17 Lecture 11-4

17.1 Chapter 7 Lagrange's Equations (con)

17.1.1 Examples of Lagrange's Equations

We will now consider four examples of Lagrange's equations. The first two are relatively simple and can be easily solved using Newton's second law. They are included to provide some experience with using the Lagrangian approach. NTL, even these simple cases show some advantages over the Newtonian formalism in that they obviate any need to consider the forces of constraint. The last two are sufficiently complex that the solution using the Newtonian approach requires considerable ingenuity. By contrast the Lagrangian approach lets us write down the equations of motion almost without thinking.

The Lagrangian formalism always (at least almost always) provides us with a straightforward means of writing down the equations of motion. However, they cannot guarantee that the resulting equations are easy to solve. But even if an analytic solution is not possible, writing down the equations of motion is an essential first step to understanding the solutions. In a worst case scenario, we can always solve the equations of motion numerically and usually solve for the positions of equilibrium very quickly.

Atwood's Machine In an Atwood machine, see figure 7.3, the two masses,

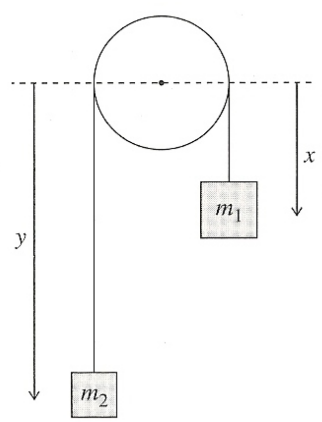

Figure 7.3 Atwood machine with constraints. Because the length of the string is fixed, the position of the whole system can be specified by the single variable x.

m_1 and m_2 are suspended by an inextensible string of length l which passes over a pulley with frictionless bearings, a radius R, and a momentum of inertia I. The kinetic energy for this system is

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 + \frac{1}{2}I\dot{\phi}^2 , \tag{1}$$

where $\dot{\phi}$ is the rate of angular rotation of the pulley. Because the length of the string is fixed the heights of the two masses, x and y, cannot vary independently. Rather $x + y + \pi R = l$. From this we see that $\dot{x} = -\dot{y}$. Additionally from a no slip condition between the string and the pulley the angular frequency of the pulley satisfies $R\dot{\phi} = \dot{x}$. We can now write the kinetic energy for the Atwood machine as

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}\frac{I}{R^2}\dot{x}^2. \tag{2}$$

The potential energy (to within a constant) is

$$U = -m_1gx - m_2gy = -(m_1 - m_2)\,gx. \tag{3}$$

Combining these expressions we find the Lagrangian to be

$$\mathcal{L} = T - U = \frac{1}{2}\left(m_1 + m_2 + I/R^2\right)\dot{x}^2 + (m_1 - m_2)\,gx. \tag{4}$$

The Lagrange equation of motion is

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \to (m_1 - m_2)\,g = \left(m_1 + m_2 + I/R^2\right)\ddot{x}. \tag{5}$$

Solving for the acceleration we find

$$\ddot{x} = (m_1 - m_2)\,g/\left(m_1 + m_2 + I/R^2\right). \tag{6}$$

By choosing m_1 and m_2 fairly close together, one can make this acceleration much less than g. Hence, the Atwood machine gave an early and reasonably accurate method for measuring g.

What is relevant here is that the corresponding Newtonian solution requires three free body equations which involves the two constaint forces, the tensions T_1 and T_2. The result is three equations with three unknowns, the tensions T_1, T_2, and the acceleration $\ddot{x}$. We can eliminate both T_1 and T_2 which reduces the three equations to the expression shown in equation (6). The Lagrangian solution of the Atwood machine is too simple to truly appreciate the advantage of this approach. NTL the Lagrangian approach did eliminate the need to reduce three Newtonian equations to the one that see in equation (6).

Particle Confined to Move on a Cylinder Consider a particle of mass m constrained to move on a frictionless cylinder of radius R, see figure 7.4. Besides the force of constraint (the normal force from the wall of the cylinder), the particle experiences a force due to a spring anchored at the origin ($\rho = z = 0$) given by $\overrightarrow{F} = -k\overrightarrow{r}$. Since the particle's radial coordinate is fixed at $\rho = R$, we can specify the position of the particle with just it height, z, and its angular

