14 Lecture 10-28

14.1 Chapter 6 Calculus of Variations (con)

Example 3 - The Brachistochrone Consider a bead that is allowed to fall in a uniform gravitational field attached to a frictionless wire from point 1, (x_1, y_1), to point 2, (x_2, y_2), where clearly $y_2 < y_1$. The problem is then to find the path of the wire that minimizes the travel time of the bead between these two points. This famous problem is called the *Brachistochrone* problem, from the Greek words *brachistos* meaning "shortest" and *chronos* meaning "time".
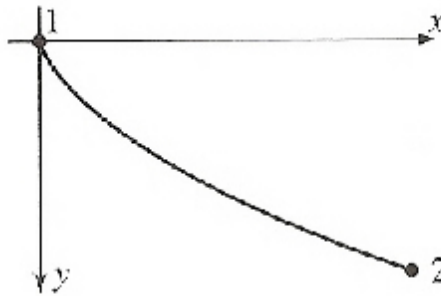

Figure 6.4 Brachistochrone problem involves finding the path which results in
the shortest time for a falling object to travel from point 1 to point 2.

The time to travel from point 1 to 2 is

$$T = \int_1^2 \frac{ds}{v}, \tag{1}$$

where v is the speed at any height y which is determined by the conservation of energy to be $v = \sqrt{2gy}$. Here by convention our initial conditions are $v = 0$ at $y = 0$ and we are measuring y positive in the downward direction. As in example 1, $ds = \sqrt{dx^2 + dy^2}$, and the integral for the total time is

$$T = \int_1^2 \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gy}}. \tag{2}$$

For this problem it is more convenient to perform the integral over y and we find

$$T = \frac{1}{\sqrt{2g}} \int_1^2 \frac{\sqrt{1 + x'^2}}{\sqrt{y}} dy. \tag{3}$$

Since this integrand only depends on x' and not on x, the Euler-Lagrange equations become

$$\frac{d}{dy}\frac{\partial f}{\partial x'} - \frac{\partial f}{\partial x} = \frac{d}{dy}\frac{\partial f}{\partial x'} = 0. \tag{4}$$

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This means that $\partial f / \partial x' = const$ and

$$\frac{\partial f}{\partial x'} = \frac{x'}{y^{1/2} \left(1 + x'^2\right)^{1/2}} = \frac{1}{\sqrt{2a}}, \tag{5}$$

or

$$\frac{x'^2}{y\left(1 + x'^2\right)} = \frac{1}{2a}, \tag{6}$$

where we have defined for reasons of convenience this constant to be $1/2a$. Solving this expression for x' leads to

$$x' = \sqrt{\frac{y}{2a - y}} \rightarrow x = \int_0^y \sqrt{\frac{y}{2a - y}} dy, \tag{7}$$

where we have defined the starting point as the origin. The substitution $y = 2a \sin^2 \theta/2$ leads to

$$x = 2a \int_0^\theta \sin^2 \theta/2 d\theta = a \int_0^\theta \left(1 - \cos \theta\right) d\theta = a \left(\theta - \sin \theta\right). \tag{8}$$

The final parametric solution to this problem is

$$x = a \left(\theta - \sin \theta\right), \quad \text{and} \quad y = a \left(1 - \cos \theta\right), \tag{9}$$

where the constant a has been chosen to that the curve pass through the point (x_2, y_2).

The curve represented by the parametric solution in equation (9) is shown in figure 6.4. The continuation of the path with the dashed line in figure 6.5
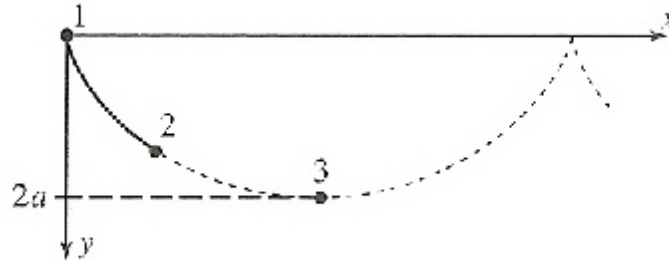

Figure 6.5 The path that gives the shortest time between points 1 and 2 is part of a cycloid. The cycloid is the curve traced out by a wheel of radius a that rolls along the underside of the x axis. Point 3 is the lowest point on the curve and resides one diameter of the wheel, $2a$, below the x axis.

shows that the solution of the brachistochrone problem happens to be a cycloid, that is, the path traced out by a point on the rim of a wheel of radius a as it rolls along the underside of the x axis.

