4.10 (a) $\rho_b = -\nabla\cdot P = -k\,\nabla\cdot\vec{r} = -3k$ for $r<R$.

$\sigma_b = \vec{P}\cdot\hat{n} = \vec{P}\cdot\hat{r} = kr = kR$

(b). No charge @ $r\to\infty$, $\Rightarrow D(r\to\infty) = 0$.

Propose $D=0$ everywhere. Check: $\nabla\cdot D = 0$ ✓, $D(r\to\infty)=0$ ✓

$\Rightarrow$ By uniqueness Thm, $D=0$ IS THE soln

$D = \varepsilon_0 E + P$

$$\Rightarrow E = -\frac{P}{\varepsilon_0} = \begin{cases} -\dfrac{k\vec{r}}{\varepsilon_0} & r<R \\ 0 & r>R \end{cases}$$

4.15 Same reasoning as 4.10 $\Rightarrow$ $D=0$ everywhere

(a). As in 4.10,

$$\rho_b = -\nabla\cdot P = -k\,\nabla\cdot\left(\frac{\hat{r}}{r}\right) = -k\left[\nabla\left(\tfrac{1}{r}\right)\cdot\hat{r} + \tfrac{1}{r}\nabla\cdot\hat{r}\right]$$

Now, $\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^2}\underbrace{\nabla(r)}_{\equiv\hat{r}} = -\frac{\hat{r}}{r^2}$,

Also, by $\nabla\cdot\vec{r} = 3 = \nabla\cdot(r\hat{r}) = \overbrace{\underbrace{(\nabla r)}_{\hat{r}}\cdot\hat{r}}^{1} + r\nabla\cdot\hat{r}$

$$\Rightarrow \nabla\cdot\hat{r} = \frac{2}{r}$$

$$\Rightarrow \rho_b = -k\left(-\frac{1}{r^2} + \frac{2}{r^2}\right) = -\frac{k}{r^2}$$

$$\sigma_b = \vec{P}\cdot\hat{n} = \frac{k}{r}\hat{r}\cdot\hat{r}\Big|_R = \frac{k}{R}$$

} all bound charge

$$E(r) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0 r^2}\displaystyle\int_0^r \rho_b\cdot 4\pi r'^2\,dr' = \dfrac{-k}{4\pi\varepsilon_0 r^2}\displaystyle\int_0^r 4\pi\,dr' \\ \qquad = \dfrac{-k}{\varepsilon_0 r}, \quad r<R \\ \dfrac{1}{4\pi\varepsilon_0} Q_{b,tot}/r^2 = 0 \quad r>R. \end{cases}$$

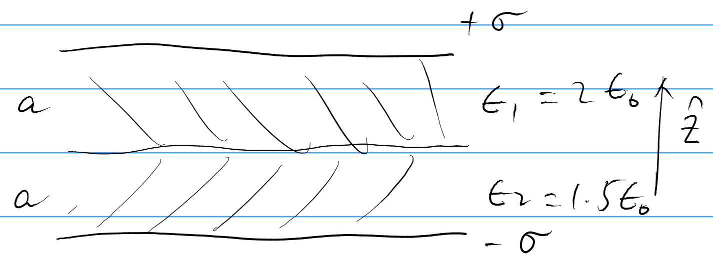
Note that by def of "bound charge",

$Q_{b,\,total} = 0$ (Any residue charge will be "free" charge)

(b). Now, $D = 0 \Rightarrow E = -\dfrac{P}{\varepsilon_0} = \begin{cases} \dfrac{k}{\varepsilon_0 r} & r<R \\ 0 & r>R \end{cases}$

same as in (a)

4.18.



(a) $D = -\sigma \hat{z}$

(b) $E = \dfrac{D}{\epsilon}$

$\Rightarrow E_1 = \dfrac{-\sigma \hat{z}}{2\epsilon_0}$

$E_2 = \dfrac{-\sigma \hat{z}}{1.5\epsilon_0}$

(c). $D = \epsilon_0 E + P \Rightarrow P = D - \epsilon_0 E = D\left(1 - \dfrac{\epsilon_0}{\epsilon}\right)$

$\Rightarrow P_1 = \frac{1}{2} D = -\frac{1}{2}\sigma \hat{z}$

$P_2 = D\left(1 - \frac{1}{1.5}\right) = \frac{1}{3} D = -\frac{1}{3}\sigma \hat{z}$

(d) $V = \int E\, dl = (E_1 + E_2)a = \dfrac{7}{6}\dfrac{\sigma a}{\epsilon_0}$

(e). Inside slabs 1 & 2: $E_1 = \text{const}$, $E_2 = \text{const}$

$\Rightarrow \rho_b^{(1)} = \rho_b^{(2)} = 0$

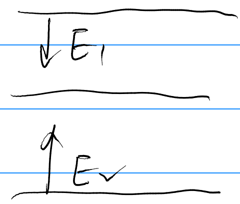
upper surface: $\sigma^{(up)} = P_1 \cdot \hat{z} = -\frac{1}{2}\sigma$.

