

25 Lecture 11-30

25.1 Chapter 11 Coupled Oscillators and Normal Modes (con)

Double Pendulum with Equal Lengths and Masses As we did with the coupled spring mass system, we will simplify our discussion by restricting ourselves to the case of equal masses $m_1 = m_2 = m$ and equal lengths $L_1 = L_2 = L$. Additionally we will simplify things with the definition $\omega_o^2 = g/L$, the frequency of a single pendulum of length L and a bob of mass m . With these restrictions and definitions the mass and spring matrix become

$$\mathbf{M} = mL^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } \mathbf{K} = mL^2 \begin{bmatrix} 2\omega_o^2 & 0 \\ 0 & \omega_o^2 \end{bmatrix}. \quad (1)$$

The matrix $(\mathbf{K} - \mathbf{M}\omega^2)$ of the eigenvalue equation is therefore

$$(\mathbf{K} - \mathbf{M}\omega^2) = mL^2 \begin{bmatrix} 2(\omega_o^2 - \omega^2) & -\omega^2 \\ -\omega^2 & \omega_o^2 - \omega^2 \end{bmatrix}. \quad (2)$$

The normal frequencies are determined by the condition $\det(\mathbf{K} - \mathbf{M}\omega^2) = 0$, which leads to

$$\omega^4 - 2(\omega_o^2 - \omega^2)^2 = (\omega^2 - \sqrt{2}(\omega_o^2 - \omega^2))(\omega^2 + \sqrt{2}(\omega_o^2 - \omega^2)) = 0,$$

or

$$\omega_1^2 = \frac{\sqrt{2}\omega_o^2}{\sqrt{2}+1} = (2 - \sqrt{2})\omega_o^2 \text{ and } \omega_2^2 = \frac{\sqrt{2}\omega_o^2}{\sqrt{2}-1} = (2 + \sqrt{2})\omega_o^2. \quad (3)$$

Now that we know the two normal frequencies we can solve for the normal modes, by solving the equation $(\mathbf{K} - \mathbf{M}\omega^2)\mathbf{a} = 0$. For the first normal mode, $\omega = \omega_1$, and

$$mL^2 \begin{bmatrix} 2(\omega_o^2 - \omega_1^2) & -\omega_1^2 \\ -\omega_1^2 & \omega_o^2 - \omega_1^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = mL^2\omega_o^2(\sqrt{2}-1) \begin{bmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0. \quad (4)$$

This implies that $a_2 = \sqrt{2}a_1$, and writing $a_1 = A_1 e^{-i\delta_1}$ we find the the first normal mode

$$\phi(t) = A_1 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \cos(\omega_1 t - \delta_1) \quad [\text{first mode}]. \quad (5)$$

With this solution we see that in the first normal mode the two pendulums oscillate in phase with the amplitude of the lower pendulum being $\sqrt{2}$ times that of the upper pendulum as shown in figure 11.7.

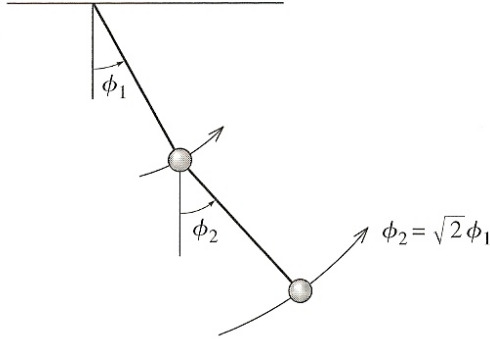


Figure 11.7 First normal mode for a double pendulum with equal masses and equal lengths. The two angles ϕ_1 and ϕ_2 oscillate in phase with the amplitude for ϕ_2 being larger by a factor of $\sqrt{2}$.

