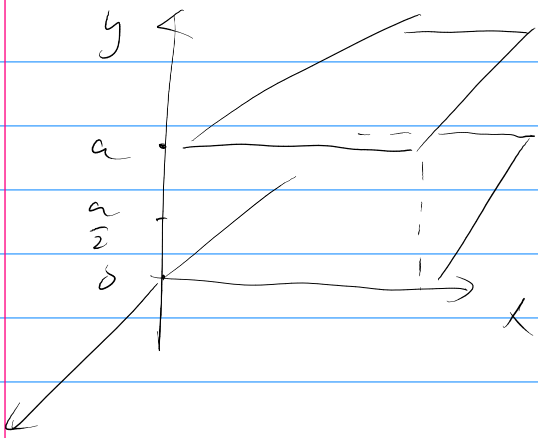
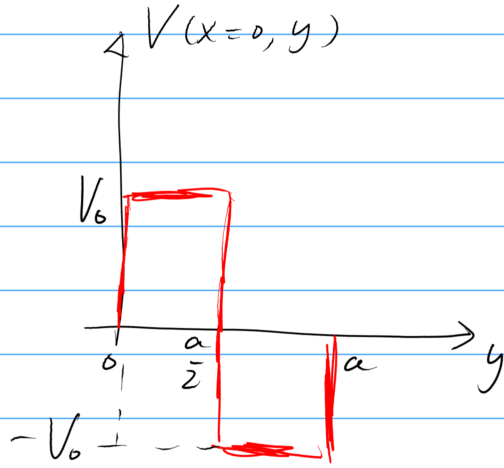
3.12





$$V = \sum_n V_n \, e^{-nkx} \sin(nky), \quad k = \frac{2\pi}{a}, \; n = 1, 2, \dots$$

note that ① for any n, $\partial_x^2 V = (nk)^2 V$, $\partial_y^2 V = -(nk)^2 V$, thus $\nabla^2 V = 0$ is satisfied for all n.

②. the boundary condition at $x=0$ demands that the y function is "a"-periodic and $=0$ @ $0, \frac{a}{2}$ & a, hence the form $\sin(nky)$.

$$V(x=0, y) = \sum_n V_n \sin(nky) = \begin{cases} V_0, & y \in [0, \frac{a}{2}] \\ -V_0, & y \in [\frac{a}{2}, a] \end{cases}$$

$$\Rightarrow V_n = \frac{\int_0^a dy \, V(x=0, y) \sin(nky)}{\int_0^a dy \, \sin^2(nky)}$$

$$= \frac{2V_0 \int_0^{\frac{a}{2}} dy \, \sin(nky)}{\int_0^a dy \, \sin^2(nky)} \underline{\underline{t = nky}} \frac{2V_0 \int_0^{\frac{nka}{2}} dt \, \sin t}{\int_0^{nka} dt \, \sin^2 t}$$

$$= \int_0^{2\pi n} \sin^2 t \, dt = \frac{1}{2} \int_0^{2\pi n} (\underbrace{\sin^2 t + \cos^2 t}_{=1}) \, dt$$

(recall $nka = n \cdot 2\pi$)

$$= \frac{2V_0 (1 - \cos(n\pi))}{n\pi} = \begin{cases} \frac{4V_0}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

2. $$V = \sum_{n_x, n_y} V_{n_x n_y} \sin(n_x k x) \sin(n_y k y) \sinh(n_z k z)$$

where $n_z \equiv \sqrt{n_x^2 + n_y^2}$, $k = \frac{\pi}{a}$

note: ① easy to verify $\nabla^2 V = 0$ for all (n_x, n_y) pair

② sinh is b/c need ~~both~~ $e^{()}$ & $e^{-()}$ as there is no $V(z \to \infty) = 0$ B.C., and for $V(x=0, y=0, z=0) = 0$, need $e^{()}$ & $e^{-()}$ have opposite coeff.

