrium is small we have:

$$\Delta U(x) \simeq \frac{1}{2}\frac{d^2U(x_o)}{dx^2}(x - x_o)^2 \text{ and } F = -\frac{dU}{dx} = -\frac{d^2U(x_o)}{dx^2}(x - x_o)$$

Oscillatory Motion and Potential Energy Functions

For small displacements from a position of stable equilibrium, $\boldsymbol{x_o}$, the potential can usually be approximated by an upright parabola.



$$F = -\frac{dU}{dx} = -\frac{d^2U(x_o)}{dx^2}(x - x_o) = -k(x - x_o) \text{ with } k = \frac{d^2U(x_o)}{dx^2}$$

This is Hook's law all over again with an effective spring constant being given by the second derivative of the potential at the point of equilibrium. So SHO is a very general phenomena. As it turns out it is also very general phenomena even in quantum mechanics. Only when the second derivative vanishes or when the displacements from equilibrium are large is this not a good approximation!

Newton's Law of Gravity

Newton realized that the motion of the falling apple and the motion of the moon around the Earth were due to the same force. They were both falling toward the Earth due to the force of gravity.

$$F_g = -\frac{GMm}{r^2}$$ Universal Gravitation

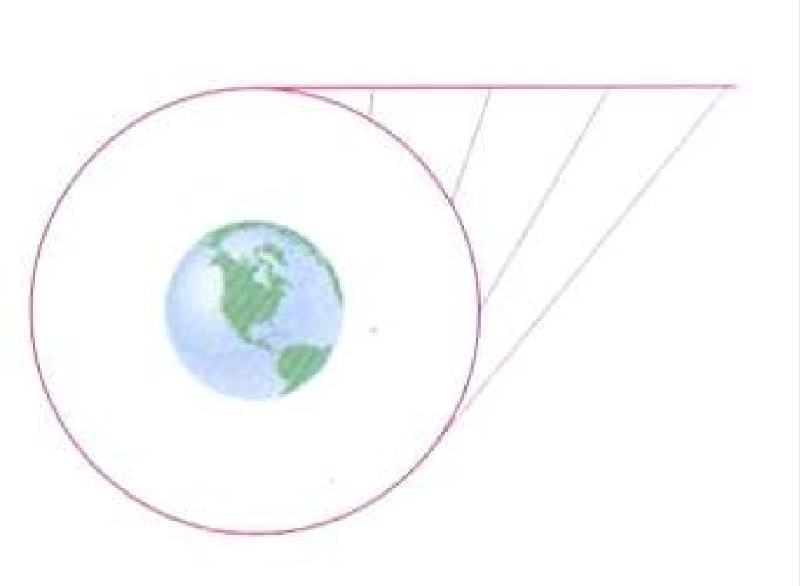
This force obeys the inverse square law. Also the minus sign indicates that this force is attractive.

G is the universal constant of gravitational attraction and is given by

$$\boldsymbol{G = 6.673\text{x}10^{-11} Nm^2/kg^2}$$

Strictly speaking it only applies to point objects. However, for spherically symmetrical objects ***r*** is the distance between their centers. As long as the size of the object is small compared to ***r***, then it is simply the distance between them.

Orbital Motion



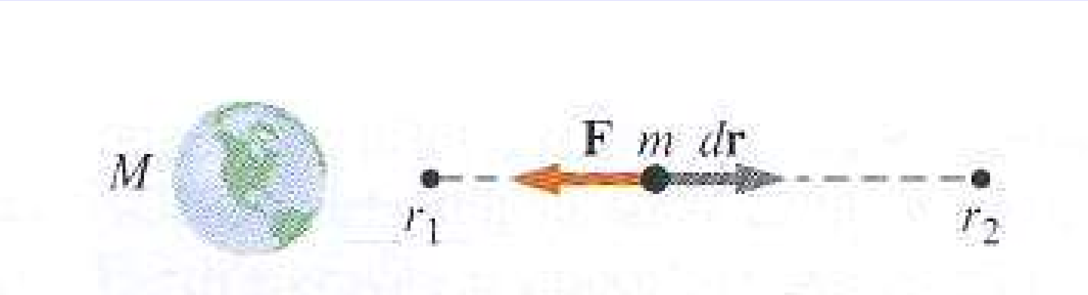
An object orbiting the Earth (or any other object orbiting a large massive object) is accelerating toward the center of the Earth. The blue lines indicate the path of an object in the absence of gravity. From our study of circular motion we know that gravity must provide the force for radial acceleration. This leads to the period for a circular orbit:

$$\frac{GMm}{r^2} = \frac{mv^2}{r} = m\omega^2 r \rightarrow \omega^2 = \frac{GM}{r^3}$$

$$\frac{4\pi^2}{T^2} = \frac{GM}{r^3} \rightarrow T^2 = \frac{4\pi^2}{GM} r^3$$

We have proved Kepler's 3rd law for circular orbits. Note that this expression is independent of the object's **mass**. This law is the primary way astronomers measure the product ***GM*** of objects throughout our galaxy.

