

15 Lecture 11-2

15.1 Chapter 6 Calculation of Variations (con)

15.1.1 More Than Two Variables

So far we have only considered problems with two variables, the independent variable (usually x) and the dependent variable (usually y). For many applications in mechanics we will find several dependent variables and fortunately still only one independent variable which is usually the time, t . For a simple example of two dependent variables we go back to the original problem of finding the shortest path between two points in a plane. We assumed that the path could be written in the form $y(x)$. However we could also write the path in parametric form (analogous to the solution for the Brachistochrone) as

$$x = x(\lambda) \quad \text{and} \quad y = y(\lambda), \quad (1)$$

where λ is any convenient variable in terms of which the curve can be parameterized (for example the length along the curve).

With this parameterization the length along the curve is

$$L = \int_1^2 \sqrt{dx^2 + dy^2} = \int_1^2 \sqrt{x'(\lambda)^2 + y'(\lambda)^2} d\lambda, \quad (2)$$

where $x'(\lambda) = dx/d\lambda$ and $y'(\lambda) = dy/d\lambda$. The job now is to find the two functions for which this integral is minimum.

The problem is more complicated than before because there are two unknown functions. Now the integral that we wish to find the minimum is of the form

$$S = \int_1^2 f(x(\lambda), y(\lambda), x'(\lambda), y'(\lambda)) d\lambda, \quad (3)$$

between two points $(x(\lambda_1), y(\lambda_1))$ and $(x(\lambda_2), y(\lambda_2))$. The problem is actually very similar to the one variable case. Basically we will need to satisfy two Euler-Lagrange equations. To show this we let the correct path be given by

$$x = x(\lambda) \quad \text{and} \quad y = y(\lambda), \quad (4)$$

and let the nearby incorrect path be of the form

$$X = x(\lambda) + \eta(\lambda) \quad \text{and} \quad Y = y(\lambda) + \varepsilon(\lambda). \quad (5)$$

Again we insist that the change in the integral due to a nearby path vanish or

$$\delta S = \int_1^2 f(X(\lambda), Y(\lambda), X'(\lambda), Y'(\lambda)) d\lambda - \int_1^2 f(x(\lambda), y(\lambda), x'(\lambda), y'(\lambda)) d\lambda = 0. \quad (6)$$

Once again using a Taylor's expansion for the quantity inside the first integral on the left results in

$$\delta S = \int_1^2 \left(\frac{\partial f}{\partial x} \eta + \frac{\partial f}{\partial y} \varepsilon + \frac{\partial f}{\partial x'} \eta' + \frac{\partial f}{\partial y'} \varepsilon' \right) d\lambda = 0. \quad (7)$$

Integrating the last two terms by parts and recognizing that by definition the curve passes through the endpoints so that $\eta(\lambda_1) = \eta(\lambda_2) = \varepsilon(\lambda_1) = \varepsilon(\lambda_2) = 0$, results in

$$\delta S = \int_1^2 \left[\left(\frac{\partial f}{\partial x} - \frac{d}{d\lambda} \frac{\partial f}{\partial x'} \right) \eta + \left(\frac{\partial f}{\partial y} - \frac{d}{d\lambda} \frac{\partial f}{\partial y'} \right) \varepsilon \right] d\lambda = 0. \quad (8)$$

In a parallel argument that we used before, namely $\eta(\lambda)$ and $\varepsilon(\lambda)$ are arbitrary functions (albeit of small magnitude), we obtain

$$\frac{\partial f}{\partial x} - \frac{d}{d\lambda} \frac{\partial f}{\partial x'} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} - \frac{d}{d\lambda} \frac{\partial f}{\partial y'} = 0. \quad (9)$$

These two equations determine a path for which the integral in equation (3) is stationary. Conversely, if the integral is stationary for some path, that path must satisfy this pair of Euler-Lagrange equations.

With this new development let's again consider the shortest distance between two points. From equation (2) we have $f = \sqrt{x'^2 + y'^2}$, and the integrand is independent of either x or y . From the Euler-Lagrange equations we then know that

$$\frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{x'^2 + y'^2}} = C_1 \quad \text{and} \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{x'^2 + y'^2}} = C_2. \quad (10)$$

Dividing these two expressions by each other and recognizing that $y'/x' = dy/dx$ allows us to conclude

$$\frac{dy}{dx} = \frac{C_2}{C_1} = m, \quad (11)$$

which is the slope of the path. One more integration and we have the equation of a line $y = mx + b$. The further generalization to an arbitrary number of dependent coordinates is a straightforward extension of this approach and shouldn't need any further details.

