18 Lecture 11-9

18.1 Chapter 7 Lagrange's Equations (con)

18.1.1 Generalized Momenta, Ignorable Coordinates, and Conservation Laws

As we have already mentioned for any system with n generalized coordinates $q_i(i = 1, \cdots, n)$, we refer to the n quantities $\partial \mathcal{L}/\partial q_i = F_i$ as *generalized forces* and $\partial \mathcal{L}/\partial \dot{q}_i = p_i$ as *generalized momenta*. With this terminology the Lagrange equation can be rewritten as

$$F_i \text{ (generalized force)} = \frac{d}{dt} p_i \text{ (generalized momentum)}. \tag{1}$$

In particular if the Lagrangian is independent of a particular coordinate q_i, the $F_i = \partial \mathcal{L}/\partial q_i = 0$ and the corresponding generalized momentum p_i is constant. In general, the generalized forces and momenta are not the same as the usual forces and momenta. As we saw in the example where we used cylindrical coordinates, the ϕ component of the generalized force was the torque, and the ϕ component of the generalized momentum was the angular momentum. In any case when the Lagrangian is independent of a coordinate q_i, the corresponding generalized momentum is conserved. So when the interaction between two particles only depends on the magnitude of the distance between them and is independent of ϕ the angular momentum is conserved. When the Lagrangian is independent of a coordinate q_i, that coordinate is often referred to as *ignorable* or *cyclic*. Obviously it is a good idea, when possible, to choose coordinates so that as many as possible are ignorable and their corresponding momenta are constant. In fact, this is perhaps the main criteria in choosing generalized coordinates for any given problem.

We can rephrase this by noting that when $\mathcal{L}$ is independent of a coordinate q_i, then $\mathcal{L}$ is unchanged, or invariant when q_i varies. So we can also say that if $\mathcal{L}$ is invariant under changes in a coordinate q_i, then the corresponding momentum is conserved. This connection between the invariance of $\mathcal{L}$ and certain conservation laws is the first of several similar results relating invariance under transformations (translations, rotations, etc.) to conservation laws. These results are known as Noether's theorem.

Conservation of Total Momentum We already know from Newtonian mechanics that the total momentum of an isolated system of N particles is conserved. Now let's examine this result from a Lagrangian point of view. One of the most prominent features of an isolated system is that it is *translationally invariant*. By this we mean that if you transport all N particles through the same distance $\overrightarrow{\epsilon}$, then nothing physically significant about the system changes. We have already seen this when we discussed the internal interaction between two particles where the interaction potential depended on $\overrightarrow{r}_1 - \overrightarrow{r}_2$. If we were to transport this system then $\overrightarrow{r}_1$ and $\overrightarrow{r}_2$ would be replaced with $\overrightarrow{r}_1 + \overrightarrow{\epsilon}$ and

$\overrightarrow{r}_2 + \overrightarrow{\epsilon}$ respectively. However the interaction which only depends on $\overrightarrow{r}_1 - \overrightarrow{r}_2$ would remain unchanged. If the system contained N particles then we would replace every position $\overrightarrow{r}_\alpha$ with $\overrightarrow{r}_\alpha + \overrightarrow{\epsilon}$, that is

$$\overrightarrow{r}_1 \to \overrightarrow{r}_1 + \overrightarrow{\epsilon}, \quad \overrightarrow{r}_2 \to \overrightarrow{r}_2 + \overrightarrow{\epsilon}, \quad \cdots, \quad \overrightarrow{r}_N \to \overrightarrow{r}_N + \overrightarrow{\epsilon} \tag{2}$$

As with the two particle interaction potential, the potential energy for the N particle system must be unaffected by this displacement, so that

$$U\left(\overrightarrow{r}_1 + \overrightarrow{\epsilon}, \overrightarrow{r}_2 + \overrightarrow{\epsilon}, \cdots, \overrightarrow{r}_N + \overrightarrow{\epsilon}, t\right) = U\left(\overrightarrow{r}_1, \overrightarrow{r}_2, \cdots, \overrightarrow{r}_N, t\right). \tag{3}$$

