PHYSICS 2B - Lecture Notes

Ch. 32: Inductance and Magnetic Energy

Preliminaries

A current in a circuit produces a magnetic flux that can extend to neighboring circuits. Changes in the current in the first circuit can induce an emf in the second circuit. This is the phenomenon we consider in this chapter.

Transformer

Consider a coil, with N_1 turns wound on an iron core. A time-varying potential V_1 produces a flux ϕ_c in the core. Since the coil has N_1 turns, the total flux through the coil is $N_1\phi_c$ and, by Faraday's law

$$V_1 = -N_1 \frac{d\phi_c}{dt}.$$

A second coil with N_2 turns is wound on the same iron core. The changing flux in the core induces a potential in second coil given by

$$V_2 = -N_2 \frac{d\phi_c}{dt}.$$

Comparing, we see that.

$$\frac{V_2}{V_1} = \frac{N_2}{N_1}.$$

This is the essence of the *transformer*; a smaller time-varying potential applied to the coil with the smaller number of turns will produce a higher potential across the coil with the higher number of turns

Mutual Inductance

A current in circuit 1 induces a magnetic flux in circuit 2. We define the *mutual inductance* of the two circuits by

$$M = \frac{\phi_2}{I_1}.$$

From this definition it follows that

$$\frac{d\phi_2}{dt} = M\frac{dI_1}{dt} \qquad \Rightarrow \qquad \mathcal{E}_2 = -M\frac{dI_1}{dt},$$

where we have used Faraday's law. If we reverse the roles of circuits 1 and 2, the value of M will remain the same.

Self Inductance

A current in a circuit will also induce a magnetic flux in that circuit. Parallel to our definition of mutual inductance, we define *self inductance* as

$$L = \frac{\phi_B}{I},$$

so that

$$\frac{d\phi_B}{dt} = L\frac{dI}{dt} \qquad \Rightarrow \qquad \mathcal{E} = -L\frac{dI}{dt}.$$

The SI unit of inductance is the *henry,* named for the American scientist Joseph Henry (1797-1878).

Solenoid

Recall that the magnetic field inside a solenoid is

$$B = \mu_0 n I,$$

where n is the number of turns per unit length of the solenoid. The flux through each turn of the solenoid is BA, where A is the cross-sectional area of the solenoid. If the length of the solenoid is ●, the total number of turns is N = n●, and the total flux is the sum of the fluxes through each turn, so that

$$\phi_B = NAB = (n\ell)A(\mu_0 n I) = \mu_0 n^2 I A \ell.$$

The self-inductance of the solenoid is then

$$L = \frac{\phi_B}{I} = \mu_0 n^2 A \ell.$$

Inductors in Circuits

Consider a series circuit with a source of emf, a resistor and an inductor. If the current is steady, the inductor does not contribute an emf to the circuit and simply acts like a conductor. But if a switch is added to the circuit and closed at $t = 0$, the inductor plays a more active role. The Kirchoff law for emfs gives

$$\mathcal{E} - RI - L\frac{dI}{dt} = 0 \qquad \Rightarrow \qquad \frac{L}{R}\frac{dI}{dt} = \frac{\mathcal{E}}{R} - I \qquad \Rightarrow \qquad \frac{dI}{\mathcal{E}/R - I} = \frac{R}{L}dt.$$

So that,

$$\ln(\mathcal{E}/R - I)\Big|_0^I = -\frac{Rt}{L} \qquad \Rightarrow \qquad I = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{Rt}{L}}\right).$$

The current, which is initially zero, rises to its steady state value as given by Ohm's law.

Magnetic Energy

We can now obtain the energy stored in an inductor with a current I passing through it. The rate at which energy is being stored in the inductor is given by the power,

$$P = VI = L\frac{dI}{dt}I \ .$$

Then, the energy stored in the inductor after a time t after which the current has reached a value I is

$$U(t) = \int_0^t VIdt = L\int_0^I IdI = \frac{LI^2}{2} \ .$$

This is energy stored in the inductor and which would be released if the current were to drop to zero.

Laboratories that use superconducting magnets must always be vigilant about the level of the liquid helium coolant. Such magnets can carry currents in excess of a thousand amperes. If the coolant is lost and the magnet reverts to a normal conductor, this stored energy is released instantaneously with potentially damaging results.

Magnetic Energy Density

The stored energy in the solenoid is

$$U = \frac{LI^2}{2} = \frac{(\mu_0 n^2 A\ell)I^2}{2} = \frac{(\mu_0 nI)^2}{2\mu_0}A\ell \ .$$

The quantity, A●, is the volume of the solenoid and the quantity is the uniform magnetic field in the solenoid, so that the stored energy per unit volume, the magnetic energy density, is

$$u_B = \frac{B^2}{2\mu_0} \ .$$

As was the case for the electrical energy density, this result, although derived for a special case, is true in general.