23 Lecture 11-20

23.1 Chapter 8 Two-Body Central Force Problem (con)

23.1.1 Changes of Orbit

Before we leave our discussion of orbits we shall discuss how to change from one orbit to another. Consider for example an Earth satellite wishing to change from one orbit about the Earth to another, perhaps from a circular orbit to an elliptical orbit that will carry it to a higher altitude. The analysis of Earth orbits is the same as for those around the Sun. Only now the total mass M is the sum of the Earth's mass plus that of the satellite which is so close to that of the Earth that will assume it to be the Earth's mass. For Earth orbits the closest points and furthest points are called the perigee and apogee instead of perihelion and aphelion for the Sun. We shall confine ourselves to bounded, elliptical orbits, for which the most general form is

$$r(\phi) = \frac{r_o}{1+\epsilon\cos(\phi-\delta)} = \frac{\ell^2/GM}{1+\epsilon\cos(\phi-\delta)}. \tag{1}$$

Note that for a single orbit we can always define our x axis so that $\delta = 0$. If we are interested in two orbits then there will at least be a phase difference that we cannot avoid.

Suppose that our spacecraft is initially in an orbit described by equation (1) with orbital parameters ℓ_1, ε_1, and phase δ_1 (if we know ε and ℓ then via equation (**??**) we know ϵ). A common way to change orbits is to fire your rockets for a short time. To a good approximation we can treat this as an impulse that occurs at a unique angle ϕ_o and causes an instantaneous change in velocity by a known amount. From the new velocity we can calculate the new energy ε_2 and angular momentum ℓ_2. Then from equation (**??**) we can calculate the new eccentricity ϵ_2. Finally, the new orbit must join onto the old one at the angle ϕ_o, i.e. $r_1(\phi_o) = r_2(\phi_o)$. We can find the new phase from the expression

$$\frac{\ell_1^2}{1+\epsilon_1\cos(\phi_o-\delta_1)} = \frac{\ell_2^2}{1+\epsilon_2\cos(\phi_o-\delta_2)}. \tag{2}$$

This calculation, though straightforward in principle, is tedious and not especially illuminating. To simplify the calculations and to better reveal important features, I shall treat just one important special case.

A Tangential Thrust at Perigee Let's consider a satellite that transfers from one orbit to another by firing its rockets in the tangential direction, forward or backward, when it is at the perigee of its initial orbit. By choice of our x axis, we can arrange it so that this occurs in the direction $\phi = 0$, so that $\phi_o = 0$ and $\delta_1 = 0$. Moreover, since the rockets are fired in the tangential direction, the velocity just after firing is still in the same direction, which is perpendicular to the radius from the Earth to the satellite. Therefore, the position at which the rockets are fired is also the perigee for the final orbit and $\delta_2 = 0$ as well. Under these conditions the continuity of the orbits, equation (2), reduces to

$$R_1(\text{per}) = \frac{\ell_1^2/GM}{1+\epsilon_1} = R_2(\text{per}) = \frac{\ell_2^2/GM}{1+\epsilon_2}. \tag{3}$$

We will denote by λ the ratio of the satellite's speeds just before and after the firing of the rockets, $v_2 = \lambda v_1$. We shall call λ the *thrust factor*: if $\lambda > 1$ then the thrust was forward and if $\lambda < 1$ then the thrust was backward. Before we proceed, it is useful to note that if the transfer were to take place at the apogee then the continuity of obits would insist that

$$R_1(\text{apo}) = \frac{\ell_1^2/GM}{1-\epsilon_1} = R_2(\text{apo}) = \frac{\ell_2^2/GM}{1-\epsilon_2}, \tag{4}$$

as the apogee occurs at $\phi = \pi$ in the orbit.

