

20 Lecture 11-13

20.1 Chapter 8 Two Body Central Force Problem

We will now discuss the motion of two bodies which exert a conservative central force on each other. We will ignore all other external forces on this two body system. This is often a very good approximation for planets orbiting a star, a binary star system, the Moon orbiting the Earth, etc. In most cases the true situation is more complicated. For example, even if we are interested in a planet orbiting the Sun, we cannot completely ignore the effects of the other planets. In a similar fashion, we cannot ignore the effects of the Sun when we analyze the Earth-Moon system.

There are other systems such as the Hydrogen atom or a diatomic molecule in which oppositely charged particles interact under an attractive conservative central force. Such atomic scale systems must be treated with a quantum mechanical approach. None-the-less some of the ideas developed in this chapter (concept of a reduced mass and relative coordinate) play a critical role in the quantum mechanical two-body problem. Thus for many reasons it is a worthwhile goal to understand the two-body central force problem. However from a mechanics point of view we will only consider the scenario in which two bodies interact via the gravitational interaction.

20.1.1 Statement Of The Problem

Let us consider two particles with masses m_1 and m_2 which we will consider to be point particles. We will denote their positions to be \vec{r}_1 and \vec{r}_2 and the only forces are the forces \vec{F}_{12} and \vec{F}_{21} of their mutual interaction which we will assume to be central and conservative. In the case of two astronomical bodies the gravitational force is an attractive force of magnitude $Gm_1m_2/|\vec{r}_1 - \vec{r}_2|^2$. The corresponding potential energy is

$$U(r_{12}) = -\frac{Gm_1m_2}{r_{12}} \quad \text{where} \quad r_{12} = |\vec{r}_1 - \vec{r}_2|. \quad (1)$$

We should note that not only is this interaction translationally invariant but it is also conservative as it only depends on the magnitude of $\vec{r}_1 - \vec{r}_2$. To take advantage of this property we shall define the relative position of the two objects as

$$\vec{r} = \vec{r}_1 - \vec{r}_2, \quad (2)$$

which has a magnitude $|\vec{r}| = r$ (this is the same as r_{12} in equation (1)). In terms of this relative coordinate the potential energy takes the form $U(r)$.

We can now state the mathematical problem that we want to solve is the motion of two bodies, say the Earth (or any other planet) and the Sun, whose Lagrangian is

$$\mathcal{L} = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 - U(r). \quad (3)$$

Of course we could state this problem from the point of view of Newtonian mechanics (and when it is more convenient to do so we will use the language of Newtonian mechanics) but for the time being the Lagrangian formalism is more convenient.

20.1.2 CM and Relative Coordinates, and the Reduced Mass

From our discussions about Lagrangian mechanics, it should be clear that the first thing we need to do is determine the best generalized coordinates to use to solve our problem. We have already found that the potential energy is most easily expressed in terms of the relative coordinate \vec{r} . The question then is what to choose for the other variable. The best choice as it turns out is the center of mass position \vec{R} . For a two particle system this coordinate is defined as

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{M}, \quad (4)$$

where clearly M is the total mass given by $M = m_1 + m_2$. As we saw in chapter 3, the CM lies on the line joining the two particles with the distances between the CM and the two particles m_1 and m_2 being in the ratio, m_1/m_2 . In particular if $m_2 \gg m_1$, then the CM lies very close to body 2. In figure 8.1,

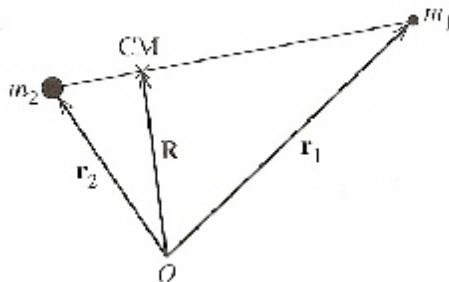


Figure 8.1. The center of mass lies at position $\vec{R} = (m_1 \vec{r}_1 + m_2 \vec{r}_2) / M$ on the line joining the two bodies.

the ratio of m_1/m_2 is approximately 1/3 so the CM is a quarter of the way from m_2 to m_1 .