For the second normal mode, $\omega = \omega_2$, and

$$mL^2 \begin{bmatrix} 2(\omega_o^2 - \omega_2^2) & -\omega_2^2 \\ -\omega_2^2 & \omega_o^2 - \omega_2^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -mL^2\omega_o^2(\sqrt{2} + 1) \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0. \quad (6)$$

The solution to this set of algebraic equations is $a_2 = -\sqrt{2}a_1$. In an analogous manner $a_2 = A_2 e^{-i\delta_2}$ so that the second normal mode is

$$\phi(t) = A_2 \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} \cos(\omega_2 t - \delta_2) \quad [\text{second mode}]. \quad (7)$$

Here we see that the two pendulums oscillate exactly out of phase with the amplitude of the lower pendulum still being $\sqrt{2}$ times that of the upper pendulum as shown in figure 11.8.

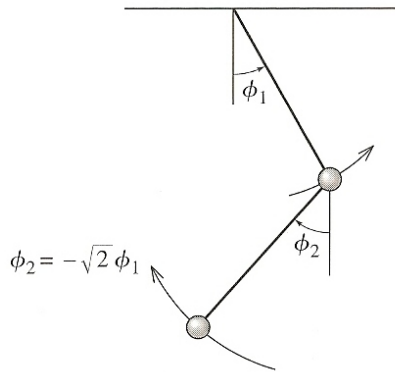


Figure 11.8 The second normal mode for a double pendulum with equal masses and equal lengths. The two angles ϕ_1 and ϕ_2 oscillate exactly out of phase, with the amplitude of ϕ_2 again being larger by a factor of $\sqrt{2}$.

The general solution is a linear combination of these two normal modes.

25.1.1 The General Case

We have now studied in great detail the normal modes of two systems, a pair of masses attached to springs and a double pendulum. With this as a background we are now ready to discuss the general case of a system with n degrees of freedom that is oscillating about a point of stable equilibrium. Since the system has n degrees of freedom, its configuration can be specified by n generalized coordinates (we are assuming holonomic systems here), q_1, \dots, q_n . In a notation consistent with our previous notation we shall designate the set of all n coordinates with a single bold faced \mathbf{q} . For example in our previous examples $\mathbf{q} = (x_1, x_2)$ for the coupled springs and $\mathbf{q} = (\phi_1, \phi_2)$ for the coupled double pendulum. It should be noted that in general \mathbf{q} is an n dimensional vector in the n -dimensional space of the generalized coordinates q_1, \dots, q_n .

We shall assume that the system is conservative, so that the potential energy is

$$U(q_1, \dots, q_n) = U(\mathbf{q}) \quad (8)$$

and the Lagrangian is $\mathcal{L} = T - U$. The kinetic energy in Cartesian coordinates is of course $T = \sum_{\alpha} \frac{1}{2} m_{\alpha} \dot{\vec{r}}_{\alpha}^2$, where the sum over α ranges over all N particles that comprise the system. This must be rewritten in terms of the generalized coordinates $\mathbf{q} = (q_1, \dots, q_n)$ using the relation between the Cartesian coordinates \vec{r}_{α} and the generalized coordinates

$$\vec{r}_{\alpha} = \vec{r}_{\alpha}(q_1, \dots, q_n), \quad (9)$$

where we shall take it for granted that this relation has no explicit time dependence. As we saw when we derived the expression for the Hamiltonian

$$\dot{\vec{r}}_{\alpha} = \sum_i \frac{\partial \vec{r}_{\alpha}}{\partial q_i} \dot{q}_i, \quad (10)$$

so that the kinetic energy is

$$T = \frac{1}{2} \sum_{\alpha} m_{\alpha} \sum_j \frac{\partial \vec{r}_{\alpha}}{\partial q_j} \dot{q}_j \sum_k \frac{\partial \vec{r}_{\alpha}}{\partial q_k} \dot{q}_k,$$

or after regrouping terms

$$T = \frac{1}{2} \sum_j \sum_{\alpha} m_{\alpha} \frac{\partial \vec{r}_{\alpha}}{\partial q_j} \cdot \sum_k \frac{\partial \vec{r}_{\alpha}}{\partial q_k} \dot{q}_j \dot{q}_k = \frac{1}{2} \sum_{jk} A_{jk}(\mathbf{q}) \dot{q}_j \dot{q}_k, \quad (11)$$

where

$$A_{jk}(\mathbf{q}) = \sum_{\alpha} m_{\alpha} \frac{\partial \vec{r}_{\alpha}}{\partial q_j} \cdot \sum_k \frac{\partial \vec{r}_{\alpha}}{\partial q_k}. \quad (12)$$

