

13 Lecture 10-23

13.1 Chapter 6 Calculus of Variations

As we have seen, the expressions for the components of the acceleration in non-Cartesian coordinates can be quite messy and the situation can get much worse as we move on to more complicated systems. We need an alternative equation of motion (although equivalent) that works well in any coordinates. This required alternative is provided by Lagrange's equations.

The proof of Lagrange's equations comes from the "variational principle". It has proved possible to formulate almost every branch of physics; classical mechanics, quantum mechanics, optics, electromagnetism, etc. in variational terms. Because they allow a similar formulation of so many different subjects, variational methods have given a unity to physics and have played a crucial role in physical theory. For this reason we will start by introducing variational methods in a reasonably general setting. Ergo this chapter.

13.1.1 Two Examples

Shortest Path Between Two Points The first example is that of finding the shortest distance between two points in a plane. While I am sure that you already know the answer, I am also sure that unless you have already studied the calculus of variations you have not seen a proof of this fact. Consider two points in the plane (x_1, y_1) and (x_2, y_2) and a general curve joining them as shown in figure 1. The problem then is to find the curve $y(x)$ that has the

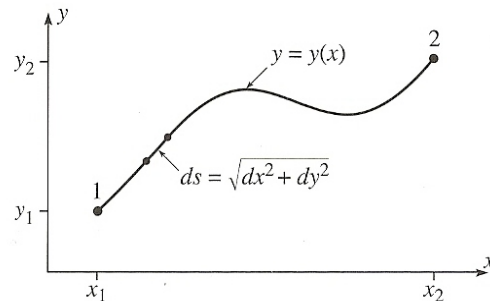


Figure 6.1 An arbitrary path connecting two points (x_1, y_1) and (x_2, y_2) , the total length of which is $L = \int_1^2 ds$ where $ds = \sqrt{dx^2 + dy^2}$.

shortest length and prove that it is in fact a straight line. A differential length of line in a plane is $ds = \sqrt{dx^2 + dy^2}$, or equivalently $ds = \sqrt{1 + (dy/dx)^2} dx$. This means that we can express the total length as

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + (dy/dx)^2} dx. \quad (1)$$

The problem now is to find the function $y(x)$ for which the integral in equation (1) is minimized. The student should note the difference between this problem and finding the minimum of a known function $y(x)$. It should be obvious that this problem is more difficult than the old more familiar one.

Fermat's Principle A similar problem is to find the path that light will follow between two points. If the refractive index is a constant, then the path is again a straight line. However, if the index varies or if we insert a mirror or lens then the path is not so obvious. The French mathematician Fermat discovered that the required path is the path for which the travel time is minimized. Now the travel time over a differential length is $dt = ds/v = ds/(c/n) = nds/c$. Hence, using similar arguments to those used just above, the travel time is found from the integral

$$T = \frac{1}{c} \int_1^2 nds = \frac{1}{c} \int_{x_1}^{x_2} n(x, y) \sqrt{1 + (dy/dx)^2} dx. \quad (2)$$

From Fermat's principle then, we need to minimize this slightly more involved integral. Of course if $n(x, y)$ is a constant, then the problem is reduced to finding the shortest distance, i.e. that of a straight line. In general, the refractive index can vary and again we must find a path that minimizes an integral.

This integral, equation (2), is more involved due to the factor $n(x, y)$. As we shall learn, similar integrals arise in many other problems. In fact there will be times when we want to find a path that results in an integral being a maximum, and other times where we are interested in both a minima and a maxima. In elementary calculus we know that the necessary condition for a function $f(x)$ to be a minimum or a maximum is simply $df/dx = 0$. However this condition is not necessarily enough to guarantee a minimum or a maximum, e.g. if $d^2f/dx^2 = 0$ then it is an inflection point. We do know that whenever $df/dx = 0$ at a point $x = x_o$ the function $f(x)$ is stationary since an infinitesimal displacement of x from x_o leaves $f(x)$ unchanged. We define such a point as a *stationary point*.

The method that we are about to describe finds paths that makes integrals like those in equations (1) and (2) stationary, in the sense that an infinitesimal variation of the path from the stationary path does not change the value of the integral. Since our concern is how infinitesimal variations of a path change an integral, the subject is called the *calculus of variations*. For these same reasons, the methods that we shall develop are called variational methods.