To state this more succinctly,

$$\delta U = 0, \tag{4}$$

where δU denotes the change in U under the translation in equation (3). Clearly, the velocities are unchanged and $\delta T = 0$ under this translation as well. Therefore

$$\delta \mathcal{L} = 0 \tag{5}$$

under this translation. Since this result is true for any displacement $\overrightarrow{\epsilon}$, then it must be true for an infinitesimal translation in the x direction. Under that translation all of the x coordinates $x_1, x_2, \cdots, x_N$ increase by ϵ, while the y and z coordinates remain unchanged. For this translation

$$\delta \mathcal{L} = \epsilon \frac{\partial \mathcal{L}}{\partial x_1} + \cdots + \epsilon \frac{\partial \mathcal{L}}{\partial x_N} = 0. \tag{6}$$

This implies that

$$\sum_{\alpha=1}^{N} \frac{\partial \mathcal{L}}{\partial x_\alpha} = 0. \tag{7}$$

Now using Lagrange's equation we can rewrite each derivative as

$$\frac{\partial \mathcal{L}}{\partial x_\alpha} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_\alpha} = \frac{d}{dt} p_{\alpha x},$$

where $p_{\alpha x}$ is the x component of the momentum for particle α. Thus equation (7) becomes

$$\sum_{\alpha=1}^{N} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_\alpha} = \frac{d}{dt} \sum_{\alpha=1}^{N} p_{\alpha x} = \frac{d}{dt} P_x = 0, \tag{8}$$

where P_x is the x component of the total momentum defined by $\overrightarrow{P} = \sum_{\alpha=1}^{N} \overrightarrow{p}$. By choosing the small displacement $\overrightarrow{\epsilon}$ successively in the y and z directions, we can prove the same result for the y and z components. Thus we can reach the conclusion that, provided the Lagrangian is unchanged by the translation in equation (3), the total momentum of the N particles is conserved. This result is an example of Noether's theorem.

Conservation of Energy   As time advances the Lagrangian, $\mathcal{L}\left(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n, t\right)$, evolves in time, both because of any explicit dependence on t and the implicit dependence of t through the q's and $\dot{q}$'s. Thus via the chain rule

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial t} + \sum_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial q_i} \dot{q}_i + \sum_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \ddot{q}_i. \tag{9}$$

Now from Lagrange's equation we note that

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{d}{dt} p_i = \dot{p}_i,$$

where p_i is the generalized momentum associated with the coordinate q_i. Substituting p_i or $\dot{p}_i$ where appropriate in equation (9) results in

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial t} + \sum_{i=1}^{n} \dot{p}_i \dot{q}_i + \sum_{i=1}^{n} p_i \ddot{q}_i,$$

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial t} + \frac{d}{dt} \sum_{i=1}^{n} p_i \dot{q}_i. \tag{10}$$

For many interesting systems, the Lagrangian does not depend explicitly on t so that $\partial \mathcal{L}/\partial t = 0$. When this is the case we have

$$\frac{d}{dt} \left(\sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L} \right) = 0, \tag{11}$$

and the quantity $\sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L}$ is conserved. The student should recognize this as equivalent to the first integral to the Euler Lagrange equation that we investigated in our study of variational calculus. Now this quantity is so important that it has its own symbol,

$$\mathcal{H} = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L}, \tag{12}$$

and is called the **Hamiltonian**. With these definitions we can now proclaim that

> If the Lagrangian $\mathcal{L}$ does not depend explicitly
> on time, then the Hamilitonian $\mathcal{H}$ is conserved.

The discovery of any conservation law is a significant event and is enough to justify saying that the Hamiltonian is an important quantity. In fact, it goes much further than this. The Hamiltonian $\mathcal{H}$ is the basis of the Hamiltonian formulation of mechanics (as well as quantum mechanics, many body formalisms

in the field of condensed matter physics, etc.), in just the same way that $\mathcal{L}$ is the basis of Lagrangian mechanics.

