24 Lecture 11-23

24.1 Chapter 11 Coupled Oscillators and Normal Modes (con)

The General Solution We have now found two normal mode solutions, which we can rewrite as

$$\mathbf{x}_1(t) = A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_1 t - \delta_1) \quad \text{and} \quad \mathbf{x}_2(t) = A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_2 t - \delta_2), \quad (1)$$

where ω_1 and ω_2 are the normal frequencies. Both of these solutions satisfy the equation of motion $\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$ for any values of the four real constants A_1, A_2, δ_1, and δ_2. Since the equation of motion is linear and homogeneous, the sum of these two solutions is also a solution:

$$\mathbf{x}(t) = A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_1 t - \delta_1) + A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_2 t - \delta_2). \quad (2)$$

Because the equation of motion is really two second order differential equations for the two variables $x_1(t)$ and $x_2(t)$, its general solution has four constants of integration. Therefore the solution (2) with its four arbitrary constants, is in fact the general solution. Any solution can be written in this form with the four arbitrary constants determined by the initial conditions.

The general solution (2) is hard to visualize and describe. The motion of each mass is a mixture of the two frequencies, ω_1 and ω_2. Since $\omega_2 = \sqrt{3}\omega_1$ the motion never repeats itself, except in the special case when either A_1 or A_2 is zero and the solution is just one of the normal modes.

Normal Coordinates We have seen that in any possible motion of our two-cart system, both of the coordinates $x_1(t)$ and $x_2(t)$ vary with time. In the normal modes, their time dependence is simple sinusoidal, but it is still true that both vary. This is because they are coupled and one mass cannot move without the other moving. It is possible to introduce alternative, so-called normal coordinates which, although less physically transparent, have the property that they can move independent of each other. This statement is true for any system of coupled oscillators and is especially easy to see in the present case of two equal masses joined by three identical springs.

Instead of the coordinates $x_1(t)$ and $x_2(t)$, we will characterize the positions of the masses by the two normal coordinates

$$\xi_1 = \frac{1}{2}(x_1 + x_2), \quad \text{and } \xi_2 = \frac{1}{2}(x_1 - x_2). \quad (3a)$$

The physical significance of the original variables $x_1(t)$ and $x_2(t)$ is more transparent, but $\xi_1(t)$ and $\xi_2(t)$ serve just as well to define the motion of the system. Moreover we can now characterize the first normal mode as

$$\left.\begin{array}{rcl} \xi_1(t) & = & A_1 \cos(\omega_1 t - \delta_1) \\ \xi_2(t) & = & 0 \end{array}\right\} \text{[first normal mode]}, \quad (4)$$

whereas the second normal mode is given by

$$\left.\begin{array}{rcl} \xi_1(t) & = & 0 \\ \xi_2(t) & = & A_2 \cos(\omega_2 t - \delta_2) \end{array}\right\} \text{[second normal mode]}. \tag{5}$$

In the first normal mode only the new variable ξ_1 oscillates while ξ_2 remains stationary. In the second mode it is the other way around, ξ_2 oscillates while ξ_1 remains stationary. In this sense these new coordinates, the normal coordinates, are independent of each other and one can oscillate without the other. The general solution is a supposition of both modes in which case both ξ_1 and ξ_2 oscillate. However, ξ_1 oscillates at ω_1 only while ξ_2 oscillates at ω_2 only. In some more complicated problems, these new coordinates, the normal coordinates, represent a considerable simplification.

24.1.1 Two Weakly Coupled Oscillators

So far we have discussed the oscillations of two equal masses joined by three equal springs. For this system, the two normal modes were easy to understand and visualize, but for the nonnormal or general oscillations that was not the case. A system where some of the nonnormal oscillations are readily visualized is two identical masses that are connected by identical springs of spring constant k to the walls and to each other by a much weaker spring with spring constant $k_2 << k$ as shown in figure 11.4.

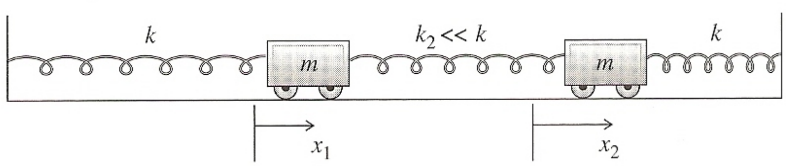

Figure 11.4. Two weakly coupled oscillating masses wth the middle spring much weaker than the outer two springs.