Before we proceed we should define a *functional*. A function is a mathematical object that given a variable (in general a complex number or set of numbers) it returns a number (or set of numbers if the function forms the components of a vector). A functional is a mathematical object that takes an entire function and returns a number. The integrals mentioned above do just that. Given functions $y(x)$ and $y'(x)$ the integrals for the path length or transit time can be performed and return a number, either L or T . Our goal then is to find the functions (paths) that result in a minimum, maximum, or at least a stationary value of the integral or functional of interest.

13.1.2 The Euler-Lagrange Equation

The two examples that we considered were of the general form of the variational problem. The integrals that we wish to consider at this point are of the form

$$S = \int_{x_1}^{x_2} f(y(x), y'(x), x) dx, \quad (3)$$

where $y(x)$ is an as-yet unknown curve joining two points (x_1, y_1) and (x_2, y_2) as in figure 6.2. Among all of the curves joining the points 1 and 2 we have to find the one that minimizes (or maximizes or at least finds stationary) the integral S . Notice that the function f inside the integral is a function of three variables, $f = f(y, y', x)$, but since the integral follows the path $y(x)$ the integrand

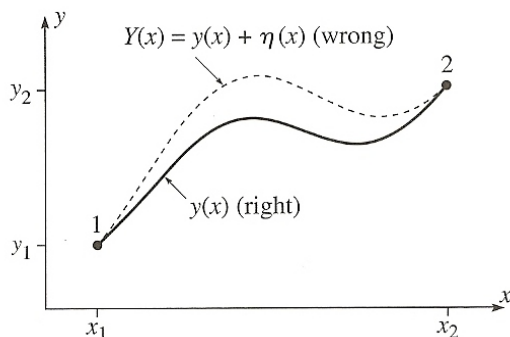


Figure 6.2 The function $y(x)$ is a stationary path for the integral S . The curve $Y(x) = y(x) + \eta(x)$ differs from $y(x)$ by a small arbitrary amount given by $\eta(x)$.

$f(y(x), y'(x), x)$ is actually just a function of the one variable x .

We will assume that the correct solution to the problem is given by $y(x)$. Then any neighboring curve given by $Y(x) = y(x) + \eta(x)$ does not give the stationary result that we are looking for. Now we are going to assume that $\eta(x)$ is a small arbitrary deviation from the correct curve. Since $Y(x)$ must pass through the endpoints (x_1, y_1) and (x_2, y_2) , any deviation must satisfy the constraints

$$\eta(x_1) = \eta(x_2) = 0. \quad (4)$$

We can express the change in the integral S due to any deviation as

$$\delta S = \int_{x_1}^{x_2} f(y(x) + \eta(x), y'(x) + \eta'(x), x) dx - \int_{x_1}^{x_2} f(y(x), y'(x), x) dx. \quad (5)$$

We have assumed that $y(x)$ is a stationary path. This means then any small deviation from this path will result in the integral being unchanged or $\delta S = 0$. Since the deviations, $\eta(x)$, from the correct curve (i.e. the stationary one) are

small we can expand $f(y(x) + \eta(x), y'(x) + \eta'(x), x)$ in a Taylor's series with the result

$$f(y(x) + \eta(x), y'(x) + \eta'(x), x) = f(y(x), y'(x), x) + \frac{\partial f(y(x), y'(x), x)}{\partial y} \eta(x) + \frac{\partial f(y(x), y'(x), x)}{\partial y'} \eta'(x). \quad (6)$$

Conveniently, in the expression for δS the second term on the right hand side of equation (5) cancels the first term on the right hand of equation (6) and we are left with

$$\delta S = \int_{x_1}^{x_2} \frac{\partial f(y(x), y'(x), x)}{\partial y} \eta(x) dx + \int_{x_1}^{x_2} \frac{\partial f(y(x), y'(x), x)}{\partial y'} \eta'(x) dx. \quad (7)$$

To further simplify the expression for δS we simply integrate the second term by parts with the result

$$\int_{x_1}^{x_2} \frac{\partial f(y(x), y'(x), x)}{\partial y'} \eta'(x) dx = - \int_{x_1}^{x_2} \frac{d}{dx} \frac{\partial f(y(x), y'(x), x)}{\partial y'} \eta(x) dx + \left[\frac{\partial f(y(x), y'(x), x)}{\partial y'} \eta(x) \right]_{x_1}^{x_2}. \quad (8)$$