We also found out in chapter 3 that the total momentum of the two bodies is the same as if the total mass M were concentrated at the CM such that

$$\vec{P} = M \dot{\vec{R}}. \quad (5)$$

We also know for a system that is translationally invariant ($U = U(|\vec{r}_1 - \vec{r}_2|)$ guarantees this) that the total momentum of the system is conserved. This means that if we choose an inertial reference frame in which the CM is at rest,

then $\dot{\vec{R}} = 0$ and will remain so for all time. That means that the *CM frame* is an extremely convenient frame in which to analyze the motion.

It is a straightforward exercise to combine equations (2) and (4) algebraically and solve for \vec{r}_1 and \vec{r}_2 . The result is

$$\vec{r}_1 = \vec{R} + \frac{m_2}{M} \vec{r} \quad \text{and} \quad \vec{r}_2 = \vec{R} - \frac{m_1}{M} \vec{r}. \quad (6)$$

In terms of \vec{R} and \vec{r} the kinetic energy can be expressed as

$$\begin{aligned} T &= \frac{1}{2} m_1 \dot{\vec{r}}_1^2 + \frac{1}{2} m_2 \dot{\vec{r}}_2^2 \\ T &= \frac{1}{2} m_1 \left(\dot{\vec{R}} + \frac{m_2}{M} \dot{\vec{r}} \right)^2 + \frac{1}{2} m_2 \left(\dot{\vec{R}} - \frac{m_1}{M} \dot{\vec{r}} \right)^2 \\ T &= \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \left(\frac{m_1 m_2^2}{M^2} + \frac{m_2 m_1^2}{M^2} \right) \dot{\vec{r}}^2 \\ T &= \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \frac{m_1 m_2}{M} \dot{\vec{r}}^2. \end{aligned} \quad (7)$$

With this result it is natural to define the *reduced mass* as

$$\mu = \frac{m_1 m_2}{M} = \frac{m_1 m_2}{m_1 + m_2}. \quad (8)$$

From this expression you can easily check that μ is always less than either m_1 or m_2 , hence the name. In fact if $m_2 \gg m_1$ then $\mu \simeq m_1$. This the case when considering the planets orbiting the Sun. The reduced mass is very nearly equal to the mass of the planet. On the other hand if $m_2 = m_1 = m$ then the reduced mass is exactly $m/2$.

Now we can rewrite the kinetic energy in terms of μ as

$$T = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2. \quad (9)$$

This rather remarkable result shows that the kinetic energy is the same as that of two “fictitious particles”, one with mass M moving at the speed of the CM and the other of mass μ moving with the speed of the relative coordinate \vec{r} . This result also allows us to write the Lagrangian as

$$\mathcal{L} = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2 - U(r), \quad (10a)$$

$$\mathcal{L} = \mathcal{L}_{CM} + \mathcal{L}_{rel}. \quad (10b)$$

We now see that by using the CM and relative positions as the generalized coordinates, we can separate the Lagrangian into two separate parts each of which only involves one of the generalized coordinates. This means that we can

solve for the motions of \vec{R} and \vec{r} as two separate problems. Also we have an additional simplification, in the language of Lagrangian mechanics we would say that \vec{R} is an ignorable coordinate. Since $\dot{\vec{R}}$ is a constant, we can choose the CM as the origin of our preferred inertial reference frame in which $\dot{\vec{R}} = 0$, in this way the ignorable coordinate is truly “ignored”. Now the Lagrangian reduces to only \mathcal{L}_{rel} , which further simplifies the problem to that of a one body problem. Thus, with a judicious choice of generalized coordinates, what started out as a two-body problem has been reduced to a one-body problem.

20.1.3 Equations of Motion

The Lagrange equation for the relative coordinate \vec{r} is just Newton’s second law for a reduced mass

$$\mu \ddot{\vec{r}} = -\nabla U(r), \quad (11)$$

which can easily be verified with the use of Cartesian coordinates. With our definition of the reduced mass the interaction itself takes on a slightly different form namely

$$U(r) = -\frac{Gm_1m_2}{r} = -\frac{G\mu M}{r}. \quad (12)$$

So from the point of view of the reduced mass, the gravitational interaction is an attraction to a mass equal to the total mass of both particles, M , and the acceleration of the relative coordinate is independent of μ !