We can easily solve for the normal modes of this system. The mass matrix $\mathbf{M}$ is the same as before. Referring to equation (**??**) we see that the spring matrix takes on a somewhat different form

$$\mathbf{K} = \begin{bmatrix} k + k_2 & -k_2 \\ -k_2 & k + k_2 \end{bmatrix}, \tag{6}$$

so that the combination $\mathbf{K} - \omega^2 \mathbf{M}$ is

$$\mathbf{K} - \omega^2 \mathbf{M} = \begin{bmatrix} k + k_2 - \omega^2 m & -k_2 \\ -k_2 & k + k_2 - \omega^2 m \end{bmatrix}. \tag{7}$$

To find the normal frequencies we again set the determinant of the matrix $\mathbf{K} - \omega^2 \mathbf{M}$ to zero and find

$$\left(k + k_2 - \omega^2 m\right)^2 - k_2^2 = \left(k - \omega^2 m\right)\left(k + 2k_2 - \omega^2 m\right) = 0. \tag{8}$$

Hence the two normal frequencies are

$$\omega_1^2 = k/m \quad \text{and} \quad \omega_2^2 = (k + 2k_2)/m. \tag{9}$$

The first frequency is exactly the same as in the previous example. The reason for this (as you can easily check) is that the normal mode solution is in exactly the form as the equal spring case. Both of the masses again move together exactly in phase and the middle spring is undisturbed, see figure 11.2. Hence we get the same frequency for this mode independent of the strength of the middle spring.

As it turns out (due to symmetry reasons) the motion for the second mode is also the same as that for the corresponding mode of the equal spring example. In this mode the two masses are exactly out of phase with one moving inward while the other moves outward at any point in time, see figure 111.3. In this mode the strength of the middle spring is of course relevant, hence the second normal mode frequency is depends on k_2. For this example ω_2 is very close to ω_1. To take advantage of this closeness, it is convenient to define ω_o to be the average of the normal mode frequencies

$$\omega_o = \frac{\omega_1 + \omega_2}{2}. \tag{10}$$

Since ω_1 and ω_2 are very close to each other, ω_o is close to either. To show this effect we will define a small frequency ϵ via

$$\omega_1 = \omega_o - \epsilon \quad \text{and} \quad \omega_2 = \omega_o + \epsilon. \tag{11}$$

That is the small number ϵ is one half the difference between the two normal frequencies.

The two normal modes for this weakly coupled system can now be written as

$$\mathbf{z}_1(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i(\omega_o - \epsilon)t} \text{ and } \mathbf{z}_2(t) = C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i(\omega_o + \epsilon)t}. \tag{12}$$

Both of these satisfy the equation of motion for any values of the complex numbers C_1 and C_2. Since the equation of motion is linear and homogeneous the sum of these two solutions is also a solution,

$$\mathbf{z}(t) = \mathbf{z}_1(t) + \mathbf{z}_2(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i(\omega_o - \epsilon)t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i(\omega_o + \epsilon)t}. \tag{13}$$

Since the C's are complex with a magnitude and a phase, there are 4 undetermined real constants that are fixed by the initial conditions, i.e. the positions and velocities of the two masses at $t = 0$. Hence this is the general solution for the problem with the real part describing the actual motion of the system.

To see some general features it is helpful to factor out the $e^{i\omega_o t}$ term,

$$\mathbf{z}(t) = \mathbf{z}_1(t) + \mathbf{z}_2(t) = \left\{ C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-i\epsilon t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\epsilon t} \right\} e^{i\omega_o t}. \tag{14}$$

We have now expressed our solution as a product of two terms. The term in braces, $\{\cdots\}$, is a 2×1 column matrix which depends on time. However, since ϵ is very small, it varies very slowly compared to the second factor, $e^{i\omega_o t}$. Over any reasonably short time interval $(\Delta t << 2\pi/\epsilon)$, the first factor is essentially constant and the solution behaves like $\mathbf{z}(t) = \mathbf{a}e^{i\omega_o t}$, with $\mathbf{a}$ being a constant 2×1 column matrix. That is over any short interval, the two masses will oscillate sinusoidally with angular frequency ω_o. But if we wait long enough the 'constant' $\mathbf{a}$ will vary and the details of the motion of the two masses will change.