From the endpoint constraints on $\eta(x)$ the last term in the equation above vanishes and the expression for δS becomes

$$\delta S = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \eta(x) dx = 0, \quad (9)$$

where for the sake of brevity we have dropped the explicit reference to the dependences of f on y , y' , and x . This condition must be satisfied for any choice of the function $\eta(x)$ (as long as η is small), therefore, as long as the quantity inside the parenthesis is continuous, it must vanish for all x :

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0 \quad (\text{Euler-Lagrange Equation}) \quad (10)$$

for all x in the range $x_1 \leq x \leq x_2$. This equation is so-called *Euler-Lagrange equation*, named after the Swiss mathematician Leonhard Euler, and the Italian-French physicist and mathematician Joseph Lagrange. It is this equation whose solution allows us to find the path for which the integral S is stationary.

Some discussion is order here. Equation (9) has the form $\int g(x) \eta(x) dx = 0$. In general this does not imply that $g(x) = 0$. However equation (9) holds for any choice of the function $\eta(x)$, and if $\int g(x) \eta(x) dx = 0$ for any $\eta(x)$ then we can conclude that $g(x) = 0$ (at least in the range of the integration). To see how we can prove this, let's assume that the contrary is true, i.e. $g(x)$ is nonzero over some interval inside the range of the integration. We will then choose $\eta(x)$ so that it is of the same sign as $g(x)$ over this same range. If we also assume that

the integrand is continuous, then over this range $g(x)\eta(x) \geq 0$, and is nonzero at least in this chosen interval. Under these conditions $\int g(x)\eta(x)dx$ cannot be zero. This contradiction implies that $g(x)$ does in fact equal zero over the entire range of the integral. There is one caveat here and that is we assumed that the integrand is continuous or more specifically that $g(x)$ is continuous. Physicists are notorious for implicitly making these types of assumptions and it can drive mathematicians crazy. They would point out for example that a noncontinuous $g(x)$ could be zero everywhere except for isolated points where it is nonzero. These points would not contribute to the integral and allow for $g(x) \neq 0$ at these isolated points. We shall not concern ourselves with these issues and carry on as physicists (and engineers) and assume that all functions of interest are continuous.

13.1.3 Applications of the Euler-Lagrange Equation

Example 1 - Shortest Path Between Two Points We will start with the problem that we began with, namely finding the shortest path between two points in a plane. Thus we wish to find the path that makes the length stationary or equivalently $\delta L = 0$. As we saw in equation (1) the length L is given by $L = \int_{x_1}^{x_2} \sqrt{1 + (dy/dx)^2} dx$. This is in the standard form, equation (3), with the function f given by

$$f = \left(1 + (y')^2\right)^{1/2}. \quad (11)$$

Evaluating the two partial derivatives that are required by the Euler-Lagrange equation yields

$$\frac{\partial f}{\partial y} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y'} = \frac{y'}{\left(1 + (y')^2\right)^{1/2}}. \quad (12)$$

Substituting these results into the Euler-Lagrange equation, we find

$$\frac{d}{dx} \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = \frac{d}{dx} \frac{y'}{\left(1 + (y')^2\right)^{1/2}} = 0. \quad (13)$$

In other words

$$\frac{y'}{\left(1 + (y')^2\right)^{1/2}} = \sin \alpha, \quad (14)$$

where α is a constant. Although the expression on the left in this equation is less than one somewhat justifying our choice of the constant, $\sin \alpha$, we have chosen this constant to be $\sin \alpha$ for reasons of convenience. Solving for y' we find that

$$y' = \tan \alpha. \quad (15)$$

A simple integration leads to

$$y = x \tan \alpha + \beta, \quad (16)$$

which is the equation of a straight line with a slope of $\tan \alpha$, justifying our choice of the integration constant.

So now what we have really proved by finding the curve that satisfies the Euler-Lagrange equations is that a straight line satisfies $\delta L = 0$. However the physics (actually the geometry) of the problem makes it clear that this solution is in fact the shortest distance between two points.

