hw1

Solutions for 1.7, 1.12, 1.25

1.7

source: (2,8,7), field: (4,6,8), so

$$\vec{r} = (4-2, 6-8, 8-7) = (2,-1,1)$$
$$r = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$$
$$\hat{r} = \vec{r}/r = \frac{1}{\sqrt{6}}(2,-1,1)$$

1.12

$h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$

(a) and (b): necessary condition of hill top is $\partial h/\partial x = 10(2y - 6x - 18) = 0$ and $\partial h/\partial y = 10(-8y + 2x + 28) = 0$

solve, $\Rightarrow x = -2, y = 3$, at which point $h(x,y) = 720$

(c): The gradient at arbitrary point (x,y) is:

$\nabla h = 10(2y - 6x - 18, -8y + 2x + 28)$

The slope along direction $\hat{n}$ is $\hat{n} \cdot \nabla h$. At the point $(1,1)$, the gradient is $(-220, 220)$, thus the slope along an arbitrary direction $\hat{n} = (n_x, n_y)$ is $\hat{n} \cdot \nabla h = -220n_x + 220n_y$

The steepest slope is along the direction of the gradient, $\nabla h/|\nabla h| = (-1,1)/\sqrt{2}$, i.e., toward northwest.

1.25

Below, $\nabla^2 \equiv \Delta$

(a). $\nabla T_a = \big(2(x+y), 2x, 3\big)$, $\Delta T_a = 2$

(b). $\partial^2 T_b/\partial x^2 = -T_b$, etc., $\Rightarrow \Delta T_b = -3T_b = -3\sin x \sin y \sin z$

(c). $\partial_x^2 \to (-5)^2 = 25, \partial_y^2 \to -(4^2) = -16, \partial_z^2 \to -(3^2) = -9$, thus

$\Delta T_c = (25 - 16 - 9)T_c = 0$

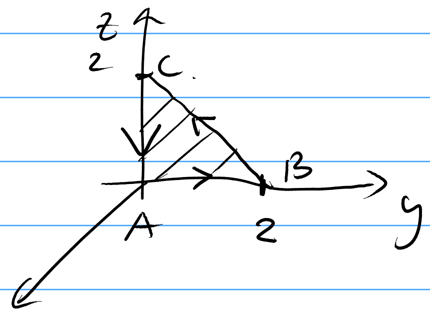
(d). $\Delta v_x = 2, \Delta v_y = 6x, \Delta v_z = 0$, thus

$\Delta \vec{v} = 2\hat{x} + 6x\hat{y}$

1.33. $\nabla \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ xy & 2yz & 3zx \end{vmatrix} = -(2y, 3z, x)$

Stokes: $\oint \vec{v} \cdot d\vec{l} = \int \nabla \times \vec{v} \cdot d\vec{a}$

LHS :



$$\oint = \int_{A\to B} + \int_{B\to C} + \int_{C\to A}$$

$$\int_{A\to B} \vec{v} \cdot d\vec{l} = \int_0^2 \vec{v} \cdot \hat{y} \, dy \Big|_{x=z=0}$$

$$= 0 \quad \left(\because \vec{v} \cdot \hat{y} \Big|_{z=0} = 0 \right)$$

$$\int_{C\to A} \vec{v} \cdot d\vec{l} = \int_2^0 \vec{v} \cdot \hat{z} \, dz \Big|_{x=y=0} = 0. \quad (\because v_z = 0)$$

$$\int_{B\to C} \vec{v} \cdot d\vec{l} = \int \left(\underbrace{v_z}_{=0 \because x=0} \hat{z} + \underbrace{v_y}_{=2yz = 2y(2-y)} \hat{y} \right) \Big|_{\substack{x=0, \\ y+z=2}} \cdot dl$$

$$= \int_2^0 \underbrace{(4y - 2y^2)}_{v_y} \, dy$$

$$= \frac{2}{3} y^3 - 2y^2 \Big|_0^2 = \frac{16}{3} - 8 = -\frac{8}{3}$$

direction of $d\vec{a}$ by right hand rule.

RHS: $(\nabla\times\vec{v})\cdot\hat{x} = -2y$.

$$\Rightarrow \int(\nabla\times\vec{v})\cdot d\vec{a} = \int_0^2 dy\int_0^{2-y} dz\,(-2y)$$

$$= \int_0^2 dy\cdot(2y^2-4y) = -\frac{8}{3}$$

LHS = RHS. Verified.

1.42 $\vec{v} = v_s \hat{s} + v_\varphi \hat{\varphi} + v_z \hat{z}$,

$v_s = s(2 + \sin^2\varphi)$,

$v_\varphi = s\sin\varphi\cos\varphi = \frac{s}{2}\sin 2\varphi$

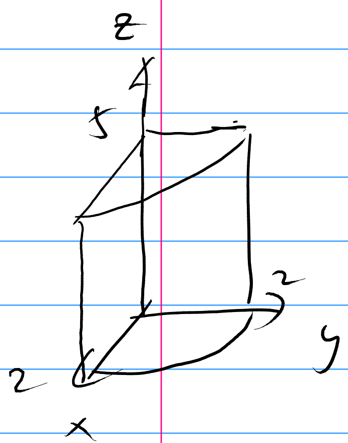
$v_z = 3z$.