Now, at $z = a$, denote $A_{n_x n_y} \equiv V_{n_x n_y} \sinh(n_z \overbrace{ka}^{=\pi})$,

$$\Rightarrow V(z=a) = \sum_{n_x n_y} A_{n_x n_y} \sin(n_x k x) \sin(n_y k y) = V_0$$

$$\Rightarrow A_{n_x n_y} = \frac{\iint_0^a V_0 \sin(n_x k x) \sin(n_y k y)\, dx\, dy}{\int_0^a \sin^2(n_x k x)\, dx \int_0^a \sin^2(n_y k y)\, dy}$$

$$\left(t \equiv n_x k x \Rightarrow \frac{\int_0^{n_x \pi} \sin t\, dt}{\int_0^{n_x \pi} \sin^2 t\, dt} = \frac{1 - \cos n_x \pi}{\frac{1}{2} n_x \pi} = \begin{cases} \frac{4}{n_x \pi}, & n_x \text{ odd} \\ 0 & \text{even} \end{cases} \right)$$

$$\Rightarrow A_{n_x n_y} = \begin{cases} \frac{16 V_0}{n_x n_y \pi^2} & n_x \text{ \& } n_y \text{ both odd} \\ 0 & \text{otherwise} \end{cases}$$

$$(a+b)^3 = a^3 + b^3 + 3ab(a+b)$$

3.18. $$\cos 3\theta = \frac{e^{i3\theta} + e^{-i3\theta}}{2} = \frac{(e^{i\theta} + e^{-i\theta})^3 - 3(e^{i\theta} + e^{-i\theta})}{2}$$

$$= 4(\cos\theta)^3 - 3\cos\theta$$

$$= 4x^3 - 3x, \qquad x \equiv \cos\theta$$

recall: $P_0 = 1$, $P_1 = x$, $P_2 = \frac{3x^2-1}{2}$, $P_3 = \frac{5x^3-3x}{2}$

$$\Rightarrow \cos 3\theta = 4 \cdot \left(\frac{2P_3 + 3P_1}{5}\right) - 3P_1$$

$$= \frac{8}{5} P_3 - \frac{3}{5} P_1$$

① Inside: $$V(r,\theta) = \sum_l A_l r^l P_l(\cos\theta)$$

$$= A_3 r^3 P_3 + A_1 r P_1$$

$$\underline{\underline{r=R}} \quad A_3 R^3 P_3 + A_1 R P_1$$

$$\Rightarrow A_3 = \frac{8k}{5R^3}, \quad A_1 = -\frac{3k}{5R}$$

② Outside: $$V(r,\theta) = \sum_l \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$= \frac{B_3}{r^4} P_3 + \frac{B_1}{r^2} P_1$$

$$\underline{\underline{r=R}} \quad \frac{B_3}{R^4} P_3 + \frac{B_1}{R^2} P_1$$

$$\Rightarrow B_3 = \frac{8}{5} kR^4, \quad B_1 = -\frac{3}{5} kR^2$$

③ Gauss: $$E_\perp(R^+) - E_\perp(R^-) = \frac{\sigma}{\epsilon_0}$$

$$E_\perp(R^+) = -\frac{\partial V}{\partial R^+} = -\frac{\partial V_{out}}{\partial R} = \frac{4B_3}{R^5} P_3 + \frac{2B_1}{R^3} P_1$$

$$= \frac{k}{R}\left(\frac{32}{5} P_3 - \frac{6}{5} P_1\right)$$

$$E_\perp(R^-) = -\frac{\partial V_{in}}{\partial R} = -3A_3 R^2 P_3 - A_1 P_1$$

$$= \frac{k}{R}\left(-\frac{24}{5} P_3 + \frac{3}{5} P_1\right)$$

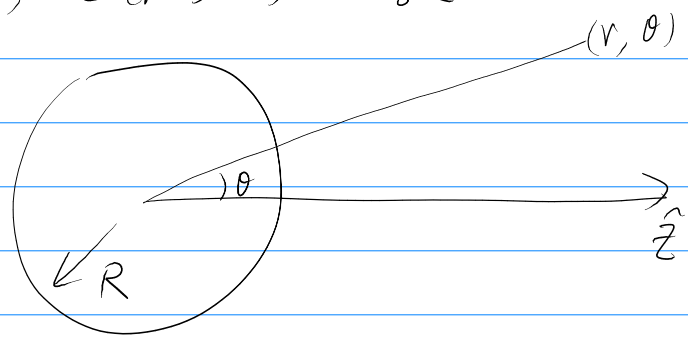
$$\Rightarrow \sigma(\theta) = \epsilon_0\left(E_\perp(R^+) - E_\perp(R^-)\right)$$