At perigee (or apogee) the angular momentum is just $\ell = rv$. The value of r does not change during the impulse, and I shall assume that the firing of the rockets changes the satellites mass by a negligible amount. Under these assumptions, then angular momentum changes by the same factor as the speed, $\ell_2 = \lambda\ell_1$.Thus the new eccentricity is

$$\epsilon_2 = \lambda^2\epsilon_1 + \lambda^2 - 1. \tag{5}$$

This expression for the new eccentricity contains almost all the interesting information about the new orbit. For example if $\lambda > 1$ then the new orbit has $\epsilon_2 > \epsilon_1$. The new orbit has the same perigee as the old one, but a greater eccentricity and so lies outside the old orbit as shown in figure 8.12(a).

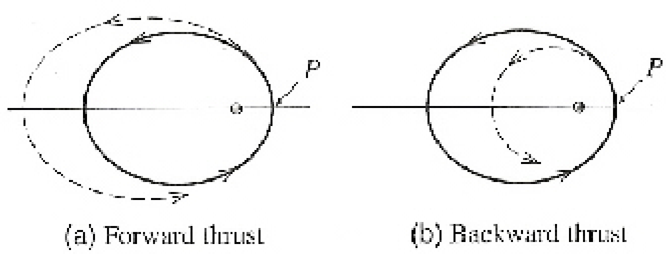

Figure 8.12. Changing orbits. The rockets are fired at at the perigee P. (a) A forward thrust moves the satellite to the larger dashed elliptical orbit. (b) A backward thrust moves the satellite to the smaller dashed elliptical orbit.

If we make λ large enough then the new eccentricity can be greater than one, which means that the new orbit is a hyperbola and the satellite escapes the Earth.

If we choose the thrust factor $\lambda < 1$, then the new eccentricity is less than the old and the new orbit is inside the old as shown in figure 8.12(b). As we make λ steadily smaller we can move the satellite into a circular orbit, i.e. $\epsilon_2 = 0$. If continue to make λ still smaller then $\epsilon_2 < 0$. This means that the old perigee is now the apogee as $\phi_o \to \pi + \phi_o$.

From this discussion if a satellite's crew wishes to change from one circular orbit to another, then it will require two successive boosts. The first boost will increase the eccentricity from zero to a positive value. The second boost must come at the apogee. There the boost will increase the negative eccentricity back to zero.

As an example of how this would work consider an initial circular orbit with radius $r = R_1$ and eccentricity $\epsilon_1 = 0$ as shown in figure 8.13.

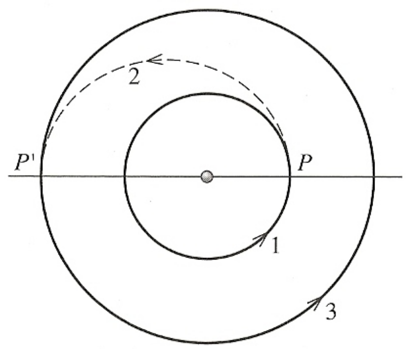

Figure 8.13. Two successive boosts, at P and P', transfers a satellite from orbit 1 to the transfer orbit 2 followed by a transfer to orbit 3.

The final orbit is to have a radius of R_3. After an initial boost at point P in the figure, the satellite is in a transfer orbit with $\ell_2 = \lambda\ell_1$. From equation (3) we know that

$$R_1 = \ell_1^2/GM = R_2 = \frac{\ell_2^2/GM}{1+\epsilon_2} = \frac{\lambda^2\ell_1^2/GM}{1+\epsilon_2} = \frac{\lambda^2}{1+\epsilon_2}R_1, \tag{6}$$

or $\epsilon_2 = \lambda^2 - 1$. By the time the satellite reaches the point P' the satellite it has achieved a radius of R_3. Since the point P' is the apogee of the transfer orbit equation and the third orbit is a circular orbit, equation (3) now takes the form

$$\frac{\ell_2^2/GM}{1-\epsilon_2} = \frac{\lambda^2\ell_1^2/GM}{2-\lambda^2} = \frac{\lambda^2}{2-\lambda^2}R_1 = R_3. \tag{7}$$

This is easily solved for λ to give

$$\lambda^2 = \frac{2R_3}{R_1+R_3}. \tag{8}$$

For example if $R_3 = 2R_1$, then the boost must be $\lambda = \sqrt{4/3} \simeq 1.15$, and the satellite must increase its speed by 15% so that its apogee is twice its perigee.