We will now examine the behavior of equation (14) for some simple values of C_1 and C_2. As we have already noted if either C_1 or C_2 vanish then the solution reverts to one of the normal modes. A more interesting case occurs when C_1 and C_2 are of equal magnitude. To simplify our analysis we will assume that they have the same phase as well. This means that by defining $t=0$ appropriately we can eliminate this phase and set both of them to be real as well or

$$C_1 = C_2 = A/2. \tag{15}$$

where A is a real constant. In this case equation (14) becomes

$$\mathbf{z}(t) = \frac{A}{2}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-i\epsilon t} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\epsilon t}\right\} e^{i\omega_o t} = A\begin{bmatrix} \cos\epsilon t \\ -i\sin\epsilon t \end{bmatrix} e^{i\omega_o t}. \tag{16}$$

To find the actual behavior of the system we must take the real part of this matrix equation and find

$$x_1(t) = A\cos\epsilon t\cos\omega_o t \quad \text{and} \quad x_2(t) = A\sin\epsilon t\sin\omega_o t. \tag{17}$$

The solution (17) has a simple interpretation. First notice that at time zero, $x_1 = A$ whereas $\dot{x}_1 = x_2 = \dot{x}_2 = 0$. So our solution describes the motion when mass m_1 is pulled to the right a distance A and released at $t=0$ with mass m_2 stationary at its original equilibrium position. Now because ϵ is small there is an appreciable interval, namely $0 \le t \le 2\pi/\epsilon$, during which the sine and cosine functions involving ϵt in equation (17) remain unchanged. During this initial period the positions are given by

$$x_1(t) \approx A\cos\omega_o t \quad \text{and} \quad x_2(t) \approx 0 \quad (t\approx 0). \tag{18}$$

So initially mass m_1 oscillates at frequency ω_o while mass m_2 remains stationary.

This state of affairs cannot last indefinitely. As soon as m_1 starts to move it interacts (albeit weakly) with m_2 through the middle spring. Eventually m_2 starts to oscillate, also at frequency ω_o with an amplitude of $A\sin\epsilon t$. Meanwhile the amplitude of the oscillations of m_1 are starting to decrease via the relation $A\cos\epsilon t$. This process continues until $t = \pi/2\epsilon$ at which point $\sin\epsilon t = 1$ and $\cos\epsilon t = 0$. Now we have the reverse situation that we had originally,

$$x_1(t) \approx \quad \text{and} \quad x_2(t) \approx A\sin\omega_o t \quad (t\approx \pi/2\epsilon). \tag{19}$$

Now mass m_2 is oscillating at maximum amplitude and mass m_1 is stationary. However soon via the weak coupling m_2 starts to drive m_1 until after a time $t = \pi/\epsilon$ the situation reverts to the initial motion. This process, in which the two masses pass energy back and forth from each other, continues indefinitely (ignoring dissipative forces). It is illustrated in figure 11.5.

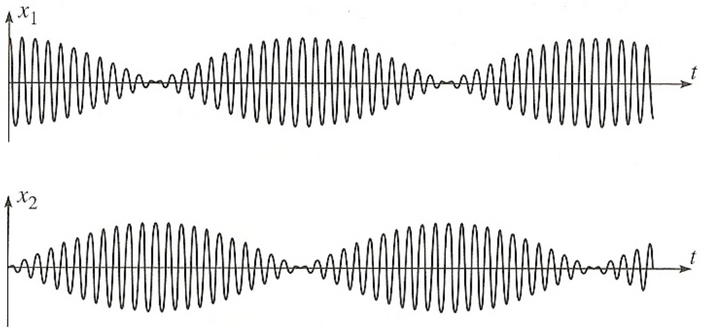

Figure 11.5. The positions of $x_1(t)$ and $x_2(t)$ for two weakly coupled oscillators if mass 1 is released from rest at $x_1 = A > 0$ and mass 2 is at $x_2 = 0$.

If you have any familiarity with beats you probably notice the similarity of figure 11.5 to a plot of beats. Beats are the result of superposition of two waves with nearly equal frequencies. Because of the small difference in frequencies, the two waves move regularly in and out of phase. This means that the resulting interference is alternating constructive and destructive. To understand what is beating in the case of our two masses, we need to consider the two normal coordinates of equation (3a), $\xi_1 = (x_1 + x_2)/2$ and $\xi_2 = (x_1 - x_2)/2$. Making use of trigonometric identities the normal coordinates can be expressed as

$$\xi_1(t) = \frac{A}{2}(\cos\epsilon t\cos\omega_o t + \sin\epsilon t\sin\omega_o t) = \frac{A}{2}\cos(\omega_o - \epsilon)t = \frac{A}{2}\cos\omega_1 t \tag{20a}$$

$$\xi_2(t) = \frac{A}{2}(\cos\epsilon t\cos\omega_o t - \sin\epsilon t\sin\omega_o t) = \frac{A}{2}\cos(\omega_o + \epsilon)t = \frac{A}{2}\cos\omega_2 t \tag{20b}$$