$$= \frac{\epsilon_0 k}{R}\left(\frac{56}{5} P_3 - \frac{9}{5} P_1\right)$$

3.20. Consider an uncharged metal sphere first, then one can claim that

$$V(r,\theta) = \left(\frac{A}{r^2} - E_0 r\right)\cos\theta$$

b/c this gives the right behavior at $r \to \infty$, i.e., $E(r\to\infty) = E_0 \hat{z}$



Now, $V(R,\theta) = \text{const} = \left(\frac{A}{R^2} - E_0 R\right)\cos\theta$

$\Rightarrow A = E_0 R^3$ (otherwise $V(r=R)$ has θ dependence)

A charged sphere will generate an extra $V_Q = \frac{kQ}{r}$ on top of the above potential,

Thus

$$V = \underbrace{\frac{kQ}{r}}_{①} + \underbrace{E_0\left(\frac{R^3}{r^2} - r\right)\cos\theta}_{②}$$

① gives $\int \sigma \, dA = Q$

② sol^n of $\nabla^2 V = 0$

and $V(r \to \infty, \theta = \frac{\pi}{2}) = 0$.

Clearly, $V(R) = \frac{kQ}{R}$. One may as well choose $r = R$ to be the zero level of potential, in which case

$$V = -\frac{kQ}{R} + \frac{kQ}{r} + E_0\left(\frac{R^3}{r^2} - r\right)\cos\theta$$

(i.e., $V(r,\theta) = \text{Const} + \frac{A_0}{r^1} P_0 + \left(\frac{A_1}{r^2} + B_1\right) P_1$)

3.22 $V_{out} = \sum_l \frac{A_l}{r^{l+1}} P_l \Rightarrow E(R^+) = \sum_l \frac{(l+1) A_l}{R^{l+2}} P_l$

$V_{in} = \sum_l B_l r^l P_l \Rightarrow E(R^-) = -\sum_l l B R^{l-1} P_l$.

$V_{out}(R) = V_{in}(R) \Rightarrow A_l = B_l R^{2l+1}$

$$\sigma = \epsilon_0 \left(E(R^+) - E(R^-) \right)$$

$$= \epsilon_0 \sum_l \left[\frac{(l+1) A_l}{R^{l+2}} + l B R^{l-1} \right] P_l$$

$$= \epsilon_0 \sum_l \underbrace{(2l+1) B_l \cdot R^{l-1}}_{\equiv C_l} P_l = \begin{cases} \sigma_0, & x = \cos\theta \in [0,1] \\ -\sigma_0, & x \in [-1, 0]. \end{cases}$$

Clearly, $\sigma(x)$ is odd in x, thus only $l =$ odd survives,

$$C_l = \begin{cases} \dfrac{\sigma_0}{\epsilon_0} \cdot \dfrac{\int_0^1 P_l(x)\, dx}{\int_0^1 P_l^2(x)\, dx} = \dfrac{(2l+1)\sigma_0}{\epsilon_0} \int_0^1 P_l(x)\, dx, & l \text{ odd} \\ 0 & l \text{ even} \end{cases}$$

$$\Rightarrow B_l = \frac{C_l}{(2l+1) R^{l-1}} = \frac{\sigma_0}{\epsilon_0 \cdot R^{l-1}} \int_0^1 P_l(x)\, dx, \quad l \text{ odd}$$

or 0, l even

$$\Rightarrow B_1 = \frac{\sigma_0}{\epsilon_0} \cdot \frac{1}{2}; \quad B_3 = -\frac{\sigma_0}{\epsilon_0 R^2} \frac{1}{8}; \quad B_5 = \frac{\sigma_0}{\epsilon_0 R^4} \cdot \frac{1}{16}$$

$(B_2 = B_4 = B_6 = 0, \quad A_l = B_l R^{2l+1})$