At this point it is worthwhile to take a moment to consider what the motion looks like in the CM frame (our frame of choice). In figure 8.2,

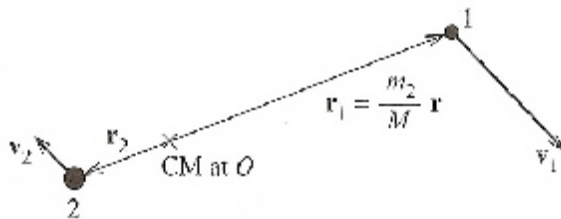


Figure 8.2 In the CM frame $\dot{\vec{R}} = 0$. The relative position \vec{r} is the position of particle 1 relative to particle 2.

we note that the CM is a stationary point which we take to be the origin. Both particles are moving albeit with equal and opposite momenta. If m_2 is much greater than m_1 , as is often the case, then the CM is close to m_2 which has a relatively small velocity. It is important to note that \vec{r} is the position of particle 1 relative to particle 2 and *NOT* the actual position of either particle. As shown in figure 8.2, the position of particle 1 is $\vec{r}_1 = (m_2/M) \vec{r}$. However in the case where $m_2 \gg m_1$ then the CM is very close to particle 2 and $\vec{r}_1 \simeq \vec{r}$

The equation of motion in the CM frame is precisely the same as the equation for a single particle of mass μ in a fixed central force field of potential energy $U(r)$. We will continue to make use of the mass μ to remind you that the equations apply to the relative motion of two bodies. However, you may find it easy to visualize a single body (of mass μ) orbiting a fixed force center. In particular when $m_2 \gg m_1$, these two problems are for all practical purposes the same. In any event, any solution for the relative coordinate \vec{r} always gives the motion of particle 1 relative to particle 2.

Conservation of Angular Momentum With a central force, we already know that the total angular momentum is conserved. However, like so many things, this condition takes an especially simple form in the CM frame. In any frame, the total angular momentum is

$$\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 \quad (13)$$

$$\vec{L} = m_1 \vec{r}_1 \times \dot{\vec{r}}_1 + m_2 \vec{r}_2 \times \dot{\vec{r}}_2. \quad (14)$$

In the CM frame we know from equation (6) that

$$\vec{r}_1 = \frac{m_2}{M} \vec{r} \quad \text{and} \quad \vec{r}_2 = -\frac{m_1}{M} \vec{r}. \quad (15)$$

Substituting this result into the expression for the angular momentum we find

$$\begin{aligned} \vec{L} &= \mu \vec{r} \times \dot{\vec{r}}_1 - \mu \vec{r}_2 \times \dot{\vec{r}}_2 \\ \vec{L} &= \vec{r} \times \mu \dot{\vec{r}}. \end{aligned} \quad (16)$$

So the total angular momentum in the CM frame is exactly the same as the angular momentum of a single particle with mass μ and position \vec{r} .

For our present purposes, the important point is that, because angular momentum is conserved, we see that $\vec{r} \times \dot{\vec{r}}$ is conserved. This implies that both \vec{r} and $\dot{\vec{r}}$ remain in a fixed plane. That is, in the CM frame, the whole motion remains in a fixed plane which we will take to be that normal to the z axis, the $x - y$ plane. So we have yet another reduction in the complexity of the two-body problem with central conservative forces, it is reduced to a two-dimensional problem.

The Two Equations of Motion To set up the equations of motion for the remaining two-dimensional problem, we need to choose coordinates in the plane of motion. The obvious choice is the polar coordinates r and ϕ (equivalent to the usual spherical polar coordinates with $\theta = \pi/2$). In these coordinates the Lagrangian is

$$\mathcal{L} = \frac{1}{2} \mu \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right) - U(r). \quad (17)$$

Since this Lagrangian is independent of ϕ , the ϕ coordinate is ignorable, and the Lagrange equation for ϕ is just

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu r^2 \dot{\phi} = L, \quad (18)$$

where L is conserved. Since $\mu r^2 \dot{\phi}$ is the angular momentum L normal to the plane of motion, the ϕ equation and is a statement of the conservation of angular momentum for this problem.