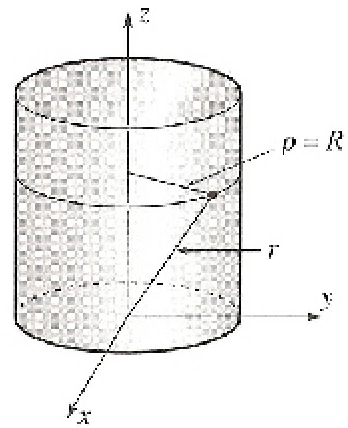

Figure 7.4 A mass m is confined to the surface of a cylinder, $\rho = R$, and subject to Hook's law, $\overrightarrow{F} = -k\overrightarrow{r}$.

coordinate ϕ. Hence the kinetic energy of the particle is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{z}^2 + R^2\dot{\phi}^2\right). \tag{7}$$

The potential energy of the spring is

$$U = \frac{1}{2}kr^2 = \frac{1}{2}k\left(\rho^2 + z^2\right) = \frac{1}{2}k\left(R^2 + z^2\right), \tag{8}$$

and the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\left(\dot{z}^2 + R^2\dot{\phi}^2\right) - \frac{1}{2}k\left(R^2 + z^2\right). \tag{9}$$

Since the system has two degrees of freedom, there are two equations of motion. The equation for the z coordinate is

$$\frac{\partial \mathcal{L}}{\partial z} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}} \rightarrow -kz = m\ddot{z}. \tag{10}$$

The ϕ equation is even simpler as the Lagrangian does not depend on ϕ, and the ϕ equation is

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mR^2\ddot{\phi} = 0. \tag{11}$$

The z equation tells us that the particle undergoes simple harmonic motion in the z direction, $z = A\cos(\omega t - \delta)$. The ϕ equation tells that the quantity $mR^2\dot{\phi}$ is constant, i.e. the angular momentum about the z axis is conserved. Since there is no torque on the particle, this is a result that we should have expected. The entire motion of the particle is that of moving around the cylinder at a constant angular velocity while oscillating in the z direction about $z = 0$ at angular frequency $\omega = \sqrt{k/m}$.

Block sliding on a Wedge Consider the block and wedge shown below,

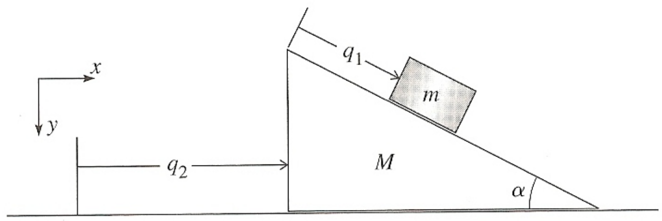

Figure 7.5 A block of mass m slides down an incline of mass M that resides on a frictionless horizontal table.

in figure 7.5. The block of mass m is free to slide on the wedge, and the wedge of mass M is free to slide on the horizontal table, both with negligible friction. The block is released from the top of the wedge while both are initially at rest. The wedge has an angle α and the length of its slope is l, and we are interested in determining how long it takes the block to reach the bottom.

The system has two degrees of freedom, one for the block and one for the wedge. A reasonable choice for the generalized coordinates is shown in figure 7.5, and they are the distance of the block from the top of the wedge, q_1, and the distance of the wedge from some fixed point on the table, q_2.

The kinetic energy of the wedge is simply $T_M = \frac{1}{2}M\dot{q}_2^2$, however the kinetic energy of the block is a bit more complicated. The block has a velocity $\dot{q}_1$ down the wedge, but that velocity is relative to the wedge not the table. Remembering that we need to write the Lagrangian in a nonaccelerating frame requires that we express the kinetic energy of the block in the inertial frame of the table. The x and y components for the velocity of the block are

$$v_x = \dot{q}_1 \cos\alpha + \dot{q}_2 \quad \text{and} \quad v_y = \dot{q}_1 \sin\alpha. \tag{12}$$

Thus the kinetic energy of the block is

$$T_m = \frac{1}{2}m\left(v_x^2 + v_y^2\right) = \frac{1}{2}m\left(\dot{q}_1^2 + \dot{q}_2^2 + 2\dot{q}_1\dot{q}_2\cos\alpha\right). \tag{13}$$

The potential energy of the wedge is constant while that of the block is $-mgy$, where $y = q_1 \sin\alpha$. Therefore the potential energy of the system is simply

$$U = -mgq_1 \sin\alpha, \tag{14}$$

and the Lagrangian is

$$\mathcal{L} = \frac{1}{2}M\dot{q}_2^2 + \frac{1}{2}m\left(\dot{q}_1^2 + \dot{q}_2^2 + 2\dot{q}_1\dot{q}_2\cos\alpha\right) + mgq_1\sin\alpha. \tag{15}$$

Once we have found the Lagrangian, all that is left is two write down the two Lagrange equations, one for q_1 and the other for q_2, and then of course solve them. The q_2 equation is

$$\frac{\partial \mathcal{L}}{\partial q_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q_2}}. \tag{16}$$

Since the Lagrangian is independent of q_2, we know that the generalized momentum $\partial \mathcal{L}/\partial \dot{q_2}$ is constant,

$$M\dot{q_2} + m\left(\dot{q_2} + \dot{q_1}\cos\alpha\right) = const. \tag{17}$$

This is the sum of the total momentum in the x direction and something you could have written down without any help from the Lagrangian.