Example 2 - Snell's Law Consider a lifeguard on the beach at point (x_1, y_1) who wants to reach a struggling swimmer in the water at point (x_2, y_2) , where the interface between the beach and the water lies along $x = 0$. What path does the lifeguard take to minimize the time to reach the swimmer? It is not a straight line because the lifeguard's speed on the sand, v_1 , is greater than his speed, v_2 , in the water. The optimal path is shown in figure 6.3 which consists

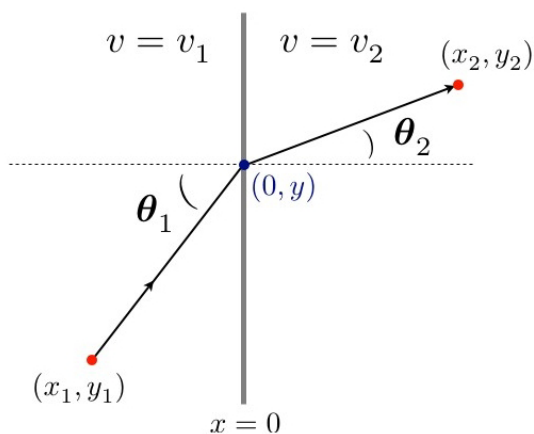


Figure 6.3 The optimal path that results in the minimum time for the lifeguard to reach the swimmer.

of two separate line segments. The total time is a function of y ;

$$T(y) = \frac{1}{v_1} \sqrt{x_1^2 + (y - y_1)^2} + \frac{1}{v_2} \sqrt{x_2^2 + (y_2 - y)^2}. \quad (17)$$

To minimize the time we set $dT/dy = 0$ with the result

$$\begin{aligned} \frac{dT}{dy} &= \frac{1}{v_1} \frac{y - y_1}{\sqrt{x_1^2 + (y - y_1)^2}} + \frac{1}{v_2} \frac{y - y_2}{\sqrt{x_2^2 + (y - y_2)^2}} \\ \frac{dT}{dy} &= \frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2} = 0. \end{aligned} \quad (18)$$

Hence the optimal path satisfies

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}, \quad (19)$$

which is known as Snell's law.

Snell's law is well known in geometrical optics, where the speed of light in a dielectric medium is $v = c/n$ and n is the index of refraction. In terms of the index Snell's law is written

$$n_1 \sin \theta_1 = n_2 \sin \theta_2. \quad (20)$$

So essentially the quantity $n \sin \theta = \alpha$ which is a constant across the interface.

Now imagine that there are many such interfaces between regions of very small thicknesses. We can then regard n and θ as continuous functions of the coordinate x . From Snell's law

$$n(x) \sin \theta(x) = \alpha$$

must still be satisfied across any interface. Only for this case the interfaces occur continuously as the light propagates through the medium. For a continuous medium we can state this condition as

$$\frac{d}{dx} (n \sin \theta) = 0. \quad (21)$$

Now consider the integral in equation (2) for the transit time. From Fermat's principle the path of light is that corresponding to a minimum transit time. This path is found from the solution of the Euler Lagrange equation with $f = n(x) \sqrt{1 + y'^2}$. Since this function is independent of y we find

$$\frac{d}{dx} \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = \frac{d}{dx} \left(n \frac{y'}{\sqrt{1 + y'^2}} \right) = 0. \quad (22)$$

Now from figure 6.3 we have $y' = \tan \theta$, and the Euler Lagrange equation for minimum transit time becomes

$$\frac{d}{dx} \left(n \frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}} \right) = \frac{d}{dx} (n \tan \theta \cos \theta) = \frac{d}{dx} (n \sin \theta) = 0. \quad (23)$$

Making use of Fermat's principle we have again found Snell's law, albeit for an index that changes continuously as a function of x .

Example 3 - The Brachistochrone Consider a bead that is allowed to fall in a uniform gravitational field attached to a frictionless wire from point 1, (x_1, y_1) , to point 2, (x_2, y_2) , where clearly $y_2 < y_1$. The problem is then to find the path of the wire that minimizes the travel time of the bead between these two points. This famous problem is called the *Brachistochrone* problem, from the Greek words *brachistos* meaning "shortest" and *chronos* meaning "time".