The second transfer is into another circular orbit ($\epsilon_3 = 0$) and requires an additional boost of λ' or

$$\begin{aligned}\frac{\ell_2^2/GM}{1-\epsilon_2} &= R_3 = \ell_3^2/GM = \lambda'^2\ell_2^2/GM, \\ \frac{1}{2-\lambda^2} &= \lambda'^2. \end{aligned} \tag{9}$$

Solving for λ' we easily find

$$\lambda'^2 = \frac{1}{2-\lambda^2} = \frac{R_1+R_3}{2R_1}. \tag{10}$$

Using the same example, $R_3 = 2R_1$, then the second boost must be $\lambda' = \sqrt{3/2} = 1.22$, and the satellite must increase its speed by 22% to enter a circular orbit from its apogee.

It would be tempting to think that the overall change in speed is just the product $\lambda\lambda'$, but that would neglect that the speed of the satellite changes as it moves around in its transfer orbit. Since it is conserving angular momentum in its orbit the ratio of speeds in that orbit must satisfy $v_2\,(\text{apo})\,R_3 = v_2\,(\text{per})\,R_1$. This implies that the overall gain in speed is

$$v_3 = \lambda'\frac{v_2\,(\text{apo})}{v_2\,(\text{per})}\lambda v_1 = \sqrt{\frac{R_1+R_3}{2R_1}}\frac{R_1}{R_3}\sqrt{\frac{2R_3}{R_1+R_3}} = \sqrt{\frac{R_1}{R_3}}v_1. \tag{11}$$

Again using our example of $R_3 = 2R_1$ we find that the velocity has been reduced by a factor of $\sqrt{2}$. This should have been anticipated as it is easy to show that the for circular orbits $v \propto 1/\sqrt{R}$.

23.2 Chapter 11 Coupled Oscillators and Normal Modes

The interest here is a system of masses that can oscillate and are connected to each other in some way, a system of coupled oscillators. A single oscillator has a single natural frequency at which, in the absence of driving forces, it will oscillate forever. In this chapter we shall find that two or more coupled oscillators have several natural , "normal", frequencies at which it can oscillate. The general motion is a combination of these frequencies. Here we shall have to learn how to diagonalize matrices in order to determine these normal frequencies as well as the normal modes that accompany these oscillations. Throughout we will assume that all of the forces with which we are concerned obey Hooke's law and hence all of the equations of motion are linear. While this makes up a special case (linear) it is a very important special case and occurs throughout physics.

23.2.1 Two Masses and Three Springs

Consider two masses connected via a spring that are also attached individually with springs along a line to adjacent fixed walls as shown in figure 11.1.

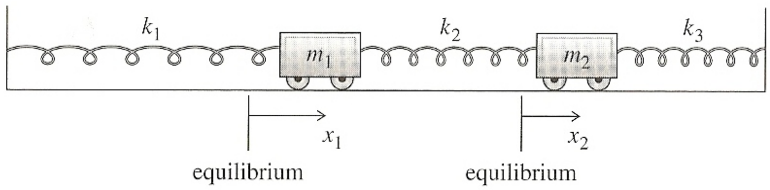

Figure 11.1 Masses m_1 and m_2 are attached to walls by springs k_1 and k_3 and to each other by spring k_2. The positions of the masses are measured from their respective equilibrium positions.

In the absence of spring 2, the middle spring, the masses would oscillate independently of each other. So it is spring 2 that couples the two oscillating masses. In fact it is this spring that makes it impossible for one of the masses to move without the other moving as well.