In general then A_{jk} can depend on the generalized coordinates. Under our current assumptions we have determined that the Lagrangian has the general form $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = T(\mathbf{q}, \dot{\mathbf{q}}) - U(\mathbf{q})$.

Our final assumption on the system is that it is making small amplitude oscillations about a configuration of stable equilibrium. By redefining the coordinates, if necessary, we will define equilibrium to occur at $\mathbf{q} = 0$, i.e. $(q_1 = q_2, \dots, = q_n = 0)$. Now, since we are only interested in small values of the coordinates \mathbf{q} , we can make Taylor's expansions for both T and U about $\mathbf{q} = 0$.

This is a particularly simple expansion for the kinetic energy as the sum for T already contains the factor $\dot{q}_j \dot{q}_k$ which is already second order in $\dot{\mathbf{q}}$. Therefore we can ignore everything but the first term in the Taylor's expansion. We define this term to be $M_{ij} = A_{ij}(\mathbf{q} = 0)$. In this limit the kinetic energy is expressed as

$$T(\dot{\mathbf{q}}) = \frac{1}{2} \sum_{jk} M_{jk} \dot{q}_j \dot{q}_k. \quad (13)$$

The expansion for the potential energy is only slightly more complicated and is of the form

$$U(\mathbf{q}) = U(\mathbf{q} = 0) + \sum_j \frac{\partial U(\mathbf{q} = 0)}{\partial q_j} q_j + \frac{1}{2} \sum_{jk} \frac{\partial^2 U(\mathbf{q} = 0)}{\partial q_j \partial q_k} q_j q_k,$$

where we have terminated this expansion at second order in \mathbf{q} . Since the first term is merely a constant we can redefine the potential energy to include this basically irrelevant term. Next we have assumed that the system is in equilibrium so that the second term in this expansion vanishes. Defining the second derivatives $\partial^2 U / \partial q_j \partial q_k = K_{jk}$ (which we should note is symmetric in the indices j, k) reduces the expansion of U to

$$U(\mathbf{q}) = \frac{1}{2} \sum_{jk} K_{jk} q_j q_k, \quad (14)$$

and the Lagrangian is

$$\mathcal{L} = T - U = \frac{1}{2} \sum_{jk} M_{jk} \dot{q}_j \dot{q}_k - \frac{1}{2} \sum_{jk} K_{jk} q_j q_k \quad (15)$$

Notice that the approximations used to obtain this expression correspond to those used in the analysis of the double pendulum. Just as in that case, the kinetic energy is a homogeneous quadratic function of $\dot{\mathbf{q}}$ and the potential energy is a homogeneous quadratic functions of \mathbf{q} . Just as with the double pendulum this will guarantee that the equations of motion are solvable linear equations.

Example - Bead on a Wire As an example of our discussion on the general case consider the case of a bead of mass m on a frictionless wire. The wire is bent into the shape $y = f(x)$ as shown in figure 11.9.

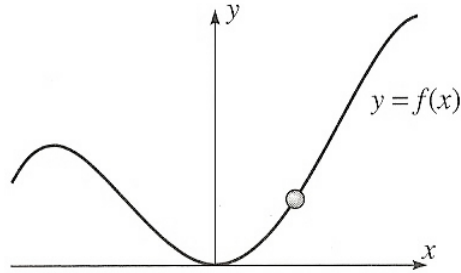


Figure 11.9. A bead on a frictionless wire in the shape of $f(x)$.

