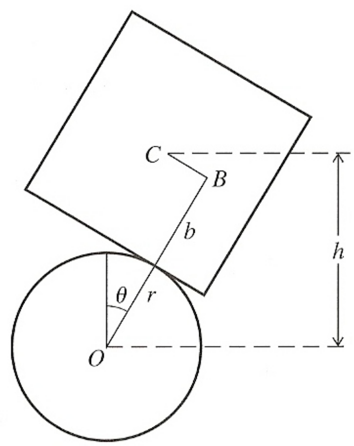
Lagrangian Problems

1. Cube on Top of a Cylinder Consider the figure below which shows a cube of mass m with a side length of $2b$ sitting on top of a fixed rubber horizontal cylinder of radius r. The cube cannot slip on the cylinder, but it can rock from side to side. Assume that the cube was initially balanced on the cylinder with its center of mass, C, directly above the center of the cylinder, O. (a) Find the Lagrangian for this system. (b) Find the Lagrange equation of motion. (c) Find any possible positions of equilibrium. If any of these positions of equilibrium are stable find the frequency of small oscillations about equilibrium.



A cube, of side $2b$ and center C, is placed on a fixed horizontal cylinder of radius r and center O.

(a) The Cartesian coordinates for the location of the center of mass relative to the center of the cylinder are

$$\begin{aligned} y &: \quad (r+b)\cos\theta + r\theta\sin\theta \\ x &: \quad (r+b)\sin\theta - r\theta\cos\theta. \end{aligned}$$

To find the translational kinetic energy of the cube we first find the sum of $\dot{x}^2 + \dot{y}^2$:

$$\begin{aligned} \dot{x} &= (r+b)\cos\theta\dot{\theta} - r\dot{\theta}\cos\theta + r\theta\sin\theta\dot{\theta} = (b\cos\theta + r\theta\sin\theta)\,\dot{\theta} \\ \dot{x} &= -(r+b)\sin\theta\dot{\theta} + r\dot{\theta}\sin\theta + r\theta\cos\theta\dot{\theta} = (-b\sin\theta + r\theta\cos\theta)\,\dot{\theta} \\ \dot{x}^2 + \dot{y}^2 &= \left(b^2 + r^2\theta^2\right)\dot{\theta}^2 . \end{aligned}$$

The moment of inertia for a cube about its center of mass is $I = 2mb^2/3$. Hence the Lagrangian for the cube is

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}m\left(b^2 + r^2\theta^2\right)\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2 - mgy, \\
\mathcal{L} &= \frac{1}{2}m\left(5b^2/3 + r^2\theta^2\right)\dot{\theta}^2 - mg\left((r+b)\cos\theta + r\theta\sin\theta\right).
\end{aligned}$$

(b) To determine the Lagrange equation of motion we find the derivatives:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \theta} &= mr^2\theta\dot{\theta}^2 - mg\left(-(r+b)\sin\theta + r\sin\theta + r\theta\cos\theta\right) = mr^2\theta\dot{\theta}^2 - mg\left(-b\sin\theta + r\theta\cos\theta\right) \\
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= \frac{d}{dt}\left(m\left(5b^2/3 + r^2\theta^2\right)\dot{\theta}\right) = m\left(5b^2/3 + r^2\theta^2\right)\ddot{\theta} + 2mr^2\theta\dot{\theta}^2
\end{aligned}$$

Equating these two derivatives yields the equation of motion,

$$-mg\left(-b\sin\theta + r\theta\cos\theta\right) = m\left(5b^2/3 + r^2\theta^2\right)\ddot{\theta} + mr^2\theta\dot{\theta}^2.$$

(c) The positions of equilibrium occur when $\dot{\theta} = \ddot{\theta} = 0$. Hence

$$-b\sin\theta_{eq} + r\theta_{eq}\cos\theta_{eq} = 0 \rightarrow \theta_{eq} = 0.$$

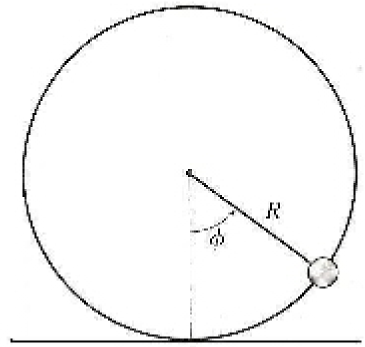
For small fluctuations about equilibrium $\theta = \theta_{eq} + \delta\theta = \delta\theta$. Keeping only first order terms in $\delta\theta$ the equation of motion becomes