The Lagrangian for this system of springs and masses is

$$\mathcal{L} = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}\left(k_1x_1^2 + k_2\left(x_2 - x_1\right)^2 + k_3x_2^2\right), \tag{12}$$

where x_1 and x_2 are the displacements of the respective masses from equilibrium. With this Lagrangian the equations of motion are

$$m_1\ddot{x}_1 = -k_1x_1 - k_2\left(x_1 - x_2\right) = -\left(k_1 + k_2\right)x_1 + k_2x_2 \tag{13a}$$

$$m_2\ddot{x}_2 = -k_3x_2 - k_2\left(x_2 - x1\right) = k_2x_1 - \left(k_2 + k_3\right)x_2. \tag{13b}$$

Before we try to solve these two coupled equations, notice that they can be written in matrix form as

$$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}. \tag{14}$$

To obtain this rather elegant compact notation we have defined the column matrix or column vector as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x2 \end{bmatrix} \tag{15}$$

which labels the configuration of our system. It is a 2×1 column matrix with two independent components because this system has two degrees of freedom. If we had a system with n degrees of freedom then it would be an $n \times 1$ column matrix. We have also defined two square matrices,

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}. \tag{16}$$

The "mass matrix" $\mathbf{M}$ is (at least in this simple case) a diagonal matrix with the masses m_1 and m_2 for diagonal elements. The "spring-constant matrix" $\mathbf{K}$ has nonzero off-diagonal elements which reflects the coupling in this system between x_1 and x_2. Notice that the matrix equation (14) is a very natural generalization of a single mass on a spring. With just one degree of freedom all three of the matrices $\mathbf{x}$, $\mathbf{M}$, and $\mathbf{K}$ would be 1×1 matrices, i.e. ordinary numbers. The matrix equation then reduces to the usual $m\ddot{x} = -kx$. Notice also that both $\mathbf{M}$ and $\mathbf{K}$ are symmetric matrices which will be true of all corresponding matrices throughout this chapter.

In an attempt to solve the matrix equation of motion we might make the guess that both masses oscillate with the same frequency ω, that is

$$x_1 = \alpha_1 \cos(\omega t - \delta_1) \quad \text{and} \quad x_2 = \alpha_2 \cos(\omega t - \delta_2). \tag{17}$$

If there is a solution of this form then there certainly also be a solution of the form

$$y_1 = \alpha_1 \sin(\omega t - \delta_1) \quad \text{and} \quad y_2 = \alpha_2 \sin(\omega t - \delta_2). \tag{18}$$

As we did in chapter 5 we can combine these two solutions (the system is linear after all) and try solutions of the form

$$z_1 = \alpha_1 e^{i(\omega t - \delta_1)} \quad \text{and} \quad z_2 = \alpha_2 e^{i(\omega t - \delta_2)}. \tag{19}$$

Then defining the complex constants $a_1 = \alpha_1 e^{-i\delta_1}$ and $a_2 = \alpha_2 e^{-i\delta_2}$ we finally have as trial solutions

$$z_1 = a_1 e^{i\omega t} \quad \text{and} \quad z_2 = a_2 e^{i\omega t}. \tag{20}$$

We are not claiming that these complex numbers represent the actual motion of the two masses, but (as we shall see) for the right choices of a_1, a_2, and ω the two complex numbers in equation (20) are solutions to the equation of motion. It is then their real parts that describe the actual motion of the system. The great advantage of the complex numbers is that they both have the same time dependence given by the factor $e^{i\omega t}$. If they have any phase differences those will be contained in the constants a_1 and a_2. The complex solutions can be combined into a 2×1 matrix of the form

$$\mathbf{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega t} = \mathbf{a} e^{i\omega t}. \tag{21}$$

When we substitute these trial solutions into the matrix EOM, $\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$, we obtain

$$-\omega^2 \mathbf{M}\mathbf{a} e^{i\omega t} = -\mathbf{K}\mathbf{a} e^{i\omega t}. \tag{22}$$