Gravitational Potential Energy



The change in gravitational potential that occurs when moving an object of mass ***m*** from $\boldsymbol{r_1}$ to $\boldsymbol{r_2}$ is:

$$\Delta U = -\int_{r_1}^{r_2} \vec{F} \cdot d\vec{r} = \int_{r_1}^{r_2} \frac{GMm}{r^2} dr = -GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

$$\Delta U = GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

So the potential energy increases as an object's distance from the Earth (or any massive object) is increased. By convention the potential energy is defined to be zero when the object is infinity far from the gravitational source. Hence setting $\boldsymbol{r_1}$ to infinity we find

$$U(r) = -\frac{GMm}{r}$$

Gravitational Energy

The total energy for an object in a gravitational field is:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

For a circular orbit there is only a tangential velocity and it is gravity that provides the radial acceleration:

$$m\frac{v^2}{r} = \frac{GMm}{r^2} \rightarrow \frac{1}{2}mv^2 = \frac{1}{2}\frac{GMm}{r}$$

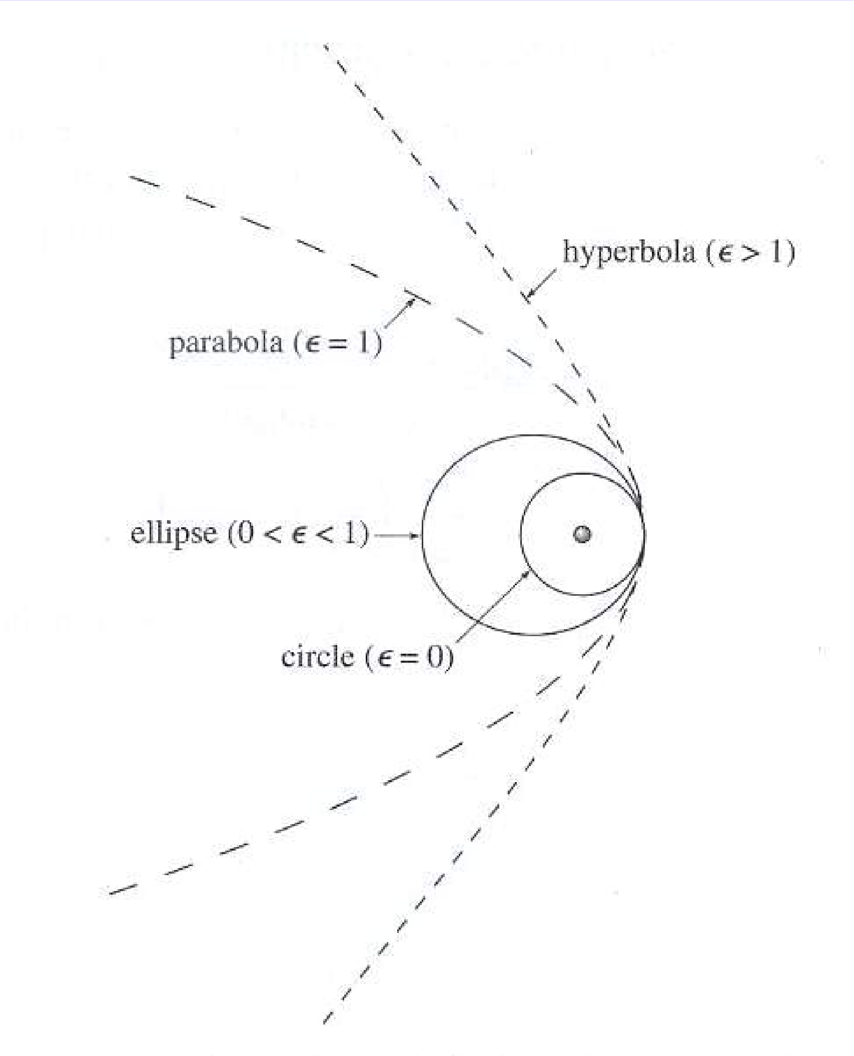
The total energy is then:

$$E = \frac{1}{2}\frac{GMm}{r} - \frac{GMm}{r}$$

$$E = -\frac{1}{2}\frac{GMm}{r} = \frac{1}{2}U = -K$$

Higher kinetic energy corresponds to a lower total energy! To get into a faster circular orbit a spacecraft must *lose* energy! To get into a slower circular orbit a spacecraft must *gain* energy!