The main significance, for now at least, of the Hamiltonian is that in many situations it is in fact just the total energy of the system. Specifically, provided the relation between the generalized coordinates and the Cartesian coordinates is time independent

$$\vec{r}_\alpha = \vec{r}_\alpha (q_1, \cdots, q_n), \tag{13}$$

then the Hamiltonian is just the total energy,

$$\mathcal{H} = T + U. \tag{14}$$

To prove this let us, we will first express the total kinetic energy $T = \frac{1}{2} \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha^2$ in terms of the generalized coordinates $q_1, \cdots, q_n$. Using the chain rule we find

$$\dot{\vec{r}}_\alpha = \sum_{i=1}^{n} \frac{\partial \vec{r}_\alpha}{\partial q_i} \dot{q}_i. \tag{15}$$

Forming the scaler product of this equation with itself we find

$$\dot{\vec{r}}_\alpha^2 = \sum_{i=1}^{n} \frac{\partial \vec{r}_\alpha}{\partial q_i} \dot{q}_i \cdot \sum_{j=1}^{n} \frac{\partial \vec{r}_\alpha}{\partial q_j} \dot{q}_j, \tag{16}$$

where we used a different index for the second sum to avoid confusion. The kinetic energy is now a triple sum which we can reorganize as

$$T = \frac{1}{2} \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha^2 = \frac{1}{2} \sum_\alpha m_\alpha \sum_{i=1}^{n} \frac{\partial \vec{r}_\alpha}{\partial q_i} \dot{q}_i \cdot \sum_{j=1}^{n} \frac{\partial \vec{r}_\alpha}{\partial q_j} \dot{q}_j,$$

$$T = \frac{1}{2} \sum_{i,j} \left[\sum_\alpha m_\alpha \frac{\partial \vec{r}_\alpha}{\partial q_i} \cdot \frac{\partial \vec{r}_\alpha}{\partial q_j} \right] \dot{q}_i \dot{q}_j = \frac{1}{2} \sum_{i,j} A_{ij} \dot{q}_i \dot{q}_j, \tag{17}$$

where we have defined

$$A_{ij} = A_{ij} (q_1, \cdots, q_n) = \sum_\alpha m_\alpha \frac{\partial \vec{r}_\alpha}{\partial q_i} \cdot \frac{\partial \vec{r}_\alpha}{\partial q_j}. \tag{18}$$

We can now evaluate the generalized momentum p_i by differentiating equation (17) with respect to $\dot{q}_i$,

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i} \frac{1}{2} \sum_{lk} A_{lk} \dot{q}_l \dot{q}_k,$$

$$p_i = \frac{1}{2} \sum_{lk} A_{lk} \dot{q}_k \delta_{li} + \frac{1}{2} \sum_{lk} A_{lk} \dot{q}_l \delta_{ki} = \frac{1}{2} \sum_k A_{ik} \dot{q}_k + \frac{1}{2} \sum_l A_{li} \dot{q}_l. \tag{19}$$

where δ_{ij} is the Kronecker delta with the property that $\delta_{ij} = 1$ when $i = j$ and is zero otherwise. Now the dummy summation indices can be set to j.so that

$$p_i = \frac{1}{2}\sum_j A_{ij}\dot{q}_j + \frac{1}{2}\sum_j A_{ji}\dot{q}_j = \sum_j A_{ij}\dot{q}_j, \tag{20}$$

where this last step is possible as can be seen from the defining equation (18) A_{ij} is symmetric in its indices, $A_{ij} = A_{ji}$. Now returning to the definition of the Hamiltonian we can rewrite the sum over $p_i\dot{q}_i$ as

$$\sum_i p_i\dot{q}_i = \frac{1}{2}\sum_{,j} A_{ij}\dot{q}_j\dot{q}_i = 2T. \tag{21}$$

Therefore

$$\mathcal{H} = \sum_i p_i\dot{q}_i - \mathcal{L} = 2T - (T - U) = T + U. \tag{22}$$

Thus we see that, provided the transformation between the Cartesian coordinates and the generalized coordinates is time independent, the Hamiltonian $\mathcal{H}$ is just the total energy of the system.

We have already proved that if the Lagrangian is independent of time then the Hamiltonian is conserved. So we now see that the time independence of the Lagrangian, together with the time independence of the transformation between the Cartesian coordinates and the generalized coordinates, implies the conservation of energy. We can rephrase the time independence of the Lagrangian $\mathcal{L}$ by saying that $\mathcal{L}$ is unchanged by translations in time, $t \to t + \epsilon$. Thus the result that we have just proved is that the invariance of $\mathcal{L}$ under a time translation is related to the conservation of energy in much the same way that the invariance of $\mathcal{L}$ under a spatial translation is related to the conservation of momentum. Both of these results are manifestations of Noether's famous theorem.