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Another remarkable feature of this curve, which the reader will have the opportunity to prove in problem 6.25, is that if we release the bead from rest at point 2 and let it fall until it reaches point 3, the time is the same independent of the location of point 2. This means that the oscillations of the bead falling back and forth along the cycloid shaped curve are exactly *isochronous* (period is independent of the amplitude). This is in contrast with the oscillations of a simple pendulum, which are only approximately isochronous, to the extent that the amplitude is small.

14.1.1   A First Integral of the Euler-Lagrange Equation

We have previously shown that for a conservative force in one dimension, the expression for the conservation of energy is the first integral of Newton's equation of motion. As it turns out there is an analogous first integral for the Euler-Lagrange equation. Consider an integral of interest where we wish to find a *stationary path* that is of the form,

$$S = \int_1^2 f\left(y, y'\right) dx, \tag{10}$$

where $f\left(y, y'\right)$ has no explicit dependence on the independent variable of integration x. For this functional form of f we apply the chain rule and find

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' = \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'', \tag{11}$$

as we have assumed that $\partial f / \partial x$ vanishes. From the Euler-Lagrange equation we know that

$$\frac{\partial f}{\partial y} = \frac{d}{dx} \frac{\partial f}{\partial y'}. \tag{12}$$

A simple substitution of this result into equation (11) results in

$$\frac{df}{dx} = \left(\frac{d}{dx} \frac{\partial f}{\partial y'}\right) y' + \frac{\partial f}{\partial y'} \frac{dy'}{dx} = \frac{d}{dx} \left(\frac{\partial f}{\partial y'} y'\right), \tag{13}$$

or

$$\frac{d}{dx} \left(f - y' \frac{\partial f}{\partial y'}\right) = 0. \tag{14}$$

This leads to the first integral of the Euler-Lagrange equation

$$f - y' \frac{\partial f}{\partial y'} = const. \tag{15}$$

This first integral can often simplify calculations and when the integrand is independent of the independent variable it is a useful starting point.

To obtain this first integral we required that the function describing the path had no explicit dependence on the variable of integration. This is exactly analogous to the situation in mechanics where to perform the first "time integration"

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of Newton's equations we insisted that the net force on the object was conservative. That required among other things that the force and subsequently the potential did not depend on time. When we apply the Euler-Lagrange equation to the laws of motion for particles in mechanics we will find an analogous requirement. That is, if the quantity which will satisfy the Euler-Lagrange equations (which we will call the Lagrangian) is independent of time, then what we will define to be the "Hamiltonian" is conserved. For many problems the Hamiltonian is equal to the total mechanical energy. This concept will be discussed in the next chapter. It is this conservation of "energy" that is again the first integral of the Euler-Lagrange equations.

The Brachistochrone Problem Again The integral that we minimized in the Brachistochrone problem was

$$T = \int_1^2 \frac{ds}{v} = \frac{1}{\sqrt{2g}} \int_1^2 \frac{\sqrt{dx^2 + dy^2}}{\sqrt{y}}. \tag{16}$$

Initially we considered the function $f = \sqrt{1 + x'^2}/\sqrt{y}$ where $x' = dx/dy$. The advantage for the Brachistochrone problem with that choice is it results in $\partial f/\partial x = 0$ which lead to the simplification $\partial f/\partial x' = 1/\sqrt{2a}$. This time however we will consider the integral

$$T = \frac{1}{\sqrt{2g}} \int_1^2 \frac{\sqrt{dx^2 + dy^2}}{\sqrt{y}} = \frac{1}{\sqrt{2g}} \int_1^2 \frac{\sqrt{1 + y'^2}}{\sqrt{y}} dx. \tag{17}$$