Cancelling the common exponential factor and rearranging we find

$$\left(\mathbf{K} - \omega^2 \mathbf{M}\right) \mathbf{a} = 0. \tag{23}$$

This is in the form of an eigenvalue equation with eigenvalue ω^2 where we have the matrix $\mathbf{M}$ instead of the unit matrix $\mathbf{I}$. It can be solved in almost exactly the same way as the case where $\mathbf{M}$ is replaced by $\mathbf{I}$. If the matrix $\mathbf{K} - \omega^2\mathbf{M}$ has a nonzero determinant, then the only solution of (23) is the trivial one, $\mathbf{a} = 0$, which corresponds to no motion at all. If on the other hand,

$$\det\left(\mathbf{K} - \omega^2 \mathbf{M}\right) = 0, \tag{24}$$

then there is certainly a nontrivial solution of (23) and hence a solution of our equations of motion with the assumed sinusoidal motion. In the presence case, the matrices $\mathbf{K}$ and $\mathbf{M}$ are 2×2 matrices so equation (24) is a quadratic equation for ω^2. This implies that in general there are two frequencies at which the masses can oscillate in pure sinusoidal motion.

The two frequencies at which our system can oscillate, the normal frequencies, are determined by the quadratic equation (24) for ω^2. While the general case is perfectly straightforward, it is not especially illuminating and somewhat messy. So we will consider two special cases where we can understand more easily what is going on. First we shall start with the case where all three springs are identical and likewise the two masses.

23.2.2 Identical Springs and Equal Masses

We will now examine the case where both masses are equal, $m_1 = m_2 = m$, and similarly the three spring constants, $k_1 = k_2 = k_3 = k$. The matrices $\mathbf{M}$ and $\mathbf{K}$ simplify to

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}. \tag{25}$$

The matrix $\mathbf{K} - \omega^2\mathbf{M}$ becomes

$$\mathbf{K} - \omega^2\mathbf{M} = \begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{bmatrix}, \tag{26}$$

and its determinant is

$$\det\left(\mathbf{K} - \omega^2\mathbf{M}\right) = \left(2k - m\omega^2\right)^2 - k^2 = \left(k - m\omega^2\right)\left(3k - m\omega^2\right). \tag{27}$$

Since this determinant must vanish, the two normal frequencies are

$$\omega = \sqrt{\frac{k}{m}} = \omega_1 \quad \text{and} \quad \omega = \sqrt{\frac{3k}{m}} = \omega_2. \tag{28}$$

These two normal frequencies are the ones at which the two masses can oscillate in purely sinusoidal motion. Notice that the first frequency, ω_1, is precisely the frequency of a single mass m on a spring with constant k. We shall see the reason for this apparent coincidence in a moment.

Equation (28) gives us the two possible frequencies of the system, but still have not described the corresponding motions. The actual motion is given by the column matrix $\mathbf{x}(t) = \operatorname{Re}\mathbf{z}(t)$ where $\mathbf{z}(t) = \mathbf{a}e^{i\omega t}$ and is composed of the two constant numbers,

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \tag{29}$$

which must satisfy the eigenvalue equation

$$\left(\mathbf{K} - \omega^2\mathbf{M}\right)\mathbf{a} = 0. \tag{30}$$

Now that we know the possible normal frequencies, we must solve this equation for the vector **a** for each normal frequency in turn. The sinusoidal motion with any of one of the normal frequencies is called a *normal mode*. We shall start with the first normal mode.

The First Normal Mode If we choose $\omega = \omega_1$ the first normal frequency given by $\omega_1^2 = k/m$, then the matrix $\mathbf{K} - \omega^2 \mathbf{M}$ becomes

$$\mathbf{K} - \omega_1^2 \mathbf{M} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}, \tag{31}$$

which you should notice has a vanishing determinant. Hence for this case the eigenvalue equation becomes

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0. \tag{32}$$

There is only one independent equation here,

$$a_1 - a_2 = 0, \tag{33}$$

hence $a_1 = a_2 = Ae^{-i\delta}$. The complex column matrix for **z** is therefore

$$\mathbf{z}(t) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega_1 t} = \begin{bmatrix} A \\ A \end{bmatrix} e^{i(\omega_1 t - \delta)}. \tag{34}$$

The corresponding actual motion is the real column matrix $\mathbf{x}(t) = \mathrm{Re}\,\mathbf{z}(t)$ or

$$\mathbf{x}(t) = \begin{bmatrix} A \\ A \end{bmatrix} \cos(\omega_1 t - \delta). \tag{35}$$

Thus our first normal mode is described by

$$x_1(t) = A\cos(\omega_1 t - \delta) \quad \text{and} \quad x_2(t) = A\cos(\omega_1 t - \delta). \tag{36}$$

Examining this normal mode we see that the two masses oscillate in phase and with the same amplitude as shown in figure 11.2.