(a) $\nabla\cdot\vec{v} = \frac{1}{s}\partial_s(s v_s) + \frac{1}{s}\partial_\varphi v_\varphi + \partial_z v_z$

$= 2(\sin^2\varphi + 2) + \cos 2\varphi + 3$

$= 8$

(b) $\int \nabla\cdot\vec{v}\, dV = 8 \times \frac{1}{4} \times (\pi\cdot 2^2) \times 5 = 40\pi$



$$\oint \vec{v}\cdot d\vec{A} = \underbrace{\int_0^5 dz \int_0^{\frac{\pi}{2}} d\varphi\, (\vec{v}\cdot\hat{s})\, s \Big|_{s=2}}_{\text{cylindrical wall}}$$

$$+ \int_0^{\frac{\pi}{2}} d\varphi \int_0^2 s\,ds \left(\underbrace{v_z\Big|_{z=5}}_{\text{upper}} - \underbrace{v_z\Big|_{z=0}}_{\text{lower}} \right)$$

$$+ \int_0^5 dz \int_0^2 ds \left[\underbrace{\vec{v}\cdot(-\hat{\varphi})}_{\text{left}} \right] \Big|_{\varphi=0}$$

$$+ \int_0^5 dz \int_0^2 ds \left[\vec{v}\cdot\hat{\varphi} \right] \Big|_{\varphi=\frac{\pi}{2}}$$

$$= 25z + 15z + 0 + 0 = 40z.$$

$$\Rightarrow \int \nabla \cdot \vec{v}\, dV = \oint_{\partial V} \vec{v} \cdot d\vec{A}.$$

(c). $\frac{1}{s}\partial_\varphi v_z - \partial_z v_\varphi = 0 - 0 = 0$

$$\partial_z v_s - \partial_s v_z = 0 - 0 = 0$$

$$\frac{1}{s}\left[\partial_s(s v_\varphi) - \partial_\varphi v_s\right] = \frac{1}{s}\left[2v_\varphi - 2s\sin 2\varphi\right] = 0$$

$$\Rightarrow \nabla \times \vec{v} = 0$$

1.44 (a) $\int_{-2}^{2} (2x+3)\, \delta(3x)\, dx$

$= \frac{1}{3} \int_{-2}^{2} (2x+3)\, \delta(x)\, dx$

$= 1$

(b) $\int_{0}^{2} (x^3 + 3x + 2)\, \delta(1-x)\, dx$

$= (x^3 + 3x + 2)\Big|_{x=1}$

$= 6$

(c) $\int_{-1}^{1} 9x^2\, \delta(3x+1)\, dx$

$= \frac{1}{3} \int 9x^2\, \delta(x+\frac{1}{3}) = \frac{1}{3}\, 9x^2\Big|_{x=-\frac{1}{3}}$

$= \frac{1}{3}$

(d) $\int_{-\infty}^{a} \delta(x-b)\, dx = \begin{cases} 1 & a>b \\ 0 & a<b \end{cases}$

$= \theta(a-b)$

1.47 (a) $\int (r^2 + \vec{r}\cdot\vec{a} + a^2)\, \delta^3(\vec{r}-\vec{a})\, d\tau$

$= (r^2 + \vec{r}\cdot\vec{a} + a^2)\Big|_{\vec{r}=\vec{a}}$

$= 3a^2$

(b) $\int_V |\vec{r}-\vec{b}|^2 \delta^3(5\vec{r})\, d\tau$

$\delta^3(5r) = \delta(5x)\,\delta(5y)\,\delta(5z)$

$= \frac{1}{5}\delta(x)\,\frac{1}{5}\delta(y)\,\frac{1}{5}\delta(z)$

$= \frac{1}{125}\delta(\vec{r})$

$= \frac{1}{125}|\vec{r}-\vec{b}|^2\Big|_{\vec{r}=0}$

$= \frac{b^2}{125}$

(c) $\int_V (r^4 + r^2(\vec{r}\cdot\vec{c}) + c^4)\, \delta^3(\vec{r}-\vec{c})\, d\tau$.

$\vec{c} = (5,3,2) \Rightarrow c = \sqrt{25+9+4} = \sqrt{38} > 6$,

i.e., $\vec{c}$ lies outside V.

$\Rightarrow \int \cdots = 0$.

(d) $\int_V \vec{r}\cdot(\vec{d}-\vec{r})\, \delta^3(\vec{e}-\vec{r})\, d\tau$

$\vec{e} = (3,2,1)$. Let $\vec{c} = (2,2,2)$ be the center of sphere

$\Rightarrow |\vec{e}-\vec{c}| = |(1,0,-1)| = \sqrt{2} < 1.5$,

$\Rightarrow \vec{e}$ inside the sphere

$\Rightarrow \int \cdots = \vec{r}\cdot(\vec{d}-\vec{r})\Big|_{\vec{r}=\vec{e}} = (3,2,1)\cdot(-2,0,2)$

$= -4$.