The q_1 equation

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q_1}} \tag{18}$$

is more complicated, since neither derivative vanishes. Performing the partial derivatives we find

$$mg\sin\alpha = \frac{d}{dt}m\left(\dot{q_1} + \dot{q_2}\cos\alpha\right) = m\left(\ddot{q_1} + \ddot{q_2}\cos\alpha\right). \tag{19}$$

We now have two equations and wish to solve for $\ddot{q_1}$. Differentiating equation (17) and solving for $\ddot{q_2}$ we find

$$(M+m)\ddot{q_2} = -m\ddot{q_1}\cos\alpha \rightarrow \ddot{q_2} = -\frac{m}{M+m}\ddot{q_1}\cos\alpha. \tag{20}$$

Substituting this result into equation (19) yields

$$\begin{aligned} mg\sin\alpha &= m\left(1 - \frac{m}{M+m}\cos^2\alpha\right)\ddot{q_1}, \\ \ddot{q_1} &= \frac{(M+m)\,g\sin\alpha}{M+m\sin^2\alpha}. \end{aligned} \tag{21}$$

Since the acceleration of q_1 is constant, we can integrate it immediately and find

$$q_1 = \frac{1}{2}\frac{(M+m)\,g\sin\alpha}{M+m\sin^2\alpha}t^2, \tag{22}$$

so that the time, τ, to reach the end of the wedge ($q_1 = l$) is

$$\tau = \sqrt{2l\left(M+m\sin^2\alpha\right)/\left((M+m)\,g\sin\alpha\right)}. \tag{23}$$

As a check we can determine if this expression agrees with several different limits. In the limit of $\alpha = \pi/2$ $(\sin\alpha = 1)$, we have $\ddot{q_1} = g$, which is as it should be. The next limit of interest is when $M \rightarrow \infty$. In that limit we have $\ddot{q_1} = g\sin\alpha$, which is again correct. The limit when $M \rightarrow 0$ is left as an exercise for the student, problem 7.19.

Bead on a Spinning Wire Hoop A bead of mass m is attached to a frictionless wire hoop of radius R. The hoop lies in a vertical plane, which is forced to rotate about the hoop's vertical diameter with a constant angular velocity, $\dot{\phi} = \omega$, as shown in figure 7.6. The bead position on the hoop is specified by the angle θ measured up from the vertical while the entire system is in a uniform

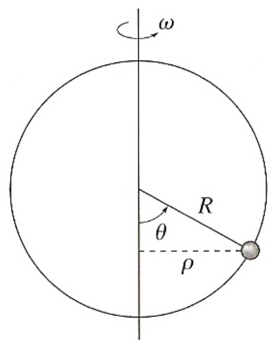

Figure 7.6 A bead is free to move around a frictionless wire hoop, which is spinning at a fixed rate ω.

gravitational field. The kinetic energy of the bead is $T = \frac{1}{2}mR^2\left(\dot{\theta}^2 + \sin^2\theta\omega^2\right)$. The gravitational potential energy is (as measured from the bottom of the hoop) is identical to that for the pendulum and is $U = mgR\left(1 - \cos\theta\right)$. Hence the Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}mR^2\left(\dot{\theta}^2 + \sin^2\theta\omega^2\right) - mgR\left(1 - \cos\theta\right). \tag{24}$$

There is only one generalized coordinate and therefore only one Lagrange equation,

$$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} \rightarrow mR^2\sin\theta\cos\theta\omega^2 - mgR\sin\theta = mR^2\ddot{\theta}. \tag{25}$$