Example - Bead on a Spinning Wire Hoop (again) At this point, the student may be tempted to ask, "When is the Hamiltonian conserved and the energy isn't?" It is a fair question, and an excellent example to consider
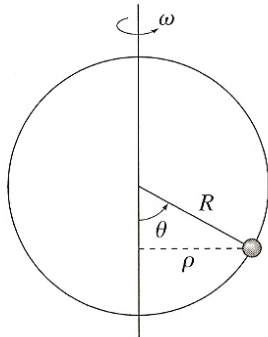
Figure 7.6 A bead is free to move around a frictionless wire hoop, which is spinning at a fixed rate ω.

again is the bead on a spinning hoop in a uniform gravitational field as shown in figure 7.6. We previously noted that the kinetic energy and potential energies are

$$T = \frac{1}{2}mR^2 \left(\dot{\theta}^2 + \sin^2\theta\omega^2 \right) \quad \text{and} \quad U = mgR\left(1 - \cos\theta\right). \tag{23}$$

The Lagrangian is simply $\mathcal{L} = T - U$ and from equation (23) it has no explicit dependence on time. This means that the Hamiltonian must be conserved. Now the Hamiltonian for this system is

$$\mathcal{H} = p_\theta \dot{\theta} - \mathcal{L}, \tag{24}$$

where

$$p_\theta = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = mR^2\dot{\theta}. \tag{25}$$

Simply substituting this result into the expression for the Hamiltonian yields

$$\mathcal{H} = mR^2\dot{\theta}^2 - \frac{1}{2}mR^2 \left(\dot{\theta}^2 + \sin^2\theta\omega^2 \right) + mgR\left(1 - \cos\theta\right),$$

$$\mathcal{H} = \frac{1}{2}mR^2 \left(\dot{\theta}^2 - \sin^2\theta\omega^2 \right) + mgR\left(1 - \cos\theta\right). \tag{26}$$

This expression, the Hamiltonian, is the quantity which is conserved for this system.

Now we need to compare this with the total energy for the bead on the spinning wire. The energy is given by $T + U$, or

$$E = \frac{1}{2}mR^2 \left(\dot{\theta}^2 + \sin^2\theta\omega^2 \right) + mgR\left(1 - \cos\theta\right). \tag{27}$$

These are not equivalent quantities as the term $mR^2\sin^2\theta\omega^2/2$ appears with a plus sign in the expression for the energy and a negative sign in the expression for the Hamiltonian. The physics is pretty clear, to conserve energy, $E = const$, the hoop must slow down as the bead rises through the gravitational field, and this is seen in the expression for the energy. But that is not the system described by the Hamiltonian, for the system that we have been considering the hoop is rotating at a constant rate, ω, independent of the height of the bead. What is happening here is while the motor is keeping the hoop spinning at a constant rate it is compensating for the changes in energy as the bead changes its position on the hoop. In other words the spinning hoop in this example is not an isolated system. So while the total energy of the bead on the spinning hoop combined with the energy supplied (or removed) by the motor is conserved, the energy of just the bead on the hoop is not. For this system of a bead on a spinning hoop, it is the Hamiltonian that is conserved while the energy isn't.

The proof that equated the Hamiltonian with the energy assumed that the relations between the Cartesian coordinates and the generalized coordinates are time independent

$$\overrightarrow{r}_\alpha = \overrightarrow{r}_\alpha \left(q_1, \cdots, q_n \right).$$

Whereas the relationships between the Cartesian coordinates and the single generalized coordinate for the system of interest are

$$x = R\sin\theta\cos\omega t, \quad y = R\sin\theta\sin\omega t, \quad z = R\left(1 - \cos\theta\right), \qquad (28)$$

where x and y are measured from the axis of rotation and z is measured from the bottom of the hoop. The time dependence of the relationship in equation (28) are obvious, so that the initial premise which required that such a relationship was time independent was violated. Hence the Hamiltonian and the energy did not have to be equivalent.

If the angle around the axis of rotation were a new generalized coordinate, ϕ, rather than ωt with ω fixed, then this time dependence would disappear and the Hamiltonian would be equal to the total energy. To verify this is left as an exercise for the student. From a physical point of view, this coordinate would allow for the spinning hoop to change its rate as the bead moved around the hoop thus conserving energy.