Now the function $f = \sqrt{1 + y'^2}/\sqrt{y}$ is independent of x and we can invoke the first integral of the Euler-Lagrange equation to find

$$f - \frac{\partial f}{\partial y'} y' = \frac{\sqrt{1 + y'^2}}{\sqrt{y}} - \frac{y'^2}{\sqrt{y}\sqrt{1 + y'^2}} = \frac{1}{\sqrt{2a}}, \tag{18}$$

where again we have chosen the constant (in this case $1/\sqrt{2a}$) for matter of convenience. Multiplying by $\sqrt{y}\sqrt{1 + y'^2}$, reorganizing terms, and squaring leads to

$$2a = y\left(1 + y'^2\right) \rightarrow y'^2 = \frac{2a - y}{y}. \tag{19}$$

Taking the square root of this equation and inverting results in

$$x' = \sqrt{\frac{y}{2a - y}} \rightarrow x = \int_0^y \sqrt{\frac{y}{2a - y}} dy, \tag{20}$$

which is the same integral as equation (7). So as expected we find the same cycloid solution as we did in the original approach. Depending on the integrand, one or the other of the approaches will often be preferred.

14.1.2 Maximum and Minimum versus Stationary

You may have noticed that in none of the examples that we considered have we bothered to actually check to see if the curves that we found actually resulted in a minimum for the integral of interest. Clearly a straight line between two points in a plane is the path that results in a minimum length, not a maximum or just stationary. However the Euler-Lagrange equation only guarantees to give a path that is stationary. The problem of deciding whether or not the curve is a minimum, maximum, or just stationary is generally very difficult. In many cases it is easy to see which is the case. For instance the solution of a straight line is clearly a minimum. It may not be so clear that a cycloid path results in the minimum time for a falling object, though it is.

To illustrate the problem of finding the shortest distance between two points, or **geodesic** as it is often called, consider finding a geodesic on the surface of a sphere. Using the calculus of variations, you can prove that a great circle does indeed make the distance between two points stationary. But it requires a bit of thought before deciding that this is the minimum distance since there are *different* great circle paths between any two points on a sphere. Consider for example two cities that lie on the equator (a great circle path on the Earth) that are approximately 2000 miles apart in South America. The choices on the great circle path are (i) the 2000 mile route or (ii) going the other way around the Earth which is approximately 22000 miles in length. It is obvious in this example that the shorter distance is the 2000 mile route. You might guess then that the 22000 mile route is a maximum, but alas, it is not, as it is easy to construct paths that are longer than this distance. So the long way around the equator is neither a maximum nor a minimum just merely a stationary route. From a calculus point of view, we may think of this as a local minima but not a global minima.

Another geodesic path that comes up frequently in physics is that of a geodesic path in relativity (either special or general) In that scenario, the integral of interest is called the *proper time*. The proper time is the time as measured by the observer traveling along his path in space-time. From the so-called twin paradox in special relatively, you are probably aware that the space traveler who leaves his Earth bound twin behind returns only to find that his twin has measured a greater passage in time and is now older than he is. As it turns out, the twin that traveled the geodesic path is the twin that remained behind. So we see from this example that a geodesic path in special relativity is the path that maximizes the proper time. Since general relativity is an extension of special relativity to curved space, it should not come as a surprise that a geodesic path in general relativity also maximizes the proper time.

As this discussion points out, it should be clear that the Euler-Lagrange equation finds a stationary path. A path that produces no change in the integral of interest due to small deviations along the path. However, determining what sort of stationary path; minimum, maximum, or just stationary, that the solution to the Euler-Lagrange equation produces can be tricky. Consider the soap film problem shown in figure 6.6, where we are interested in finding the