lower $\quad = \sigma^{(low)} = P_2 \cdot (\underbrace{-\hat{z}}) = \frac{1}{3}\sigma$

Notice surface's orientation

Middle: $P_1 \cdot (-\hat{z}) + P_2 \cdot (\hat{z}) = \frac{1}{6}\sigma$

(f):



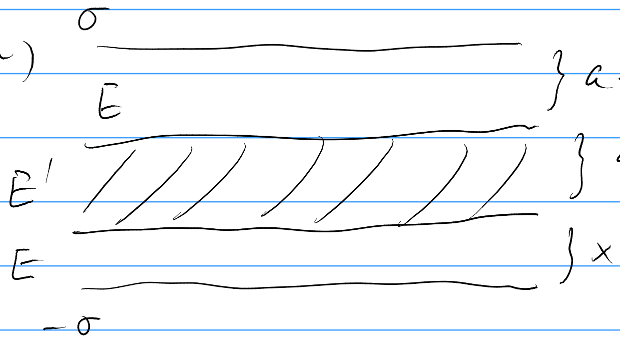
$\sigma - \frac{1}{2}\sigma = \frac{1}{2}\sigma$ $\Rightarrow E_1 = \frac{1}{2}\frac{\sigma}{\varepsilon_0}(-\hat{z})$

$\frac{1}{6}\sigma$

$-\sigma + \frac{1}{3}\sigma = -\frac{2}{3}\sigma$ $\Rightarrow E_2 = -\frac{2}{3}\frac{\sigma}{\varepsilon_0}(\hat{z})$

same as 2(b)

4.19* (a)



$D = \sigma$

$E = \frac{D}{\epsilon_0}$, $E' = \frac{D}{\epsilon_0 \epsilon_r}$

$$\Rightarrow V = \int E\, dl = E \cdot a + E' \cdot a = \frac{D}{\varepsilon_0}\left(1 + \frac{1}{\epsilon_r}\right) \cdot a$$

$$\Rightarrow C = \frac{\sigma}{V} = \frac{D}{V} = \frac{\epsilon_0 \epsilon_r}{(1+\epsilon_r)a}$$

If no dielectrics, $\Rightarrow \epsilon_r = 1$

$$\Rightarrow C_0 = \frac{\epsilon_0}{2a}$$

$$\Rightarrow \frac{C}{C_0} = \frac{2\epsilon_r}{1+\epsilon_r}$$ (check: $\epsilon_r = 1 \Rightarrow \frac{C}{C_0} = 1$)

Now, Given V, $\Rightarrow D = \frac{\epsilon_0 \epsilon_r V}{(1+\epsilon_r)a}$

$$D = \epsilon_0 \epsilon_r E$$

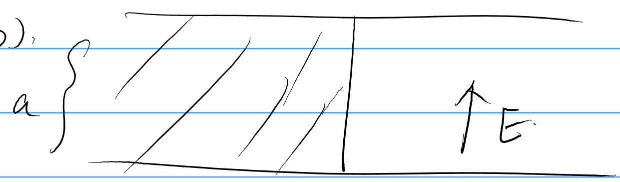
$$D = \epsilon_0 E + P = \frac{D}{\epsilon_r} + P \Rightarrow P = D\left(1 - \frac{1}{\epsilon_r}\right) = \frac{\epsilon_r - 1}{\epsilon_r} D$$

$\Rightarrow$ bound charge at surfaces of dielectrics:

up: $\sigma_{up} = \vec{P} \cdot \hat{z} = \epsilon_0 \cdot \frac{\epsilon_r - 1}{\epsilon_r + 1} \cdot \frac{V}{a}$ (assuming $\vec{D} \parallel \hat{z}$)

$\sigma_{dn} = \vec{P} \cdot (-\hat{z}) = -\sigma_{up}$

(b).