Dividing through by mR^2, we find that the angular acceleration of θ is

$$\ddot{\theta} = \left(\omega^2\cos\theta - g/R\right)\sin\theta. \tag{26}$$

Now this expression cannot be solved analytically in terms of elementary functions. NTL it can tell us a lot about the system's behavior. For starters we can determine the equilibrium position, $\theta = \theta_o$, of the bead by setting $\ddot{\theta}$ to zero (and $\dot{\theta}$ if it appeared) in equation (26). This results in

$$\left(\omega^2\cos\theta_o - g/R\right)\sin\theta_o = 0 \tag{27}$$

This equation is satisfied when either of the two factors is zero. The factor $\sin\theta_o = 0$ when $\theta_o = 0$ or π. Thus the bead can remain at the bottom or top of the hoop. The first factor vanishes when

$$\cos\theta_o = g/\omega^2 R. \tag{28}$$

Since the cosine function is less than or equal to one this expression can only be satisfied if $\omega^2 \geq g/R$. When this condition is satisfied then there are two more positions of equilibrium (although they are equivalent positions for a rotating wire) whose positions are given by

$$\theta_o = \pm\cos^{-1}\left(g/\omega^2 R\right). \tag{29}$$

We can then conclude that when the hoop is rotating slowly ($\omega^2 < g/R$) there are only two equilibrium positions, at the top and bottom of the hoop. However, when the hoop is rotating fast enough ($\omega^2 > g/R$) there are two more, symmetrically placed on either side of the bottom of the ring as given by equation (29).

So we see that at the top or the bottom of the hoop, the bead is on the axis of rotation and $\rho = 0$, where ρ is the distance from the axis of rotation. Therefore the centripetal force $m\omega^2\rho$ is zero. Also, at those locations the force of gravity is normal to the hoop, so there is no force tending to move the bead along the wire and the bead remains at rest. For the position off the axis the force of gravity has a component that is pulling the bead inward along the wire (as long as $|\theta| \leq \pi/2$) Meanwhile the centrifugal force is pushing the bead outward along the wire. At the points given by equation (29) these forces are balanced and the bead remains at rest.

All of this being said, the equilibrium points are not especially interesting unless it is a position of stable equilibrium. Using the equation of motion for θ, equation (26), we can easily address this issue. First we will start with the equilibrium at $\theta = 0$. For θ near zero the equation reduces to

$$\ddot{\theta} = \left(\omega^2 - g/R\right)\theta. \tag{30}$$

If the hoop is rotating slowly, $\omega^2 < g/R$, then the equation of motion near $\theta = 0$ is of the form $\ddot{\theta} = -k\theta$ (k is a positive constant). This expression is analogous to the spring equation where any displacement away from equilibrium induces a restoring force that pushes the object back towards equilibrium. Clearly, for this range of frequency, this is a position of stable equilibrium. However if we speed up the hoop so that $\omega^2 > g/R$, then the equation of motion is of the form $\ddot{\theta} = +k\theta$ (again k is a positive constant). Any finite value of θ induces an acceleration toward larger θ, hence this position is now unstable. Thus as we increase ω this equilibrium position goes from being stable to unstable.

For obvious reasons the position at the top of the hoop is unstable for any rate of rotation. If the bead fluctuates just a bit away from $\theta = \pi$, then the gravitational force and the centrifugal both push the bead away from this position. This is easily seen in equation (26) by letting $\theta = \pi + \delta\theta$ where $\delta\theta << 1$.

The other two equilibrium positions only exist when $\omega^2 > g/R$. To determine whether or not the position at $\theta_o = +\cos^{-1}\left(g/\omega^2 R\right)$ is stable, we will consider small deviations away from this equilibrium position. To do this we let $\theta = \theta_o + \delta\theta = \cos^{-1}\left(g/\omega^2 R\right) + \delta\theta$. This is most easily done by expanding the function $f(\theta) = \left(\omega^2 \cos\theta - g/R\right)\sin\theta$ in a Taylor's series about θ_o. Since $f(\theta_o) = 0$ we find

$$\begin{aligned}\frac{df(\theta_o)}{d\theta} &= \left(-\omega^2 \sin\theta_o\right)\sin\theta_o = -\omega^2\left(1-\cos^2\theta_o\right) \\ \frac{df(\theta_o)}{d\theta} &= -\left(\omega^2 - g^2/\omega^2 R^2\right). \end{aligned} \tag{31}$$

Hence to first order in $\delta\theta$ the equation of motion becomes

$$\delta\ddot{\theta} = -\left(\omega^2 - g^2/\omega^2 R^2\right)\delta\theta. \tag{32}$$

We see that for small oscillations about the equilibrium position, what was once a very nonlinear differential equation is now linear. This makes it easy to analyze. Since $\omega^2 > g/R$ this expression reduces to the form $\ddot{\theta} = -k\theta$ (k is a positive constant). For the reasons stated above this is a position of stable equilibrium. Since the equilibrium position at $\theta_o = -\cos^{-1}\left(g/\omega^2 R\right)$ is an equivalent position, as you might expect, you arrive at the same conclusion for fluctuations about this equilibrium position as well.

