PHYSICS 2B - Lecture Notes

Ch. 34: Maxwell's Equations and Electromagnetic Waves

Preliminaries

Arguably the most astounding result of classical electromagnetic theory was the prediction of electromagnetic waves. By far the vast majority of the information we receive about world comes in this form. From the microwave remnants of the creation of the universe, to the narrow frequency range to which our eyes are sensitive, to the gamma rays that signal that a black hole has devoured a star, all of these are electromagnetic waves that differ only in frequency.

Maxwell's Equations

These are the four equations that summarize all of classical electromagnetic theory and we have already encountered them. These are:

Gauss's Law (electrical) $\oint_A \vec{E}\cdot d\vec{A} = \frac{q_{tot}}{\varepsilon_0}$

Gauss's Law (magnetic) $\oint_A \vec{B}\cdot d\vec{A} = 0$

Faraday's Law $\oint_C \vec{E}\cdot d\vec{\ell} = -\frac{d\phi_B}{dt}$

Ampere's Law* $\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0 I_{tot}$

**** Incomplete***

Maxwell's Modification of Ampere's Law

Maxwell was troubled by Ampere's law in one particular situation. In a circuit with a capacitor, current could flow in the outer circuit even though no current was flowing in the interior of the capacitor, so that the current was not continuous around the circuit. How then to apply Ampere's law? Maxwell reasoned that in charging the capacitor, there was an effective current between the plates.

Recall that the field between the plates of a capacitor is, $E = \frac{\sigma}{\varepsilon_0}$. Multiplying by, A, the area of a plate, we have

$$EA = \frac{\sigma A}{\varepsilon_0} = \frac{q}{\varepsilon_0} = \phi_E$$

where q is the charge on one of the capacitor plates and ϕ_E is the electric flux through the area of the plates. (Recall that although Gauss's law is taken over a closed surface, the electric field is zero outside a capacitor.) Then writing

$$q = \varepsilon_0 \phi_E \qquad \text{so that} \qquad \frac{dq}{dt} = \varepsilon_0 \frac{d\phi_E}{dt}.$$

Maxwell denoted this last term the *displacement current* and included it with the true current in Ampere's law (1861). We then have,

Ampere's Law (Complete)

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 \left(I + \frac{dq}{dt} \right) = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\phi_E}{dt}.$$

Mathematical Digression: ***The Wave Equation***

Propagating waves occur in virtually every area of Physics. The simplest example is a wave that propagates with a constant speed without change of shape. In one dimension, such a wave can be written,

$$\psi(z,t) = \psi(w), \qquad \text{where} \quad w = z - ct.$$

For such a wave, we note that,

$$\frac{\partial \psi}{\partial z} = \frac{d\psi}{dw}\frac{\partial w}{\partial z} = \frac{d\psi}{dw} \times 1 \qquad \Rightarrow \qquad \frac{\partial^2 \psi}{\partial z^2} = \frac{d^2 \psi}{dw^2},$$

and,

$$\frac{\partial \psi}{\partial t} = \frac{d\psi}{dw}\frac{\partial w}{\partial t} = \frac{d\psi}{dw} \times (-c) \qquad \Rightarrow \qquad \frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{d^2 \psi}{dw^2},$$

so that

$$\frac{\partial^2 \psi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0 \quad .$$

This is the one-dimensional *wave equation*.

The quantity, c, has the dimension of *length/time* or velocity. To determine its significance, we measure the amplitude of the wave at a position z and at a time t to be

$$\psi(z,t) = \psi(w), \qquad \text{where} \quad w = z - ct.$$

After a time interval Δt, we measure the amplitude of the wave at a new position $z+\Delta z$ to be,

$$\psi(w + \Delta w) \qquad \text{where} \qquad \Delta w = \Delta z - c\Delta t.$$

Now if, $\dfrac{\Delta z}{\Delta t} = c$ then $\Delta w = 0$, and the value of ψ will be unchanged. That is, in the interval Δt, the wave will have moved a distance $\Delta z = c\Delta t$, without change in shape. Therefore, the quantity c in the wave equation is the velocity of propagation of the wave. (The Latin word for speed is *celeritas.*)

Now, back to Maxwell's equations.