minimum surface area for a surface of revolution. A soap film will naturally find this minimum surface area as it minimizes the energy associated with the surface tension in the film.
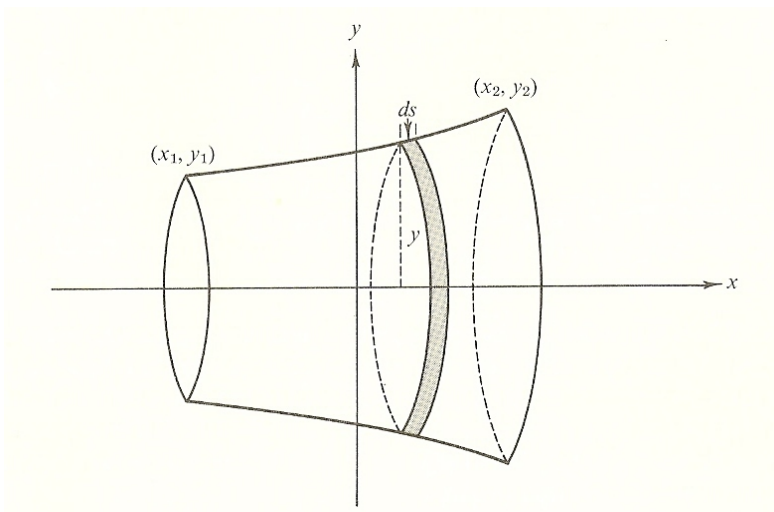

Figure 6.6 Minimum surface area for a surface of rotation, the soap film problem.

The total area for a surface of revolution about the x axis is given by

$$A = \int_1^2 2\pi y \sqrt{dx^2 + dy^2} = \int_1^2 2\pi y \sqrt{x'^2 + 1} dy. \tag{21}$$

Since the integrand is independent of x the Euler-Lagrange equation is

$$\frac{d}{dy}\left(\frac{\partial}{\partial x'} y \sqrt{x'^2 + 1}\right) = 0 \rightarrow \frac{x'y}{\sqrt{x'^2 + 1}} = y_o, \tag{22}$$

where y_o is a constant. Solving for x' results in

$$x' = y_o / \sqrt{y^2 - y_o^2} \rightarrow \frac{x - x_o}{y_o} = \int \frac{dy}{\sqrt{y^2 - y_o^2}}. \tag{23}$$

With the substitution $y = y_o \cosh u$ we find

$$\frac{x - x_o}{y_o} = \int \frac{y_o \sinh u \, du}{y_o \sinh u} = u.$$

Taking the hyperbolic cosine of both sides yields the solution for the radius of the soap film, which is called a *catenary*,

$$y = y_o \cosh \frac{x - x_o}{y_o}.$$

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These two constants can be used so that the curve passes through (x_1, y_1) and (x_2, y_2). Note that the minimum value of $y = y_{\min}$ occurs when $x = x_o$ for which $y_{\min} = y_o$.

This problem is useful to consider in a bit more detail as it demonstrates the issue of determining whether the result is a minimum, maximum, or simply a stationary solution. For simplicity we will consider the example of two equal rings of radius R that are separated by a distance R. This configuration is of interest for the surface area of a cylinder between the two rings has a surface area of $A_{cyl} = 2\pi R^2$ which is identical to the combined surface area of the two rings. If the surface area of the soap film ever exceeds this combined area of the two rings then the soap bubble tends to collapse into two separate bubbles covering the individual rings as it is energetically favorable for the soap bubble to maintain a minimum surface area. When this happens it is called the Goldschmidt discontinuous solution.

For symmetry purposes we will define the midpoint of the rings to be $x = 0$ so that the rings are located at $\pm R/2$. This symmetry sets $x_o = 0$ so that we have

$$y = y_o \cosh \frac{x}{y_o}. \tag{24}$$

The boundary conditions require that $y = R$ when $x = \pm R/2$ so that y_o, which is the minimum radius that occurs at $x = 0$, is determined by the transcendental equation