15.1.2 Generalization to Lagrangian Mechanics

The independent variable in Lagrangian mechanics is the time t . The dependent variables are the coordinates that specify the position or *configuration* of the system and are usually denoted by q_1, q_2, \dots, q_n . The number n of coordinates depends on the nature of the system. For a single particle moving unconstrained in three dimensions, n is 3, and the coordinates q_1, q_2, q_3 could just be the three Cartesian coordinates. For a double pendulum as shown in figure 6.9, there would be two coordinates q_1 and q_2 which are the two angles θ_1 and θ_2 . Because the coordinates q_1, q_2, \dots, q_n take on so many different forms they are often referred to as *generalized coordinates*. It is often useful to think of the n generalized

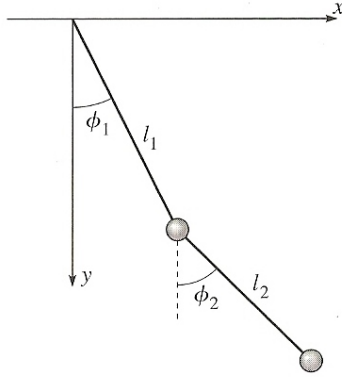


Figure 6.9 A double pendulum with generalized coordinates ϕ_1 and ϕ_2 .

coordinates as defining a point in an n -dimensional configuration space. Each of the points labels a unique position or configuration of the system.

The goal in most problems in Lagrangian mechanics is to find how the coordinates vary in time, that is to find the solutions for the n functions $q_1(t), q_2(t), \dots, q_n(t)$. One can think of these n functions as defining a path in the n -dimensional configuration space. This path is of course determined by Newton's second law, but we shall soon find that it can be characterized as the path for which a certain integral is stationary. This means it must satisfy the corresponding Euler-Lagrange equations (which we will soon just call the Lagrange equations). As it turns out it is much easier to write down these Lagrange equations than Newton's second law. In particular, Lagrange's equations, unlike Newton's second law, take exactly the same form in all coordinate systems. Additionally, Lagrange's equations only rely on writing down the kinetic and potential energies, which are scalar functions, in a set of generalized coordinates. As such they are much easier to determine than the vector quantities required for Newton's second law.

The integral S whose stationary value determines the evolution of the mechanical system is called the action integral. Its integrand is called the Lagrangian \mathcal{L} and it depends on the n coordinates $q_1(t), q_2(t), \dots, q_n(t)$, their n time derivatives $\dot{q}_1(t), \dot{q}_2(t), \dots, \dot{q}_n(t)$, and the time t ,

$$\mathcal{L} = \mathcal{L}(q_1, \dot{q}_1, \dots, q_n, \dot{q}_n, t). \quad (12)$$

Notice that since the independent coordinate is t , that the derivatives of the coordinates q_i are time derivatives and as usual with our notation time derivatives are denoted with dots.

The requirement that the action integral

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_1, \dot{q}_1, \dots, q_n, \dot{q}_n, t) dt \quad (13)$$

is stationary implies n Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_1}, \quad \frac{\partial \mathcal{L}}{\partial q_2} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_2}, \dots, \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial q_n} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_n}. \quad (14)$$

If these n equations are satisfied then the action integral is stationary; and if the action integral is stationary then these n equations are satisfied. Additionally, if the Lagrangian is independent of time then we immediately have the first integral of the Lagrange equations.

15.2 Chapter 7 Lagrange's Equations

Armed with the tools of variational calculus we can now approach the problems in mechanics from a Lagrangian point of view. Lagrange's equations have important advantages over Newton's laws of motion. First, they take the same form in any coordinate system. Second, we only need to find scalar functions (kinetic and potential energy) which is typically much easier than determining the vector quantities required in Newton's second law. Last and most important, they eliminate the forces of constraint. This greatly simplifies most problems, since the constraint forces are almost always unknown. As it turns out this simplification comes at almost no cost, since we usually do not want to know these forces anyway.