Now, if we impose const V, then $\vec{E} = $ const by def.,

$E = \frac{V}{a}$ in both region

$D_{Left} = \epsilon_0 \epsilon_r E$, $\sigma_{left} = D_{Left}$

$D_{Right} = \epsilon_0 E$, $\sigma_{Right} = D_{Right}$

$\Rightarrow C = C_{Left} + C_{Right}$ (parallel)

$$= \frac{\sigma_L}{V} + \frac{\sigma_R}{V} = \frac{\epsilon_0 E}{V}(\epsilon_r + 1)$$

$$= \frac{\epsilon_0(\epsilon_r + 1)}{a}$$

$C_0 = C(\epsilon_r \to 1) \Rightarrow \frac{C}{C_0} = \frac{\epsilon_r + 1}{2}$

Left Region: $P = D - \epsilon_0 E = \epsilon_0 E(\epsilon_r - 1)$

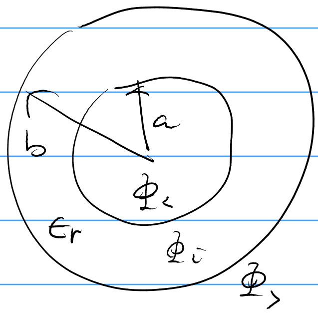
$= \epsilon_0(\epsilon_r - 1) V/a$

$\Rightarrow$ bound charge at left region surfaces.

$\sigma_{up} = \vec{P}\cdot\hat{z} = \epsilon_0(\epsilon_r - 1)\, V/a$ (assuming $\vec{E} \parallel \hat{z}$)

$\sigma_{dn} = \vec{P}\cdot(-\hat{z}) = -\sigma_{dn}$.

4.24.



$\nabla\cdot\vec{D} = 0 \Rightarrow \vec{D} = \nabla\Phi$; $D = \epsilon E \Rightarrow \Phi = -\epsilon V$

$\Rightarrow \dfrac{\Phi}{\epsilon}$ continuous.

Propose: $\Phi_> = \left(Er + \dfrac{A}{r^2}\right)\cos\theta$

$\Phi_i = \left(Br + \dfrac{C}{r^2}\right)\cos\theta,$

$\Phi_< = Fr\cos\theta$

- B.C. @ $r=a$:

$\epsilon_r \Phi_<\big|_a = \Phi_i\big|_a \Rightarrow \epsilon_r F = B + \dfrac{C}{a^3}$

$D_{<,\perp}\big|_a = D_{i,\perp}\big|_a \Rightarrow F = B - \dfrac{2C}{a^3}$

$\Rightarrow (2\epsilon_r + 1)C/a^3 = (\epsilon_r - 1)B \Rightarrow C = \dfrac{\epsilon_r - 1}{2\epsilon_r + 1} a^3 B$ ①

- B.C. @ $r=b$:

$\Phi_i\big|_b = \epsilon_r \Phi_>\big|_b \Rightarrow \underbrace{B + \dfrac{C}{b^3}}_{= B\left[1 + \frac{a^3}{b^3}\frac{\epsilon_r - 1}{2\epsilon_r + 1}\right] \equiv B(1+n)} = \epsilon_r\left(E + \dfrac{A}{b^3}\right)$

$\Rightarrow E + \dfrac{A}{b^3} = B(1+n)/\epsilon_r$ ②

$D_{i,\perp}\big|_b = D_{>,\perp}\big|_b \Rightarrow E - \dfrac{2A}{b^3} = B - \dfrac{2C}{b^3} = B(1-2n)$ ③

$$2\times ② + ③ \Rightarrow 3E = B\left[\frac{2(1+n)}{\epsilon_r} + (1-2n)\right]$$

$$\Rightarrow B = \frac{3E\cdot\epsilon_r}{2(1+n)+\epsilon_r(1-2n)}$$

$$② - ③ \Rightarrow \frac{3A}{b^3} = B\left[\frac{1+n}{\epsilon_r} - (1-2n)\right]$$

$$\Rightarrow A = \frac{b^3 B}{3\epsilon_r}\left[(1+n) - \epsilon_r(1-2n)\right]$$

$$\left(\text{recall}: n \equiv \frac{a^3}{b^3}\frac{\epsilon_r - 1}{2\epsilon_r + 1}\right)$$

4.26.

$$D(r) = \begin{cases} 0, & r<a \\ \frac{1}{4\pi}\frac{Q}{r^2} & r>a. \end{cases}, \quad E = \begin{cases} 0, & r<a \\ \frac{D}{\epsilon} & r\in(a,b) \\ \frac{D}{\epsilon_0} & r>b \end{cases}$$

$$\Rightarrow E = \int dV \cdot \frac{1}{2} D\cdot E$$

$$= \int_a^b 4\pi r^2 dr \cdot \frac{1}{2}\cdot\frac{D^2}{\epsilon} + \int_b^\infty 4\pi r^2 \cdot \frac{1}{2}\frac{D^2}{\epsilon_0}$$

$$= \frac{Q^2}{8\pi}\left[\frac{1}{\epsilon_0}\frac{1}{r}\Big|_b^a + \frac{1}{\epsilon}\frac{1}{r}\Big|_\infty^b\right], \quad \epsilon = \epsilon_0(1+\chi_e)$$

$$= \frac{1}{8\pi\epsilon}\left(\frac{1}{a} + \frac{\chi_e}{b}\right)$$