The Lagrange equation corresponding to r (usually called the radial equation) is

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}},$$

or

$$\mu r \dot{\phi}^2 - \frac{dU}{dr} = \mu \ddot{r}. \quad (19)$$

This equation has the form of Newton's second law in *one* dimension with an additional term $\mu r \dot{\phi}^2$. This term is the “fictitious” outward centrifugal force, an effective force that occurs due to the transformation to polar coordinates. In other words the particle's radial motion is exactly the same as if the particle were moving in one dimension subject to the actual force $-dU/dr$ plus the centrifugal force.

20.1.4 The Equivalent One-Dimensional Problem

The two equations that we have to solve are the ϕ equation (18) and the radial equation (19). The solution to the ϕ equation depends on the initial conditions which determines the angular momentum, L . For the radial equation it is helpful to write the centrifugal force term as a function of the conserved angular momentum, L , rather than $\dot{\phi}$ which is not a constant of the motion. With this substitution the radial equation becomes

$$\frac{L^2}{\mu r^3} - \frac{dU}{dr} = \mu \ddot{r} \quad (20)$$

We now define a potential energy, U_{cf} , associated with the “centrifugal force” so that $-dU_{cf}/dr = L^2/\mu r^3$. This potential energy term is

$$U_{cf}(r) = \frac{L^2}{2\mu r^2}. \quad (21)$$

In terms of U_{cf} the radial equation can be rewritten as

$$\mu \ddot{r} = -\frac{d}{dr} (U(r) + U_{cf}(r)) = -\frac{d}{dr} U_{eff}(r), \quad (22)$$

where the effective potential energy $U_{eff}(r)$ is the sum of the actual potential energy and the term that arises from the centrifugal force $U_{cf}(r)$:

$$U_{eff}(r) = U(r) + U_{cf}(r) = U(r) + \frac{L^2}{2\mu r^2}. \quad (23)$$

So according to equation (22), the radial motion of the particle is exactly the same as if the particle were moving in one dimension with an effective potential energy $U_{eff}(r) = U(r) + U_{cf}(r)$.

It is useful at this point to stop and review what we have accomplished. Initially we started with a two-body problem in three dimensions. Even though the interaction in this problem was a conservative central force, it still contained 6 independent coordinates. The first thing we did was to find generalized coordinates that would reduce the complexity of this problem. The first change was prompted by the dependence of the potential energy on the relative positions of the particles, $\vec{r}_1 - \vec{r}_2$. So defining a relative coordinate, $\vec{r} = \vec{r}_1 - \vec{r}_2$, along with the coordinate of the CM, \vec{R} , we found that the Lagrangian separated into two independent terms, one describing the motion of the CM and the other the motion of the relative positions of the particles. Since the potential energy only depended on the magnitude of the relative positions, $U = U(r)$, the CM coordinate was an ignorable coordinate. Defining the origin of our inertial reference frame to be the CM of the two particles allowed us to truly *ignore* this coordinate and we were left with the three coordinates of the relative position vector. For a central force, we have already found that the angular momentum is conserved, and that meant that the motion of the two particles resided in a single plane which we defined to be the $x - y$ plane. So at this point we have reduced the problem from that of 6 independent coordinates to just a two dimensional problem. Finally, again due to the conservation of angular momentum, we were able to define a potential energy, $U_{cf}(r)$, that represented the effects of the “centrifugal force” that included the conserved angular momentum. One should remain aware however that this term is actually the angular kinetic energy that, due to the conservation of angular momentum, is a function of only the radial coordinate. Adding this potential to the potential energy of the interaction force reduced the problem to just a one dimension problem, the radial dimension.