1.49 (a) • $\vec{F}_1 = x^2 \hat{z}$

$$\nabla\cdot\vec{F}_1 = \nabla(x^2)\cdot\hat{z} + x^2\,\nabla\cdot\hat{z}$$

$$= 2\vec{x}\cdot\hat{z} + 0$$

$$= 0.$$

$$\nabla\times\vec{F}_1 = \nabla(x^2)\times\hat{z} + x^2\,\nabla\times\hat{z}$$

$$= 2\vec{x}\times\hat{z} + 0$$

$$= -2x\,\hat{y}$$

$\Rightarrow \vec{F}_1 = \nabla\times\vec{A}$, where e.g., $\vec{A} = \frac{x^3}{3}\hat{y}$

• $\vec{F}_2 = x\hat{x} + y\hat{y} + z\hat{z} = \vec{r}$

$$\nabla\cdot\vec{F}_2 = \partial_x x + \partial_y y + \partial_z z = 3$$

$$\nabla\times\vec{F}_2 = \nabla\times\vec{r}$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix} = 0.$$

$\Rightarrow \vec{F}_2 = \nabla\varphi$, where e.g.,

$$\varphi = \frac{1}{2}\vec{r}\cdot\vec{r} = \frac{1}{2}(x^2+y^2+z^2)$$

(b) $\vec{F}_3 = yz\,\hat{x} + zx\,\hat{y} + xy\,\hat{z}$

$$\Rightarrow \quad \nabla\cdot\vec{F}_3 = \nabla\times\vec{F}_3 = 0.$$

- Scalar potential:

$$\varphi = xyz \Rightarrow \nabla\varphi = \vec{F}_3$$

- Vector potential.

Let's assume $\vec{A}_1 = f_1(xy)\,\hat{y}$
such that $\nabla\times\vec{A}_1 = xy\,\hat{z}$

$$\nabla\times\vec{A}_1 = \nabla f_1 \times\hat{y} + f_1 \underbrace{\nabla\times\hat{y}}_{=0}$$

$$= \partial_x f_1 \underbrace{\nabla x \times \hat{y}}_{=\hat{x}\times\hat{y} = \hat{z}}$$

$$\Rightarrow \partial_x f_1 = xy$$

$$\Rightarrow f_1 = \tfrac{1}{2}x^2y$$ to within an arbitrary function of y only.

$$\Rightarrow \vec{A}_1 = \tfrac{1}{2}x^2y\,\hat{y} = \tfrac{1}{2}x^2\vec{y}$$

Then, by cyclic permutation of xyz,
can find $\vec{A}_2 = \frac{1}{2}y^2\vec{z}$ such that $\nabla\times A_2 = yz\,\hat{x}$
and $\vec{A}_3 = \frac{1}{2}z^2\vec{x}$, $\nabla\times A_3 = zx\,\hat{y}$

So $\vec{A} = \vec{A}_1 + \vec{A}_2 + \vec{A}_3 = \frac{1}{2}(x^2\vec{y} + y^2\vec{z} + z^2\vec{x})$

1.62 (a) • Direct calculation:

$$\nabla\cdot\left(\frac{\hat{r}}{r}\right) = \nabla\left(\frac{1}{r}\right)\cdot\hat{r} + \frac{1}{r}\nabla\cdot\hat{r}$$

$$= -\frac{1}{r^2} + \frac{1}{r}\cdot\frac{2}{r}$$

$$= \frac{1}{r^2}$$

• Using Div. Thm.

$$\int^R \left(\nabla\cdot\frac{\hat{r}}{r}\right) dV = 4\pi\int^R r^2\left(\nabla\cdot\frac{\hat{r}}{r}\right) dr$$

$$= \oint \frac{\hat{r}}{r}\cdot\hat{r}\, r^2 d\Omega\Big|_{r=R} = 4\pi R$$

$$\Rightarrow \int^R r^2\left(\nabla\cdot\frac{\hat{r}}{r}\right) dr = R$$

$$\Rightarrow \nabla\cdot\frac{\hat{r}}{r} = \frac{1}{r^2}$$

• In general,

$$\nabla\cdot(r^n\hat{r}) = \nabla(r^n)\cdot\hat{r} + r^n\nabla\cdot\hat{r}$$

$$= n r^{n-1} + r^n\cdot\frac{2}{r}$$

$$= (n+2)r^{n-1}$$

For $n+2\neq 0$, $\nabla\cdot(r^n\hat{r})$ is regular, so there's no δ-f^{n} at $r=0$.

For $n+2=0$, i.e., $n=-2$, it becomes singular, and $\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta(\vec{r})$

(b) $\nabla \times (r^n \hat{r}) = \nabla(r^n) \times \hat{r} + r^n \underbrace{\nabla \times \hat{r}}_{=0}$

$= n r^{n-1} \underbrace{\hat{r} \times \hat{r}}_{=0} + 0$

$= 0.$