The system has just one degree of freedom which we will choose to be the coordinate x . With this choice the potential energy is just $U = mgy = mgf(x)$. We shall confine ourselves to small oscillations about the minimum in $f(x)$. Defining our origin to be the location of the minimum in $f(x)$, from our discuss about the general case, the potential energy is

$$U = mgf(x) \simeq mg \frac{d^2 f}{dx^2} x^2. \quad (16)$$

The kinetic energy is

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) = \frac{1}{2} m \left(\dot{x}^2 + \left(\frac{df}{dx} \dot{x} \right)^2 \right) = \frac{1}{2} m \left(1 + \left(\frac{df}{dx} \right)^2 \right) \dot{x}^2.$$

Since we are evaluating this derivative at the equilibrium point in f , the derivative df/dx vanishes and we are left with

$$T = \frac{1}{2} m \dot{x}^2. \quad (17)$$

As expected, the small amplitude oscillation approximation has reduced both U and T to homogeneous quadratic functions of x and \dot{x} .

The Equation of Motion for the General Case With our generalized approximate Lagrangian

$$\mathcal{L} = T - U = \frac{1}{2} \sum_{jk} M_{jk} \dot{q}_j \dot{q}_k - \frac{1}{2} \sum_{jk} K_{jk} q_j q_k, \quad (18)$$

we can easily write down the equations of motion. Since there are n generalized coordinates, there are n corresponding equations

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i} \quad [i = 1, \dots, n]. \quad (19)$$

In a manner analogous to our work in chapter 7, when we derived the conditions for the Hamiltonian to be equal to the energy, we must be careful when evaluating these derivatives. First we will carefully consider the partial derivative $\partial\mathcal{L}/\partial q_i$. This derivative only involves the potential energy and is expressed as

$$\frac{\partial\mathcal{L}}{\partial q_i} = -\frac{\partial U}{\partial q_i} = -\frac{1}{2}\frac{\partial}{\partial q_i}\sum_{jk}K_{jk}q_jq_k = -\frac{1}{2}\sum_{jk}K_{jk}\frac{\partial q_j}{\partial q_i}q_k - \frac{1}{2}\sum_{jk}K_{jk}\frac{\partial q_k}{\partial q_i}q_j. \quad (20)$$

The generalized coordinates form an independent set, hence $\partial q_j/\partial q_i = \delta_{ij}$, where δ_{ij} is the Kronecker delta that by definition is zero except when $i = j$ and in that case it is 1. We can now rewrite equation (20) as

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial q_i} &= -\frac{1}{2}\sum_{jk}K_{jk}\delta_{ji}q_k - \frac{1}{2}\sum_{jk}K_{jk}\delta_{ki}q_j \\ \frac{\partial\mathcal{L}}{\partial q_i} &= -\frac{1}{2}\sum_k K_{ik}q_k - \frac{1}{2}\sum_j K_{ji}q_j. \end{aligned} \quad (21)$$

Since the summation indices are dummy indices and since (as we noted previously) K_{ik} is symmetric in its indices we can simplify this expression to

$$\frac{\partial\mathcal{L}}{\partial q_i} = -\frac{1}{2}\sum_k K_{ij}q_j - \frac{1}{2}\sum_j K_{ij}q_j = -\sum_j K_{ij}q_j \quad [i = 1, \dots, n]. \quad (22)$$

A similar analysis of the kinetic energy yields analogous results and the n Lagrange equations are

$$\sum_j M_{ij}\ddot{q}_j = -\sum_j K_{ij}q_j \quad [i = 1, \dots, n]. \quad (23)$$

These n equations can be immediately grouped into a single matrix equation

$$\mathbf{M}\ddot{\mathbf{q}} = -\mathbf{K}\mathbf{q}, \quad (24)$$

where \mathbf{q} is the $n \times 1$ column matrix (vector)

$$\mathbf{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \quad (25)$$

and \mathbf{M} and \mathbf{K} are the $n \times n$ mass and spring matrices comprised of K_{ij} and M_{ij} respectively.