$$-mg\left(r-b\right)\delta\theta = m\left(5b^2/3\right)\ddot{\theta}.$$

For this position of equilibrium to be stable we must have $r - b > 0$ or $r > b$. The frequency (squared) of oscillations about $\theta = 0$ is

$$\omega^2 = g\frac{3\left(r-b\right)}{5b^2}.$$

2. Hoop Containing a Bead Consider the figure below which shows a uniform hoop of radius R and mass M which is free to roll along a horizontal track without slipping. Attached to the hoop is a bead of mass m which is free to slide without friction around the hoop in a uniform gravitational field g. (a) Find the Lagrangian for this system. (b) Find the Lagrange equations of motion. (c) Find any possible positions of equilibrium. If any of these positions of equilibrium are stable find the frequency of small oscillations about equilibrium. Exam in the limit of $M >> m$ and comment.



Uniform hoop of radius R and mass M which is free to roll without slipping along a horizontal track. It contains a bead of mass m which is free to slide without friction around the hoop.

(a) Define X to be the x coordinate of the center of the hoop. The moment of inertia of the hoop is $I = MR^2$. From the nonslip condition the kinetic energy of the hoop is

$$\begin{aligned} T_{hoop} &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}MR^2\dot{X}^2/R^2 \\ T_{hoop} &= M\dot{X}^2. \end{aligned}$$

The x and y coordinates of the bead are

$$x = X + R\sin\phi, \text{ and } y = -R\cos\phi.$$

The velocities are

$$\dot{x} = \dot{X} + R\cos\phi\dot{\phi}, \text{ and } \dot{y} = R\sin\phi\dot{\phi}.$$

The kinetic energy of the bead is

$$T_{bead} = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left(\dot{X}^2 + 2R\cos\phi\dot{X}\dot{\phi} + R^2\dot{\phi}^2\right)$$

Since the potential energy of the bead is $U = mgy = -mgR\cos\phi$, the Lagrangian for this system is

$$\mathcal{L} = M\dot{X}^2 + \frac{1}{2}m\left(\dot{X}^2 + 2R\cos\phi\dot{X}\dot{\phi} + R^2\dot{\phi}^2\right) + mgR\cos\phi.$$

(b) First we note that X is an ignorable coordinate so that P_X is conserved. The Lagrange equation of motion for X is

$$\frac{\partial \mathcal{L}}{\partial \dot{X}} = (2M + m)\,\dot{X} + mR\cos\phi\dot{\phi} = P_X.$$

This is the total linear momentum in the x direction which is conserved. The time derivative of this expression is

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} = (2M + m)\,\ddot{X} + mR\cos\phi\ddot{\phi} - mR\sin\phi\dot{\phi}^2 = 0$$

The Lagrange equation of motion for ϕ comes from the derivatives

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \phi} &= -m\left(R\sin\phi\dot{X}\dot{\phi} + gR\sin\phi\right), \\
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} &= \frac{d}{dt}m\left(R\cos\phi\dot{X} + R^2\dot{\phi}\right) = m\left(R\cos\phi\ddot{X} - R\sin\phi\dot{X}\dot{\phi} + R^2\ddot{\phi}\right).
\end{aligned}$$

The equation of motion is

$$\cos\phi\ddot{X} + R\ddot{\phi} = -g\sin\phi.$$

(c) The only positions of equilibrium occur at $\phi = 0, \pi$. The solution at $\phi = \pi$ is clearly unstable $-gR\sin(\pi + \delta\phi) = +gR\delta\phi$. Now consider the fluctuations around $\phi = 0$. Assuming that ϕ is small and only keeping first order terms in ϕ we find

$$(2M + m)\,\ddot{X} + mR\ddot{\phi} = 0, \text{ and } \ddot{X} + R\ddot{\phi} = -g\phi.$$

Substituting for $\ddot{X}$ we find

$$\begin{aligned}
\left(-\frac{mR}{2M + m} + R\right)\ddot{\phi} = \frac{2M}{2M + m}R\ddot{\phi} = -g\phi \\
\ddot{\phi} = -\frac{2M + m}{2M}\frac{g}{R}\phi \rightarrow \omega^2 = \frac{2M + m}{2M}\frac{g}{R}
\end{aligned}$$

Note that this frequency is larger than that for a simple pendulum. From the conservation of linear momentum in the x direction as the bead moves in one direction the hoop rolls in the opposite direction. This increases the effective gravitational torque, hence an increase in frequency. For $M >> m$ we find that $\omega^2 = g/R$ which is the frequency for a simple pendulum. This is what you expect for a massive hoop and a small bead.