15.2.1 Lagrange's Equations for Unconstrained Motion

Consider a particle moving unconstrained in three dimensions which is subjected to a conservative force $\vec{F}(\vec{r})$. The kinetic energy of the particle is

$$T = \frac{1}{2} m \dot{\vec{r}}^2 = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right), \quad (15)$$

and its potential energy is

$$U = U(\vec{r}). \quad (16)$$

The Lagrangian is defined as

$$\mathcal{L} = T - U. \quad (17)$$

It is important that this quantity is not the same as the total mechanical energy, $E = T + U$. It is also important to realize that the Lagrangian depends on the particle's position (x, y, z) as well as its velocity $(\dot{x}, \dot{y}, \dot{z})$, that is $\mathcal{L} = \mathcal{L}(x, y, z, \dot{x}, \dot{y}, \dot{z})$.

To obtain a better feel for this function, consider the two derivatives,

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} = F_x, \quad (18)$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m\dot{x} = p_x. \quad (19)$$

If we now differentiate the second equation with respect to time we see from Newton's second law that

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\ddot{x} = F_x = \frac{\partial \mathcal{L}}{\partial x}. \quad (20)$$

In exactly the same way we find that

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{\partial \mathcal{L}}{\partial y} \quad \text{and} \quad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{z}} = \frac{\partial \mathcal{L}}{\partial z}. \quad (21)$$

So we see that Newton's second law is exactly equivalent to what we will now define to be *Lagrange's equations*, at least in Cartesian coordinates, obtained in equations (20) and (21).

From our work in variational calculus we see that these three Lagrange equations have exactly the same form as the Euler-Lagrange equations. This means that the integral

$$\boxed{S = \int \mathcal{L} dt} \quad (22)$$

is stationary. *The observation that this integral, called the action integral, is stationary for the particle's path is called **Hamilton's principle** after its inventor, the Irish mathematician Hamilton.* So far we have only shown this to be true for a single particle in Cartesian coordinates, but we will find it to be valid for a huge class of mechanical systems and for almost any choice of coordinates.

Hamilton's principle has found generalization in many fields of physics outside of classical mechanics, quantum field theory for example, and has unified many diverse fields of physics. However, for our purposes its main significance is that Lagrange's equations hold in more-or-less any coordinate system. To prove this we will consider some other coordinates. For example they could be spherical polar coordinates (r, θ, ϕ) , cylindrical coordinates (ρ, ϕ, z) , or any set of "generalized coordinates" q_1, q_2, q_3 , with the property that each position of the position vector \vec{r} specifies a unique value of (q_1, q_2, q_3) and vice versa;

$$q_i = q_i(\vec{r}) \quad \text{for } i = 1, 2, 3, \quad (23)$$

and

$$\vec{r} = \vec{r}(q_1, q_2, q_3). \quad (24)$$

This means that we can write (x, y, z) and $(\dot{x}, \dot{y}, \dot{z})$ in terms of (q_1, q_2, q_3) and $(\dot{q}_1, \dot{q}_2, \dot{q}_3)$. It also means that we can write the dependence of the Lagrangian as

$$\mathcal{L} = \mathcal{L}(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3), \quad (25)$$

and the action integral as

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3) dt. \quad (26)$$

Now the action integral is unchanged via the change in variables. Therefore, that statement that S is stationary for variations around the correct path means that in these coordinates the Lagrange equations take the form

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad \text{for } i = 1, 2, 3. \quad (27)$$

We will find that expressing the kinetic and potential energy and subsequently the Lagrangian in generalized coordinates to be extremely useful.

There is one point that we need to be aware of in our derivation of Lagrange's equations. A crucial step was recognizing that the Lagrange equations were equivalent to Newton's second law. But Newton's law is only true in an inertial frame. So even though Lagrange's equations are true for any choice of generalized coordinates, and these coordinates may be the coordinates of a noninertial frame, we must nevertheless be careful that when we write down the Lagrangian, $\mathcal{L} = T - U$, we do so in an inertial frame.