The matrix equation is the n dimensional equivalent of the two-dimensional for the coupled mass and double pendulum that we have already considered. It is solved in exactly the same way. We first find the normal modes with the form

$$\mathbf{q}(t) = \text{Re } \mathbf{z}(t), \quad \text{where } \mathbf{z}(t) = \mathbf{a}e^{i\omega t} \quad (26)$$

and \mathbf{a} is a $n \times 1$ column matrix. This leads to the eigenvalue equation

$$(\mathbf{K} - \omega^2 \mathbf{M}) \mathbf{a} = 0, \quad (27)$$

which has a solution if and only if ω satisfies the “characteristic” or “*secular*” equation that is the result of

$$\det(\mathbf{K} - \omega^2 \mathbf{M}) = 0. \quad (28)$$

This determinant is an n^{th} degree polynomial in ω^2 , so equation (28) has n solutions, which determine the n normal frequencies of the system. With ω set equal to each of the normal mode frequencies in turn, equation (26) determines the motion of the system in the corresponding normal mode. Finally, the general motion of the system is given by an arbitrary sum of corresponding normal mode solutions.

This general procedure that we have just outlined is just a generalization of the procedure we have already discussed in some detail for the two masses and double pendulum. Before we finish our discussion of normal mode problems we will discuss a system with three degrees of freedom. Now, however, we shall discuss the general case of normal coordinates

25.1.2 Normal Coordinates

When we studied the system with two equal masses and three identical springs we found that we could replace the two coordinates x_1 and x_2 by two *normal coordinates*

$$\xi_1 = \frac{1}{2}(x_1 + x_2) \quad \text{and} \quad \xi_2 = \frac{1}{2}(x_1 - x_2). \quad (29)$$

These coordinates have the property that they always oscillate at just one of the two normal frequencies, ξ_1 at ω_1 and ξ_2 at ω_2 . As it turns out we can do the same thing for any system oscillating about a stable equilibrium (albeit for nonlinear oscillations, they must have small amplitudes). If the system has n degrees of freedom, then it is described by n generalized coordinates q_1, \dots, q_n (holonomic), and has n normal modes with frequencies $\omega_1, \dots, \omega_n$. We shall now show that each normal coordinate ξ_i oscillates at just one frequency, namely the normal frequency ω_i .

Before we proceed it is useful to review our previous discussion of normal coordinates. The two equations of motion for the case of two equal masses and three identical springs were

$$\left. \begin{aligned} m\ddot{x}_1 &= -2kx_1 + kx_2 \\ m\ddot{x}_2 &= kx_1 - 2kx_2 \end{aligned} \right\} \quad (30)$$

If we add these equations we find

$$m\ddot{\xi}_1 = -k\xi_1, \quad (31)$$

while subtracting yields

$$m\ddot{\xi}_2 = -3k\xi_2. \quad (32)$$

These two equations are uncoupled and show that each normal coordinate oscillates at a single frequency, ξ_1 at ω_1 and ξ_2 at ω_2 . In other words the normal coordinates behave just like the coordinates of uncoupled oscillators and by going over to the normal coordinates, we have uncoupled oscillations.

Just as the equations for x_1 and x_2 can be rewritten as a single matrix equation $\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$, so too can the equations for ξ_1 and ξ_2 be rewritten as $\mathbf{M}'\ddot{\boldsymbol{\xi}} = -\mathbf{K}'\boldsymbol{\xi}$. The important difference here is that the two matrices \mathbf{M}' and \mathbf{K}' are both diagonal.

$$\mathbf{M}' = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \text{ and } \mathbf{K}' = \begin{bmatrix} k & 0 \\ 0 & 3k \end{bmatrix}. \quad (33)$$

The transform from the original coordinates to the normal coordinates is said to diagonalize the matrices \mathbf{M} and \mathbf{K} . That the new matrices are diagonal is precisely equivalent to the statement that the equations for ξ_1 and ξ_2 are uncoupled and will oscillate independently.