Escape Velocity



To escape the gravitational field the spacecraft must have ***E* > 0**. It must be in a hyperbolic orbit, or at least a parabolic orbit for which ***E* = 0**. For that case:

$$E = \frac{1}{2} m v_{esc}^2 - \frac{GMm}{r} = 0$$

$$v_{esc} = \sqrt{2GM/r}$$

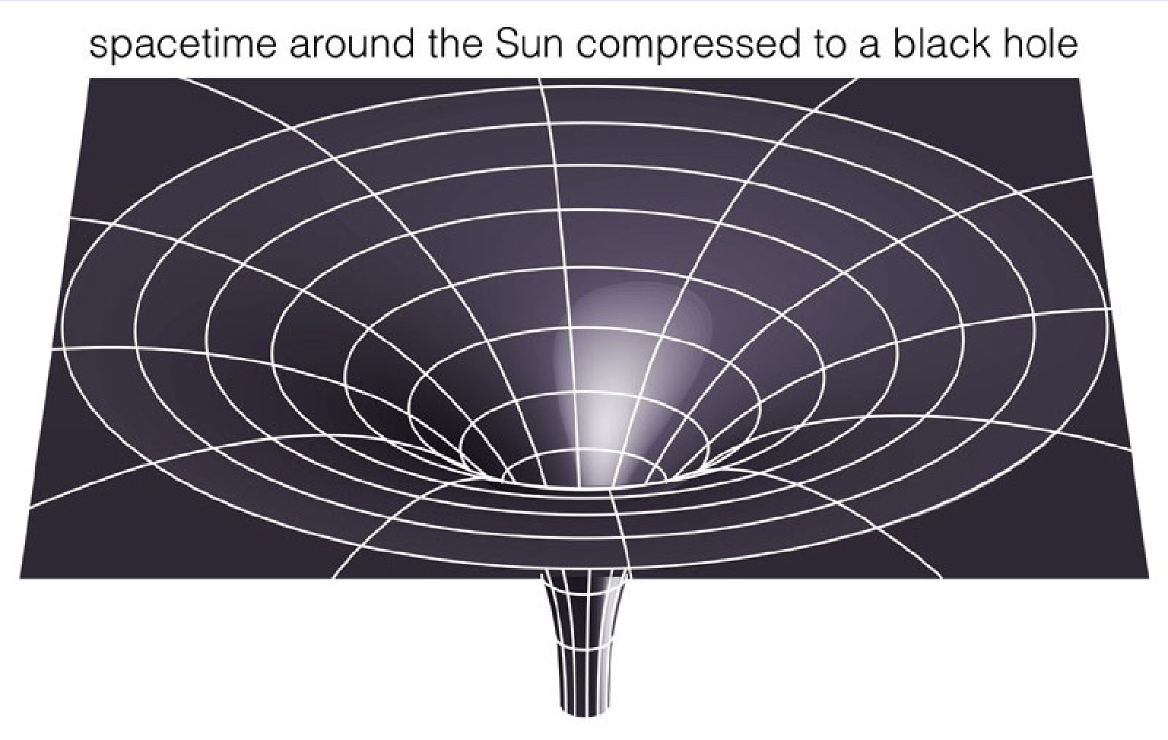
Does the direction matter? No!

At the surface of the Earth:

$$v_{esc} = \sqrt{2GM_E/R_E} = 40 km/hr$$

Black Holes

General Relativity also allows for the possibility of Black Holes. In this case the mass of a large star has collapsed to such a small volume that nothing can stop it from proceeding to infinite density. A horizon forms about this "singularity" that does not allow anything to escape even light.



The radius of an "event horizon" from which inside nothing can escape is:

$$R_{hor} = 2GM/c^2 \rightarrow R_{Sun} \simeq 3km$$