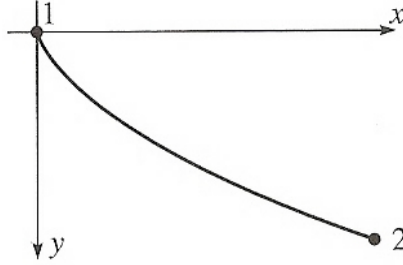


Figure 6.4 Brachistochrone problem involves finding the path which results in the shortest time for a falling object to travel from point 1 to point 2.

The time to travel from point 1 to 2 is

$$T = \int_1^2 \frac{ds}{v}, \quad (24)$$

where v is the speed at any height y which is determined by the conservation of energy to be $v = \sqrt{2gy}$. Here by convention our initial conditions are $v = 0$ at $y = 0$ and we are measuring y positive in the downward direction. As in example 1, $ds = \sqrt{dx^2 + dy^2}$, and the integral for the total time is

$$T = \int_1^2 \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gy}}. \quad (25)$$

For this problem it is more convenient to perform the integral over y and we find

$$T = \frac{1}{\sqrt{2g}} \int_1^2 \frac{\sqrt{1 + x'^2}}{\sqrt{y}} dy. \quad (26)$$

Since this integrand only depends on x' and not on x , the Euler-Lagrange equations become

$$\frac{d}{dy} \frac{\partial f}{\partial x'} - \frac{\partial f}{\partial x} = \frac{d}{dy} \frac{\partial f}{\partial x'} = 0. \quad (27)$$

This means that $\partial f / \partial x' = \text{const}$ and

$$\frac{\partial f}{\partial x'} = \frac{x'}{y^{1/2} (1 + x'^2)^{1/2}} = \frac{1}{\sqrt{2a}}, \quad (28)$$

or

$$\frac{x'^2}{y(1 + x'^2)} = \frac{1}{2a}, \quad (29)$$

where we have defined for reasons of convenience this constant to be $1/2a$. Solving this expression for x' leads to

$$x' = \sqrt{\frac{y}{2a - y}} \rightarrow x = \int_0^y \sqrt{\frac{y}{2a - y}} dy, \quad (30)$$

where we have defined the starting point as the origin. The substitution $y = 2a \sin^2 \theta/2$ leads to

$$x = 2a \int_0^\theta \sin^2 \theta/2 d\theta = a \int_0^\theta (1 - \cos \theta) d\theta = a(\theta - \sin \theta). \quad (31)$$

The final parametric solution to this problem is

$$x = a(\theta - \sin \theta), \quad \text{and} \quad y = a(1 - \cos \theta), \quad (32)$$

where the constant a has been chosen so that the curve passes through the point (x_2, y_2) .

The curve represented by the parametric solution in equation (32) is shown in figure 6.4. The continuation of the path with the dashed line in figure 6.5

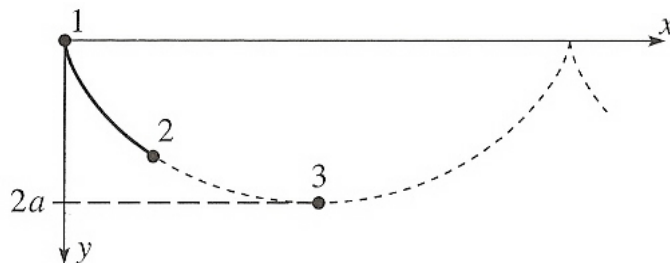


Figure 6.5 The path that gives the shortest time between points 1 and 2 is part of a cycloid. The cycloid is the curve traced out by a wheel of radius a that rolls along the underside of the x axis. Point 3 is the lowest point on the curve and resides one diameter of the wheel, $2a$, below the x axis.

shows that the solution of the brachistochrone problem happens to be a cycloid, that is, the path traced out by a point on the rim of a wheel of radius a as it rolls along the underside of the x axis.

Another remarkable feature of this curve is that if we release the bead from rest at point 2 and let it fall until it reaches point 3, the time is the same independent of the location of point 2. This means that the oscillations of the bead falling back and forth along the cycloid shaped curve are exactly *isochronous* (period is independent of the amplitude). This is in contrast with the oscillations of a simple pendulum, which are only approximately isochronous, to the extent that the amplitude is small.