Maxwell's Equations in Vacuum

In a vacuum there are no true charges or currents, so that ρ and I are both zero. Then Maxwell's equations become,

Gauss's Law (electrical) $$\oint_A \vec{E}\cdot d\vec{A} = 0$$

Gauss's Law (magnetic) $$\oint_A \vec{B}\cdot d\vec{A} = 0$$

Faraday's Law $$\oint_C \vec{E}\cdot d\vec{\ell} = -\frac{d\phi_B}{dt}$$

Ampere's Law $$\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0\varepsilon_0\frac{d\phi_E}{dt}.$$

From Maxwell's Equations to the Wave Equation

We now show that Maxwell's equations lead directly to wave-like solutions. We take the direction of the electric field to be the x-axis. With a Gaussian surface that is a flattened cylinder with its axis parallel to x-axis. The top and bottom have area A and its height is δx. Gauss's law yields

$$\oint_A \vec{E}\cdot d\vec{A} = [E_x(x+\delta x) - E_x(x)]A = \frac{\partial E_x}{\partial x}A\delta x = 0 \qquad \Rightarrow \qquad \frac{\partial E_x}{\partial x} = 0 \quad .$$

Therefore, the electric field can change only in a direction perpendicular to x, which we denote as z. A third direction, y, is chosen so that (x,y,z) form a right-handed triad. Next, we imagine a rectangular path with two sides of length, h, parallel to the x-axis and two sides of length, δz, parallel to the z-axis. Then

$$\oint_C \vec{E}\cdot d\vec{\ell} = [E_x(z+\delta z) - E_x(z)]h \qquad \text{and} \qquad \phi_B = B_y A\delta z \quad .$$

Then, by Faraday's law,

$$\oint_C \vec{E}\cdot d\vec{\ell} = -\frac{d\phi_B}{dt} \qquad \Rightarrow \qquad [E_x(z+\delta z) - E_x(z)]h = -\frac{\partial B_y}{\partial t}h\delta z \qquad \Rightarrow \qquad \frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}.$$

Similarly, a flat cylinder with its axis parallel to the y-axis serves as a Gaussian surface for the magnetic Gauss's law, yields

$$\frac{\partial B_y}{\partial y} = 0.$$

We then invoke Ampere's's law using a rectangular path with two sides of length h parallel to the y-axis and the other two of length δz parallel to the z-axis to obtain

$$\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0\varepsilon_0 \frac{d\phi_E}{dt} \quad \Rightarrow \quad [B_y(z) - B_y(z+\delta z)]h = \mu_0\varepsilon_0 \frac{\partial E_x}{\partial t} h\delta z \quad \Rightarrow \quad -\frac{\partial B_y}{\partial z} = \mu_0\varepsilon_0 \frac{\partial E_x}{\partial t}.$$

The two equations,

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \qquad \text{and} \qquad -\frac{\partial B_y}{\partial z} = \mu_0\varepsilon_0 \frac{\partial E_x}{\partial t},$$

may be combined to yield,

$$\frac{\partial^2 E_x}{\partial z^2} - \mu_0\varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} = 0 \qquad \text{and} \qquad \frac{\partial^2 B_y}{\partial z^2} - \mu_0\varepsilon_0 \frac{\partial^2 B_y}{\partial t^2} = 0 \; .$$

Therefore, the components of the electric and magnetic fields in vacuum both obey the *wave equation*.

Electromagnetic Waves

We have seen that Maxwell's equations lead to the wave equations,

$$\frac{\partial^2 E_x}{\partial z^2} - \mu_0\varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} = 0 \qquad \text{and} \qquad \frac{\partial^2 B_y}{\partial z^2} - \mu_0\varepsilon_0 \frac{\partial^2 B_y}{\partial t^2} = 0 \; .$$

A number of results follow immediately:

1. The electric and magnetic fields both obey the wave equation with a propagation velocity

$$c^2 = \frac{1}{\varepsilon_0\mu_0} \quad \Rightarrow \quad c = \frac{1}{\sqrt{\varepsilon_0\mu_0}} \quad \Rightarrow \quad c = 2.99792458\times 10^8 \text{ m/s} .$$

This is the velocity of light in vacuum, which confirmed for the first time that light is an electromagnetic wave.

2. The electric field, the magnetic field and the propagation direction form a right-handed triad. That is, the electromagnetic wave is *transverse*, with the oscillations perpendicular to the direction of propagation.

3. The electric and magnetic fields are in phase.

4. A wave can be characterized by a wavelength, λ and a frequency *f* such that $c = \lambda f$. That is, electromagnetic waves come in an enormous range of wavelengths, from gamma rays with wavelengths of a few picometers to radio waves with wavelengths of hundreds of kilometers. All propagate with the same speed, *c*.

The prediction of electromagnetic waves was confirmed by Hertz in 1888 and is regarded as one of the greatest triumphs of classical electromagnetic theory.