$$R = y_o \cosh \frac{R}{2y_o}. \tag{25}$$

Defining $u = y_o/R$ the transcendental equation reduces to

$$1 = u \cosh \frac{1}{2u}. \tag{26}$$

This transcendental equation has two solutions. One occurs at $u_1 = .8484 \rightarrow y_o = .8484R$. This is a relatively shallow curve as the minimum radius of the soap film is just slightly less than the radius of the rings. The other solution is at $u_2 = .2351 \rightarrow y_o = .2351R$. This is called the deep curve as the minimum radius of the soap film is significantly less than the radius of the rings. The general solutions for y_o/R versus the location of the rings at $\pm x_0/R$ is shown in figure 6.7 with our solution for $x_0/R = 1/2$ shown as the vertical solid line.
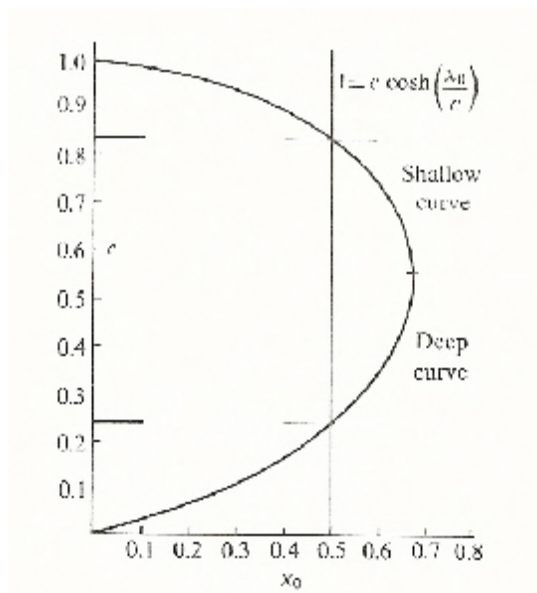
Figure 6.7. Solutions for y_o/R versus the location of the rings at $\pm x_0/R$.

Now let's consider our solutions in a bit more detail. The radius for the shallow and deep curves satisfy (respectively)

$$
\begin{aligned}
y_1 &= .8484R\cosh 1.179x/R, & (27)\\
y_2 &= .2350R\cosh 4.255x/R. & (28)
\end{aligned}
$$

So now the question is which curve has the smaller area, and just as important, are they less than the Goldschmidt discontinuous solution of $2\pi R^2$. Now the area of revolution for the soap bubble is given by

$$
A = \int_{-R/2}^{R/2} 2\pi y\sqrt{1+y'^2}dx, \tag{29}
$$

where y is given by equation (24). The result of this integral can be obtained in a straightforward manner with the result

$$
A = \pi y_o \left(R + y_o \sinh R/y_o \right). \tag{30}
$$

The total area for the first solution, the shallow curve, is

$$
A_1 = .8483\pi R^2 \left(1 + .8483\sinh 1.179 \right) = 1.907\pi R^2, \tag{31}
$$

while the area for the second solution, the deep curve, is

$$
A_2 = .2350\pi R^2 \left(1 + .2350\sinh 4.255 \right) = 2.180\pi R^2. \tag{32}
$$

We see that the area for the shallow curve is less than the area for the deep curve, hence it is the minimum area. Note that the second solution is only a

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stationary solution, and additionally it is larger than the Goldschmidt discontinuous solution. Physically due to the energy of surface tension this solution will likely never exist in nature. As shown in figure 6.8 the area of the deep curve is not only greater than the area of the shallow curve it is also greater than the Goldschmidt discontinuous solution for all values of the separation of the two rings.
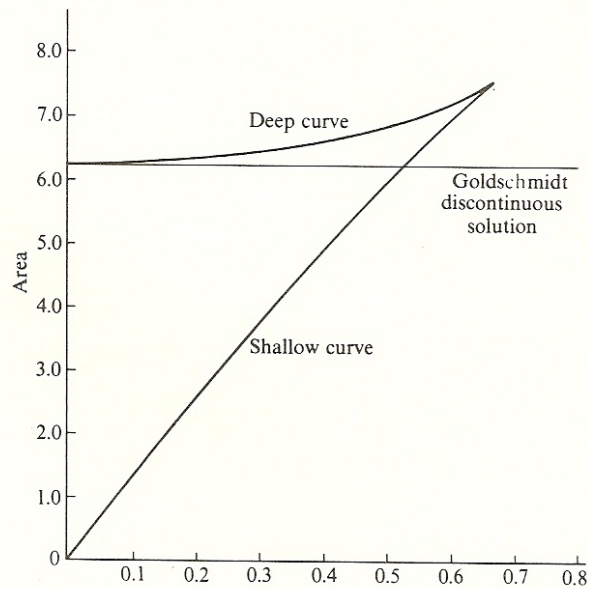

Figure 6.8. Area of the shallow and deep curves versus the Goldschmidt discontinuous solution.

The real point to be made here is that, as with a geodesic on a sphere, you always have to be careful when you interpret the solutions to the Euler-Lagrange equations as you are only guaranteed for them to be stationary solutions.