So the normal coordinates oscillate with equal amplitudes at their respective normal frequencies. Since $x_1(t) = \xi_1(t) + \xi_2(t)$, we see that the waxing and waning of $x_1(t)$ is the result of adding two signals of equal amplitude with nearly equal frequencies. A similar analysis applies to $x_2(t)$ except because $x_2(t) = \xi_1(t) - \xi_2(t)$ the moments of constructive interference for $x_1(t)$ are moments of destructive interference for $x_2(t)$ as is seen in figure 11.5.

24.1.2 The Double Pendulum

Consider a double pendulum with a mass m_1 suspended by a massless rod of length L_1, from a fixed pivot, and a second mass m_2 suspended from the first pendulum by a massless rod of length L_2. To write the Lagrangian, we will use the generalized coordinates ϕ_1 and ϕ_2 as shown in figure 11.6.

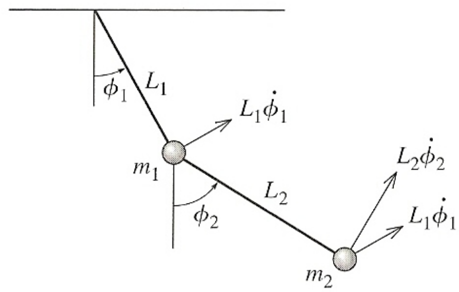

Figure 11.6. A double pendulum with generalized coordinates ϕ_1 and ϕ_2.

The potential energy as measured from the fixed pivot is for the first mass is

$$U_1 = -m_1 g L_1 \cos\phi_1. \tag{21}$$

Since the elevation (as measured from the fixed pivot) for the second mass is the sum of $-L_1\cos\phi_1$ and $-L_2\cos\phi_2$ its potential energy is given by

$$U_2 = -m_2 g\left(L_1\cos\phi_1 + L_2\cos\phi_2\right). \tag{22}$$

Summing these gives us the total potential energy

$$U\left(\phi_1,\phi_2\right) = -\left(m_1+m_2\right) g L_1\cos\phi_1 - m_2 g L_2\cos\phi_2. \tag{23}$$

The kinetic energy of the first mass is again the usual expression

$$T_1 = \frac{1}{2} m_1 L_1^2 \dot{\phi}_1^2. \tag{24}$$

The simplest way to find the kinetic energy of the second mass is to first find both its x and y coordinates in order to find the x and y components of its velocity. These are

$$x_2 = L_1\sin\phi_1 + L_2\sin\phi_2, \tag{25a}$$
$$y_2 = -L_1\cos\phi_1 - L_2\cos\phi_2. \tag{25b}$$

Thus the velocities are

$$\dot{x}_2 = L_1\cos\phi_1\dot{\phi}_1 + L_2\cos\phi_2\dot{\phi}_2, \tag{26a}$$
$$\dot{y}_2 = L_1\sin\phi_1\dot{\phi}_1 + L_2\sin\phi_2\dot{\phi}_2. \tag{26b}$$

Simply squaring and adding these two terms while taking advantage of the trigonometric relation for the addition of angles yields the kinetic energy for the second mass and is

$$T_2 = \frac{1}{2} m_2\left(L_1^2\dot{\phi}_1^2 + 2L_1L_2\cos\left(\phi_1-\phi_2\right)\dot{\phi}_1\dot{\phi}_2 + L_2^2\dot{\phi}_2^2\right). \tag{27}$$

Summing T_1 and T_2 gives us the total kinetic energy

$$T = \frac{1}{2}(m_1 + m_2) L_1^2 \dot{\phi}_1^2 + m_2 L_1 L_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2 + \frac{1}{2} m_2 L_2^2 \dot{\phi}_2^2. \tag{28}$$

We can now write down the Lagrangian $\mathcal{L} = T - U$ and then the two Lagrange equations for ϕ_1 and ϕ_2. However the two resulting equations are too complicated to be particularly illuminating and cannot be solved analytically. This is similar to the simple pendulum, whose equation of motion is also unsolvable analytically. This forces to solve it numerically or consider the situation of small amplitude oscillations around equilibrium. We are going to see that for almost all coupled oscillating systems, the exact equations are not solvable analytically, but if we confine ourselves to small oscillations about equilibrium they reduce themselves to equations that are solvable. Since this is an important special case we shall proceed with this in mind.