We can define the two normal coordinates differently, and more generally, in terms of the eigenvectors \mathbf{a} that describe the motion of the normal modes and are determined by the eigenvalue equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$. Now we wish to label each of the column vectors so that

$$\mathbf{a}_{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{a}_{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \quad (34)$$

Two important points need to be made here. The first is that each of these vectors contains an arbitrary multiplier $Ae^{-i\delta}$. However now we will change this and fix $\delta = 0$ and $A = 1$ which we have done in equation (34). Another choice and sometimes a better choice is to normalize the vectors with a factor of $1/\sqrt{2}$. In our case this does not lead to any simplification and for notational simplicity we will stay with $A = 1$. The other point is that each column is made up of two components, two different numbers, which we have labeled as a_1 and a_2 . However, now we are discussing two different columns, $\mathbf{a}_{(1)}$ and $\mathbf{a}_{(2)}$, one for each normal mode. For now we will use the parenthesis in the subscripts to emphasize this distinction. Just as we could expand the normal coordinates in terms of the generalized coordinates of the system, we can invert this to expand the generalized coordinates in terms of the normal coordinates. Specifically for the problem we have been discussing this is expressed as

$$\mathbf{x} = \xi_1\mathbf{a}_{(1)} + \xi_2\mathbf{a}_{(2)} = \begin{bmatrix} \xi_1 + \xi_2 \\ \xi_1 - \xi_2 \end{bmatrix}. \quad (35)$$

The first equality defines ξ_1 and ξ_2 in terms as the coefficients in the expansion of \mathbf{x} in terms of the eigenvectors $\mathbf{a}_{(1)}$ and $\mathbf{a}_{(2)}$. The last term in this relation shows that ξ_1 and ξ_2 are precisely the normal coordinates for this problem. That

is the normal coordinates can be defined as the coefficients in the expansion of \mathbf{x} in terms of the eigenvectors $\mathbf{a}_{(1)}$ and $\mathbf{a}_{(2)}$. We shall now see that this definition carries over naturally to the general case for coupled oscillators with n degrees of freedom.

The General Case We will now consider the case with n generalized coordinates q_1, \dots, q_n , and n normal modes. In the i^{th} mode the column vector $\mathbf{q}_{(i)}$ oscillates sinusoidally at the normal mode frequency ω_i ,

$$\mathbf{q}_{(i)} = \mathbf{a}_{(i)} \cos(\omega_i t - \delta_i)$$

where the column vector satisfies

$$\mathbf{K}\mathbf{a}_{(i)} = \omega_i^2 \mathbf{M}\mathbf{a}_{(i)}. \quad (36)$$

The columns $\mathbf{a}_{(i)}$ are n independent real $n \times 1$ column vectors and any $n \times 1$ column vector can be expanded in terms of them. That is the column vectors $\mathbf{a}_{(i)}$ form a complete set for the space of $n \times 1$ vectors. Thus any solution of the equations of motion $\mathbf{q}(t)$ can be expanded as

$$\mathbf{q}(t) = \sum_i \xi_i(t) \mathbf{a}_{(i)}. \quad (37)$$

Now the column vector $\mathbf{q}(t)$ satisfies the equation of motion

$$\mathbf{M}\ddot{\mathbf{q}} = -\mathbf{K}\mathbf{q}.$$

If we replace $\mathbf{q}(t)$ with the expansion in equation (37) the equation of motion becomes

$$\sum_i \ddot{\xi}_i(t) \mathbf{M}\mathbf{a}_{(i)} = - \sum_i \xi_i(t) \mathbf{K}\mathbf{a}_{(i)} = - \sum_i \xi_i(t) \omega_i^2 \mathbf{M}\mathbf{a}_{(i)}, \quad (38)$$

where the last step follows from equation (36). Now the n column vectors $\mathbf{a}_{(i)}$ are independent and this property is unchanged when the equation is operated on by \mathbf{M} , therefore the coefficients on each side of this equation must also be equal. That is

$$\ddot{\xi}_i = \omega_i^2 \xi_i(t). \quad (39)$$

Hence the normal coordinates for a system with n degrees of freedom do in-fact oscillate at their normal mode frequencies independently of each other.