For the simple example of a particle moving in a conservative force field in Cartesian coordinates we saw that

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} = F_x \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m\dot{x}, \quad (28)$$

with corresponding expressions for the other coordinates. When we use generalized coordinates $\{q_i\}$, we shall find that $\partial \mathcal{L}/\partial q_i$, although not necessarily a force component, plays a role in Lagrange's equations very similar to a force. Similarly, $\partial \mathcal{L}/\partial \dot{q}_i$, although not necessarily a momentum component acts like a momentum. For this reason we shall call these derivatives the *generalized force* and *generalized momentum* respectively; that is

$$\frac{\partial \mathcal{L}}{\partial q_i} = (i^{\text{th}} \text{ component of a generalized force}), \quad (29)$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = (i^{\text{th}} \text{ component of a generalized momentum}). \quad (30)$$

With these definitions, each of the Lagrange equations,

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i},$$

takes the form

$$(\text{generalized force}) = \frac{d}{dt} (\text{generalized momentum}). \quad (31)$$

To clarify these issues a bit, let's consider the example of a particle moving in two dimensions using polar coordinates. The first thing we have to do is to

write down the Lagrangian, $\mathcal{L} = T - U$, in terms of the chosen coordinates. This means that $v_r = \dot{r}$ and $v_\phi = r\dot{\phi}$, and the kinetic energy is

$$T = \frac{1}{2}m \left(\dot{r}^2 + r^2\dot{\phi}^2 \right). \quad (32)$$

Hence the Lagrangian is

$$\mathcal{L} = T - U = \frac{1}{2}m \left(\dot{r}^2 + r^2\dot{\phi}^2 \right) - U(r, \phi). \quad (33)$$

Now that we have a Lagrangian we only have to write down the two Lagrange equations.

The Lagrange equation for the radial coordinate is

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}}$$

or

$$mr\dot{\phi}^2 - \frac{\partial U}{\partial r} = \frac{d}{dt} m\dot{r} = m\ddot{r}. \quad (34)$$

Since $-\partial U/\partial r = F_r$ this equation can be rewritten as

$$F_r = m \left(\ddot{r} - r\dot{\phi}^2 \right). \quad (35)$$

You should recognize this as $F_r = ma_r$, the r component of $\vec{F} = m\vec{a}$ that we derived in chapter 1. The $-r\dot{\phi}^2$ term is the infamous centripetal acceleration. What we see here is that when we use polar coordinates the Lagrange equation for the r coordinate is just the r component of Newton's second law. The nice thing was that we were able to avoid the tedious calculation of finding the radial accelerations.

The Lagrange equation for the ϕ coordinate is

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

or

$$-\frac{\partial U}{\partial \phi} = \frac{d}{dt} \left(mr^2\dot{\phi} \right). \quad (36)$$

To interpret this equation we need to relate the left hand side to the ϕ component of the force in polar components. Since the force is found from $\vec{F} = -\nabla U$, which in polar coordinates is given by

$$\nabla U = \frac{\partial U}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial U}{\partial \phi} \hat{\phi}, \quad (37)$$

so that the ϕ component of the force is

$$F_\phi = -\frac{1}{r} \frac{\partial U}{\partial \phi}. \quad (38)$$

Thus the left hand side of equation (36) is rF_ϕ which is simply the torque Γ on the particle about the origin. Meanwhile the quantity $mr^2\dot{\phi}$ is the momentum of inertia relative to the origin for a single particle times its angular velocity, i.e. its angular momentum L about the origin. Hence the ϕ equation tells us that

$$\Gamma = \frac{dL}{dt}, \quad (39)$$

or the torque equals the rate of change of the angular momentum.

These results illustrates a wonderful feature of Lagrange's equations, that when we choose an appropriate set of generalized coordinates the corresponding Lagrange equations automatically appear in a corresponding natural form. For the ϕ equation the generalized force was the torque and the generalized momentum was the angular momentum which turn out to be the natural quantities for an angular coordinate.

This example also illustrates another feature of Lagrange's equations. The ϕ component of the generalized force, $\partial\mathcal{L}/\partial\phi$, turned out to be the torque on the particle. If the torque happens to be zero, then the corresponding generalized momentum is conserved. This is a general result. The i th component of the generalized force is $\partial\mathcal{L}/\partial q_i$. If this quantity happens to be zero, then from the Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial q_i} = \frac{d}{dt} \frac{\partial\mathcal{L}}{\partial \dot{q}_i},$$

we know that the i th component of the generalized momentum, $\partial\mathcal{L}/\partial\dot{q}_i$ is constant, i.e. it is conserved. In practice, it is often easy to notice that a Lagrangian is independent of a coordinate q_i , and if you do, then you immediately know a corresponding conservation law.