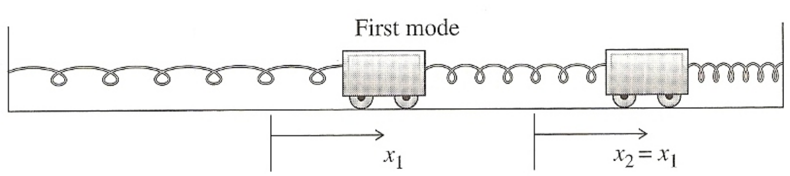

Figure 11.2. In the first normal mode the equal masses oscillate in phase with equal amplitudes so that $x_1(t) = x_2(t)$. In this mode the middle spring remains at equilibrium at all times.

A striking feature of this mode is that because $x_1(t) = x_2(t)$, the middle spring is neither stretched nor compressed during the oscillations. This means that for the first normal mode, the middle spring is actually irrelevant, and each mass oscillates just as if it were attached to a single spring (albeit in phase with the other mass). This explains why the first normal mode frequency satisfies $\omega_1^2 = k/m$.

The Second Normal Mode The second normal frequency at which our system can oscillate sinusoidally is $\omega_2^2 = 3k/m$. When this is substituted into (26) we have

$$\mathbf{K} - \omega_2^2\mathbf{M} = \begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix}. \tag{37}$$

Thus for this normal mode, the eigenvalue equation $\left(\mathbf{K} - \omega_2^2\mathbf{M}\right)\mathbf{a}$ becomes

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \tag{38}$$

which implies that $a_1 + a_2 = 0$, or $a_1 = -a_2 = Ae^{-i\delta}$. The complex column $\mathbf{z}(t)$ is then

$$\mathbf{z}(t) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega t} = \begin{bmatrix} A \\ -A \end{bmatrix} e^{i(\omega_2 t - \delta)} \tag{39}$$

and the corresponding real column $\mathbf{x}(t) = \operatorname{Re}\mathbf{z}(t)$ is

$$\mathbf{x}(t) = \begin{bmatrix} A \\ -A \end{bmatrix} \cos(\omega_2 t - \delta). \tag{40}$$

That is for the second normal mode

$$x_1(t) = A\cos(\omega_2 t - \delta) \quad \text{and} \quad x_2(t) = -A\cos(\omega_2 t - \delta). \tag{41}$$

We see that in the second normal mode the two masses oscillate with the same amplitude A but exactly out of phase as shown in figure 11.3.

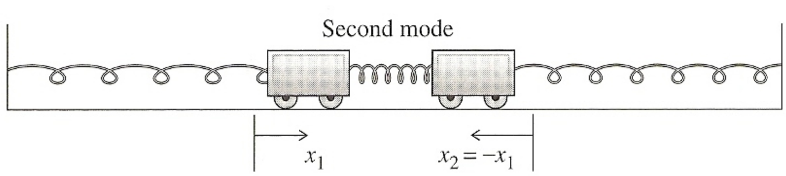

Figure 11.3. In the second normal mode the equal masses oscillate exactly out of phase with equal amplitudes so that $x_1(t) = -x_2(t)$.

Notice that in the second normal mode, when mass 1 is displaced to the right, mass 2 is displaced an equal distance to the left, and vice versa. This means that when the outer two springs are stretched, as in figure 5, the inner spring is compressed by twice that amount. This means that each mass moves as if it were attached to a single spring with force constant $3k$. This explains why the second normal frequency is $\omega_2^2 = 3k/m$.