3. Particle Confined to the Surface of a Cone A particle of mass m is confined to move on the surface of an inverted cone (pointing down) of half angle α with its axis being the vertical z axis. (a) Find the Lagrangian in terms of cylindrical polar coordinates, ρ and ϕ. (b) Find the two equations of motion. Since the ϕ coordinate is ignorable, eliminate this coordinate from the equation of motion for the radial coordinate ρ in favor of its conjugate momentum ℓ, the angular momentum about the z axis. Does the equation of motion make sense when $\ell = 0$? (c) Find the equilibrium value of ρ_o such that the particle can remain in a horizontal circular orbit. Is this orbit stable? If so determine the oscillation frequency of small oscillations about equilibrium.

(a) For a cone of half angle α, $\rho = z\tan\alpha$. The kinetic energy of a particle in cylindrical coordinates is

$$T = \frac{1}{2}m\left(\dot{z}^2 + \dot{\rho}^2 + \rho^2\dot{\phi}^2\right),$$

Since the particle is confined to the surface of the cone, $\dot{z} = \dot{\rho}/\tan\alpha$, and the kinetic energy becomes

$$T = \frac{1}{2}m\left(\dot{\rho}^2/\tan^2\alpha + \dot{\rho}^2 + \rho^2\dot{\phi}^2\right) = \frac{1}{2}m\left(\dot{\rho}^2/\sin^2\alpha + \rho^2\dot{\phi}^2\right).$$

The potential energy is

$$U = mgz = mg\rho\cot\alpha.$$

Hence we can write the Lagrangian as

$$\mathcal{L} = \frac{1}{2}m\left(\dot{\rho}^2/\sin^2\alpha + \rho^2\dot{\phi}^2\right) - mg\rho/\tan\alpha.$$

(b) The ϕ equation of motion is

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\partial\mathcal{L}}{\partial\phi} = 0 \rightarrow \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = m\rho^2\dot{\phi} = \ell \text{ (const.)}.$$

The ρ equation of motion is

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\rho} &= m\rho\dot{\phi}^2 - mg\cot\alpha = m\ddot{\rho}/\sin^2\alpha \\ m\ddot{\rho} &= -mg\sin\alpha\cos\alpha + m\rho\sin^2\alpha\dot{\phi}^2.\end{aligned}$$

Eliminating $\dot{\phi}$ in terms of the angular momentum yields

$$\begin{aligned}m\ddot{\rho} &= -mg\sin\alpha\cos\alpha + m\rho\sin^2\alpha\frac{\ell^2}{m^2\rho^4} \\ m\ddot{\rho} &= -mg\sin\alpha\cos\alpha + \frac{\ell^2\sin^2\alpha}{m\rho^3}.\end{aligned}$$

When $\ell = 0$ the equation reduces to

$$\ddot{\rho} = -g \sin\alpha \cos\alpha.$$

The downward acceleration tangent to the surface of the cone is $\ddot{r} = -g\cos\alpha$. Since $\ddot{\rho} = \ddot{r}\sin\alpha$, the answer that we obtained is as you would expect.

(c) At equilibrium $\ddot{\rho} = 0$, $\rho = \rho_o$,and the radial equation becomes

$$-mg\sin\alpha\cos\alpha + \frac{\ell^2\sin^2\alpha}{m\rho_o^3} = 0 \rightarrow \rho_o^3 = \frac{\ell^2\tan\alpha}{m^2 g}.$$

For small oscillations about equilibrium $\rho = \rho_o + \epsilon$ and the radial equation becomes

$$\begin{aligned}
\ddot{\epsilon} &= -g\sin\alpha\cos\alpha + \frac{\ell^2\sin^2\alpha}{m^2\left(\rho_o+\epsilon\right)^3} = -g\sin\alpha\cos\alpha + \frac{\ell^2\sin^2\alpha}{m^2\rho_o^3\left(1+\epsilon/\rho_o\right)^3} \\
\ddot{\epsilon} &= -g\sin\alpha\cos\alpha + \frac{\ell^2\sin^2\alpha}{m^2\rho_o^3}\left(1-3\epsilon/\rho_o\right) = -3\frac{\ell^2\sin^2\alpha}{m^2\rho_o^3}\frac{1}{\rho_o}\epsilon \\
\ddot{\epsilon} &= -3\frac{g}{\rho_o}\sin\alpha\cos\alpha\epsilon.
\end{aligned}$$

Since this expression of the form of a restoring force, ρ_o is a position of stable equilibrium. The angular frequency is given by $\omega^2 = \left(3\sin\alpha\cos\alpha\right)g/\rho_o$.