Two Unconstrained Particles Here we will discuss the situation for two particles, mainly to show the form of Lagrange's equations for $N > 1$. For two interacting particles that are not in the presence of an external force field the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\dot{\vec{r}}_1^2 + \frac{1}{2}m\dot{\vec{r}}_2^2 - U(\vec{r}_1, \vec{r}_2). \quad (40)$$

As usual the forces on the particles are $\vec{F}_1 = -\nabla_1 U$ and $\vec{F}_2 = -\nabla_2 U$. Hence Newton's second law can be expressed as

$$F_{1x} = \dot{p}_{1x}, \quad F_{1y} = \dot{p}_{1y}, \quad \dots \quad F_{2z} = \dot{p}_{2z}. \quad (41)$$

Each of these six equations is equivalent to a corresponding Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial x_1} = \frac{d}{dt} \frac{\partial\mathcal{L}}{\partial \dot{x}_1}, \quad \frac{\partial\mathcal{L}}{\partial y_1} = \frac{d}{dt} \frac{\partial\mathcal{L}}{\partial \dot{y}_1}, \quad \dots \quad \frac{\partial\mathcal{L}}{\partial z_2} = \frac{d}{dt} \frac{\partial\mathcal{L}}{\partial \dot{z}_2}. \quad (42)$$

These six equations imply that the action integral $S = \int \mathcal{L} dt$ is stationary. We can transform the integrand to any other suitable set of six coordinates, q_1, q_2, \dots, q_6 and the action integral will still be stationary. This implies that Lagrange's equations must be true with respect to the new coordinates:

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_1}, \dots, \frac{\partial \mathcal{L}}{\partial q_6} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_6}. \quad (43)$$

An example of a set of six such generalized coordinates that we shall use repeatedly when we study the two body central force problem is this: The three coordinates of the CM position $\vec{R} = (m_1 \vec{r}_1 + m_2 \vec{r}_2) / (m_1 + m_2)$, and the three coordinates of the relative position of the two particles, $\vec{r} = \vec{r}_1 - \vec{r}_2$. We will find that this choice leads to a dramatic simplification. For now however, the main point is that Lagrange's equations are automatically true in their standard form, equation (43) with respect to the new generalized coordinates.

The extension of these ideas to the case of N unconstrained particles leads to the $3N$ Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}, \quad [i = 1, 2, \dots, 3N]. \quad (44)$$

These equations are valid for any choice of the $3N$ coordinates q_1, q_2, \dots, q_{3N} needed to describe the N particles.

15.2.2 Constrained System; an Example

Since one of the great advantages of the Lagrangian approach is the effortless way in which it handles constraints. There are numerous examples of this (almost too many to count), but to get the flavor of how easily this is handled we will again consider a simple example, the plane pendulum. A bob of mass m is attached to a massless rod of length ℓ which rotates without friction in the $x - y$ plane about a point which we will take to be the origin. The bob moves in both the x and y directions, but it is constrained by the rod so that $\sqrt{x^2 + y^2} = \ell$. However only one of the coordinates is independent for as x changes y is predetermined by the constraint equation, or vice versa. Thus the system has only one degree of freedom. One way to express this is to eliminate one of the coordinates, for example we could write $y = \sqrt{\ell^2 - x^2}$ so that we could express everything in terms of x . A much simpler way to proceed is to eliminate both x and y in terms of the angle ϕ , the angle between the pendulum and its equilibrium position.

Both the kinetic and potential energy can be expressed in terms of ϕ . The kinetic energy is $T = \frac{1}{2} m \ell^2 \dot{\phi}^2$. The potential energy is given by $U = mgh$ where h is the height above equilibrium. A little trigonometry shows that this height is $h = \ell(1 - \cos \phi)$. We can now write the Lagrangian as

$$\mathcal{L} = \frac{1}{2} m \ell^2 \dot{\phi}^2 - mg\ell(1 - \cos \phi). \quad (45)$$

Now it is a fact that once a system is expressed in terms of a single generalized coordinate (for a system with only one degree of freedom), the evolution of the system again satisfies Lagrange's equation. With ϕ as our generalized coordinate Lagrange's equation is

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}}. \quad (46)$$

These derivatives are easily evaluated to give

$$-mg\ell \sin \phi = m\ell^2 \ddot{\phi}. \quad (47)$$

The quantity $-mg\ell \sin \phi$ is just the torque Γ exerted by gravity on the pendulum, while $m\ell^2$ is the pendulum's momentum of inertia. This equation is then a specific example of equation (39) in cylindrical coordinates.