There is an additional point to be made here which results in an additional simplification of the equation of motion (22). If we note that the angular momentum due to the reduced mass, $L = \mu r^2 \dot{\phi}$, is directly proportional to the reduced mass μ , then we can define the angular momentum per unit relative mass as $\ell = L/\mu$. The equation of motion then becomes

$$\begin{aligned} \mu \ddot{r} &= -\frac{d}{dr} \left(-\frac{G\mu M}{r} + \frac{\mu^2 \ell^2}{2\mu r^2} \right) = -\mu \frac{d}{dr} \left(-\frac{GM}{r} + \frac{\ell^2}{2r^2} \right) \\ \ddot{r} &= -\frac{d}{dr} \left(-\frac{GM}{r} + \frac{\ell^2}{2r^2} \right) = -\frac{GM}{r^2} + \frac{\ell^2}{r^3}. \end{aligned} \quad (24)$$

Hence we see that any resulting orbit is *independent* of the reduced mass and only depends on the gravitational field due to the total mass of the system and the angular momentum per unit reduced mass, $\ell = r^2\dot{\phi}$. Even in the case where a small satellite with a mass on the order of $100kg$ is orbiting the Earth, which has a mass of $6 \times 10^{24}kg$, both the satellite and the Earth are orbiting their common center of mass, and their orbit is described by the relative coordinate of the satellite and the Earth. In this case the center of mass lies extremely close to the center of the Earth, the total mass of the system is essentially the mass of the Earth, and the relative mass is basically the mass of the satellite. But conceptually this is no different than two nearly equal mass neutron stars orbiting each other about their common center of mass with their orbit being described by their relative coordinate which is *independent* of the reduced mass. With this in mind, equation (24) is the equation of motion that we will analyze in our discussion of the gravitational interaction between two objects. Additionally when we extend our discussion to determine the orbit equation, $r(\phi)$, we will begin with this reduced equation of motion that is independent of μ . However, when we discuss potential, kinetic, and total mechanical energy of the system we will continue to include the reduced mass.

Effective Potential for a Comet (or Planet) It is now time to examine the effective potential energy for a comet or planet moving in the gravitational field of the Sun in some detail. Since planetary motion was first described mathematically by the German astronomer Johannes Kepler, this problem of the motion of a planet or comet around the Sun is often called the Kepler problem. As we have seen the actual gravitational potential of a particle in the Sun's gravitational field is simply

$$U(r) = -\frac{Gm_1m_2}{r} = -\frac{G\mu M}{r}. \quad (25)$$

The potential energy associated with the centrifugal force is given in equation (21) so that the total effective potential is

$$U_{eff}(r) = -\frac{G\mu M}{r} + \frac{L^2}{2\mu r^2}. \quad (26)$$

The general behavior of this potential is shown in figure 8.3

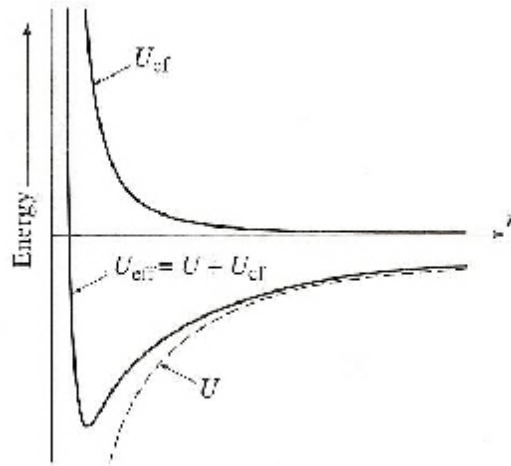


Figure 8.3. The effective potential that governs the radial motion is the sum of $U = -GM/r$ and $U_{cf} = \ell^2/2r..$

where we have plotted three potential energies $U_{eff}(r)$, $U_{cf}(r)$, and $U(r)$. For large r , the centrifugal term, $L^2/2\mu r^2$, becomes negligible compared to the gravitational term, $-G\mu M/r$, and the effective potential energy, $U_{eff}(r)$, is negative and approaching the $U = 0$ axis asymptotically with a positive slope. From the equation of motion for the radial coordinate, in this region the acceleration of a particle is always inward.

When r is small, the centrifugal term $L^2/2\mu r^2$ dominates (unless $L = 0$) and $U_{eff}(r)$ is positive with a negative slope. Hence the radial acceleration is toward larger r . Thus as an unbounded comet approaches the Sun it eventually reaches a turning point and then begins to move away from the Sun. The exception to this is when $L = 0$ which implies that $\dot{\phi} = 0$. That means $\phi = \phi_o$, so that the comet is moving exactly radially and the only potential energy it experiences is due to the attraction of the Sun. The comet is on a collision course with the Sun and will eventually be destroyed.