With this result, we arrive at the following interesting story: When the hoop is rotating slowly, there is only one position of stable equilibrium, $\theta = 0$. If we speed up the rotation, then as ω passes the critical value $\omega = \sqrt{g/R}$, this original equilibrium becomes unstable, but two new stable equilibrium points appear. They emerge from $\theta = 0$ and move out to the right and left as we continue to increase ω. This phenomenon − the disappearance of one stable equilibrium and the simultaneous appearance of two others diverging from the same point − is called a bifurcation and will be one of the principle topics in chapter 12 on chaos theory.

This example illustrates another strength of the Lagrangian method in that the generalized coordinates can be coordinates in a noninertial frame, as long as the Lagrangian itself is written in an inertial frame. In this example θ is the polar angle of the bead written in a noninertial rotating frame of the hoop. But the Lagrangian, $\mathcal{L} = T - U$, was evaluated in inertial frame in which the hoops rotates.

It may be of interest to note that a device of this example was used by James Watt as a governor for his steam engines. The device rotated with the engine, and as the engine sped up the bead rose on the hoop. When the angular velocity ω reached some maximum allowable value, the bead reached a height that caused the supply of steam to be shut off.

Oscillations of the Bead near Equilibrium With our analysis, we are well placed to determine the frequency of oscillations about equilibrium for our example of a bead on the spinning hoop. First let's consider the frequency of oscillations about the equilibrium position at $\theta = 0$. From equation (30) we found that it was a position of stable equilibrium only when $\omega^2 < g/R$. In this frequency range we can write equation (30) as

$$\ddot{\theta} = -\left(g/R - \omega^2\right)\theta = -\Omega^2\theta = 0, \tag{33}$$

where $\Omega = \sqrt{g/R - \omega^2}$. Since this equation is that of a simple harmonic oscillator, we know that the bead oscillates with a frequency Ω about $\theta = 0$ as long as $\omega^2 < g/R$.

To find the frequency of oscillations about the equilibrium about $\theta_o = \cos^{-1}\left(g/\omega^2 R\right)$, we merely have to examine equation (**??**). As long as $\omega^2 > g/R$, which is required for equilibrium to exist at θ_o, equation (**??**) can also be written in the form of a simple harmonic oscillator as

$$\delta\ddot{\theta} + \Omega'^2\delta\theta = 0, \tag{34}$$

where $\Omega'^2 = \omega^2 - g^2/\omega^2 R^2$. Therefore $\delta\theta$ oscillates about zero, which means that the bead oscillates about the equilibrium position θ_o with frequency Ω'.

So our story is now a bit more complete. When the hoop is stationary the bead is in equilibrium at $\theta = 0$ and for small amplitude oscillations, it oscillates at a frequency of $\Omega = \sqrt{g/R}$. As the hoop begins to spin, the frequency of oscillations begins to slow via the expression $\Omega = \sqrt{g/R - \omega^2}$. This slowing continues until the rate of spinning reaches the threshold $\omega_h = \sqrt{g/R}$, at which point the bead no longer oscillates at all. Unless the bead was stationary at the moment this threshold was reached (at a maximum in its oscillations), it then continues to move slowly away from the bottom of the hoop at a uniform rate. As the rate of spinning continues to increase the bead comes to equilibrium at $\theta_o = \pm\cos^{-1}\left(g/\omega^2 R\right)$, and as mentioned there are two positions of equilibrium. For frequencies just greater than the threshold ω_h these angles are close to zero, i.e. near the bottom of the hoop. The frequency of these oscillations are also quite small. Now as the spinning rate continues to increase both equilibrium positions move away from the bottom of the hoop and and the frequency of oscillations about equilibrium increases. For large spinning rates, $\omega^2 >> g/R$, the equilibrium positions approach $\pm\pi/2$ and the frequency of oscillations approaches the spinning frequency ω.

When the equation of motion has no analytic solution in terms elementary functions, in almost all cases it is still possible to examine the properties of the solutions about positions of equilibrium using the technique that we demonstrated in this example. First you find the positions of equilibrium by setting the accelerations to zero. Then you consider small fluctuations about these positions of equilibrium. This can be done via a Taylor's series expansion, or some equivalent procedure as we did here, to first order about equilibrium. If you only include first order corrections away from equilibrium, then you will have succeeded in linearizing the equation, almost always in the form of a simple harmonic oscillator.