Returning to the case of the double pendulum, we will assume that both angles and their corresponding velocities are small. With these approximations the kinetic energy reduces to

$$T = \frac{1}{2}(m_1 + m_2) L_1^2 \dot{\phi}_1^2 + m_2 L_1 L_2 \dot{\phi}_1 \dot{\phi}_2 + \frac{1}{2} m_2 L_2^2 \dot{\phi}_2^2, \tag{29}$$

and the potential energy (to within some uninteresting constants) becomes

$$U = \frac{1}{2}(m_1 + m_2) g L_1 \phi_1^2 + \frac{1}{2} m_2 g L_2 \phi_2^2. \tag{30}$$

Before we use these simplified expressions for T and U to give us equations of motion, we should stop for a minute and examine the significance of our small amplitude oscillations. The exact expression for the kinetic energy (28) was a transcendental function of the coordinates ϕ_1 and ϕ_2 and the velocities $\dot{\phi}_1$ and $\dot{\phi}_2$. The small angle approximation reduced this to a quadratic homogeneous function of the velocities only. The exact expression for the potential energy was a transcendental function of ϕ_1 and ϕ_2 and the small angle approximation reduced it to a quadratic homogeneous function of the coordinates only. We shall see (and in fact have seen in small amplitude oscillations about equilibrium approximations) that the same simplifications occur for a wide range of oscillating systems. The assumption that all oscillations are small, which reduces T to a homogeneous quadratic function of the velocities and U to a homogeneous quadratic function of the coordinates, results in the Lagrange equations being homogeneous linear functions. The equations of motion can easily be solved.

We can now use the approximate expressions for T (29) and U (30) to construct a Lagrangian, $\mathcal{L} = T - U$, and write down the two Lagrange equations of motion for ϕ_1 and ϕ_2. These are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \phi_1} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} \quad \text{or} \\ -(m_1 + m_2) g L_1 \phi_1 &= (m_1 + m_2) L_1^2 \ddot{\phi}_1 + m_2 L_1 L_2 \ddot{\phi}_2 \end{aligned} \tag{31}$$

and

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \phi_2} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_2} \quad \text{or} \\
-m_2 g L_2 \phi_2 &= m_2 L_1 L_2 \ddot{\phi}_1 + m_2 L_2^2 \ddot{\phi}_2. \qquad (32)
\end{aligned}$$

In an analogous manner to our approach with the two masses and three springs, we can write these two coupled equations as a single matrix equation

$$\mathbf{M}\ddot{\boldsymbol{\phi}} = -\mathbf{K}\boldsymbol{\phi}, \qquad (33)$$

where we have defined $\boldsymbol{\phi}$ as a 2×1 column matrix

$$\boldsymbol{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

and $\mathbf{M}$ and $\mathbf{K}$ as 2×2 matrices

$$\mathbf{M} = \begin{bmatrix} (m_1 + m_2) L_1^2 & m_2 L_1 L_2 \\ m_2 L_1 L_2 & m_2 L_2^2 \end{bmatrix} \text{ and } \mathbf{K} = \begin{bmatrix} (m_1 + m_2) g L_1 & 0 \\ 0 & m_2 g L_2 \end{bmatrix} \qquad (34)$$

When compared to our first matrix equation we see that the mass matrix $\mathbf{M}$ is not actually made up of masses, but it still plays the role of inertia in the equation of motion (33). Exactly as before any solution for $\boldsymbol{\phi}(t)$ can be written as the real part of a complex solution $\mathbf{z}(t)$ whose time dependence is $e^{i\omega t}$; that is,

$$\boldsymbol{\phi}(t) = \mathrm{Re}\, \mathbf{z}(t) \quad \text{where} \quad \mathbf{z}(t) = \mathbf{a} e^{i\omega t} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega t}.$$

Again exactly as before a function of this form satisfies the equation of motion (33) if and only if the column matrix a satisfies the eigenvalue equation $\left(\mathbf{K} - \omega^2 \mathbf{M}\right)\mathbf{a} = 0$. This equation only has a solution if and only if the determinant $\det\left(\mathbf{K} - \omega^2 \mathbf{M}\right)$ vanishes. For 2×2 matrices this leads to a quadratic equation in ω^2 which determines the two normal frequencies for the double pendulum. Knowing these two normal frequencies we can go back and find the corresponding column matrix $\mathbf{a}$ and find the normal modes. Finally, the general motion of the system is just an arbitrary superposition of these two normal modes the constants of which are determined by the initial conditions.