

## 19 Lecture 11-9

### 19.1 Chapter 7 Lagrange's Equations (con)

#### 19.1.1 Lagrange Multipliers and Constraint Forces

The powerful method of Lagrange multipliers finds applications in several areas of physics and takes on different appearances in different contexts. Here as an introduction to the concept, we shall only consider its applications in Lagrangian mechanics. Additionally to keep the discussion relatively simple, we shall consider two dimensional systems with only one degree of freedom.

One of the main strengths of Lagrangian mechanics is that we don't need to know anything about the constraint forces which often results in a dramatic simplification of the problem. However, there are times when one actually needs to know these forces. For example, the designer of a roller coaster needs to know the normal forces of the track on the car in order to design the roller coaster track safely. In a case where we need to know a constraint force we do not choose the generalized coordinates  $q_1, \dots, q_n$  all of which can be independently varied. Instead we use a larger number of coordinates and use Lagrange multipliers to handle the constraints.

To illustrate this procedure, we'll consider a system with just two rectangular coordinates  $x$  and  $y$ , which are restricted by a *constraint equation* of the form

$$f(x, y) = \text{const.} \quad (1)$$

For example, we could consider a simple pendulum with just one degree of freedom. With the standard Lagrangian approach we would use the generalized coordinate  $\phi$ , the angle between the pendulum and the vertical, and avoid any discussion of the constraints. If instead, we chose to use the Cartesian coordinates  $x$  and  $y$ , then we must recognize that these coordinates are not independent. They satisfy the constraint

$$f(x, y) = \sqrt{x^2 + y^2} = l, \quad (2)$$

where  $l$  is the length of the pendulum. The method of Lagrange multipliers allows us to accommodate this constraint, determine the time dependence of  $x$  and  $y$ , and find the tension in the rod. As a second example, consider the Atwood machine. In the standard Lagrangian approach we had one generalized coordinate  $x$ , the position of the mass  $m_1$ . However, we could instead use coordinates of both masses,  $x$  and  $y$ , where  $y$  is the position of mass  $m_2$ , while using the constraint that the length of the string is constant. This constraint is expressed as

$$f(x, y) = x + y = l - \pi R = \text{const.}, \quad (3)$$

where  $l$  is the length of the string and  $R$  is the radius of the pulley.

We again start with Hamilton's principle that the action integral

$$S = \int \mathcal{L}(x, \dot{x}, y, \dot{y}) dt \quad (4)$$

is stationary when taken along the path of the particle. If the “right” path is denoted by  $x(t)$  and  $y(t)$  then a small displacement to a neighboring “wrong” path is given by

$$x(t) \rightarrow x(t) + \delta x(t) \quad \text{and} \quad y(t) \rightarrow y(t) + \delta y(t). \quad (5)$$

Then, provided the displacement is consistent with the constraint, the action integral is unchanged and  $\delta S = 0$ . Evaluating  $\delta S$  in the usual way by making a Taylor’s series expansion about the right path we find

$$\delta S = \int \left( \frac{\partial \mathcal{L}}{\partial x} \delta x + \frac{\partial \mathcal{L}}{\partial \dot{x}} \delta \dot{x} + \frac{\partial \mathcal{L}}{\partial y} \delta y + \frac{\partial \mathcal{L}}{\partial \dot{y}} \delta \dot{y} \right) dt. \quad (6)$$

The second and fourth terms can be integrated by parts (as usual) and taking into consideration that  $\delta x$  and  $\delta y$  vanish at the endpoints we are left with

$$\delta S = \int \left( \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) \delta x dt + \int \left( \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}} \right) \delta y dt = 0. \quad (7)$$

If the displacements  $\delta x$  and  $\delta y$  were arbitrary we would have two Lagrange equations, one for  $x$  and one for  $y$ . This is completely analogous to the case for unconstrained systems or constrained systems using generalized coordinates that we considered earlier. Now however there are constraints and  $\delta x$  and  $\delta y$  are related through a constraint equation (1). Via the constraint equation  $f(x, y) = \text{const}$ , hence the displacement from the “right” path in equation (5) which is consistent with the constraint leaves  $f(x, y)$  unchanged so

$$\delta f = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y = 0. \quad (8)$$

Since this is zero, we can multiply it by an arbitrary function  $\lambda(t)$ , which we shall call the *Lagrange multiplier*, and add it to the integrand in equation (7) without changing the integral. Therefore

$$\delta S = \int \left( \frac{\partial \mathcal{L}}{\partial x} + \lambda(t) \frac{\partial f}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) \delta x dt + \int \left( \frac{\partial \mathcal{L}}{\partial y} + \lambda(t) \frac{\partial f}{\partial y} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}} \right) \delta y dt = 0. \quad (9)$$

Since our Lagrange multiplier is arbitrary we can choose it so that the first integral above is zero. Since the small displacement  $\delta x$  from the correct path is also arbitrary we know from previous arguments that

$$\frac{\partial \mathcal{L}}{\partial x} + \lambda(t) \frac{\partial f}{\partial x} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}}. \quad (10)$$

This modified Lagrange equation differs from the usual Lagrange equation only by the additional  $\lambda \partial f / \partial x$  term on the left hand side. Now since the sum of the two integrals vanishes, perforce the second integral must also vanish. Since it

is equal to zero for any choice of  $\delta y$  consistent with equation (8) we have the second slightly modified Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial y} + \lambda(t) \frac{\partial f}{\partial y} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}}. \quad (11)$$

We now have two modified Lagrange equations for the two unknown functions  $x(t)$  and  $y(t)$  as well as the Lagrange multiplier,  $\lambda(t)$ . To find three unknown functions we need three equations. Fortunately there is a third equation, namely the constraint equation

$$f(x, y) = \text{const.}$$

These three equations are, in principle at least, sufficient to determine the coordinates  $x(t)$  and  $y(t)$  along the particle's path and the Lagrange multiplier,  $\lambda(t)$ .

As an additional note on the constraint, not only is  $\delta f = 0$ , but so is the quantity  $f(x, y) - \text{const} = 0$ . So you might be tempted to ask why don't we just add the quantity  $\lambda(t)(f(x, y) - \text{const})$  to the Lagrangian initially, as that is also zero and won't effect the action integral, yet still provide the necessary constraint between the deviations  $\delta x$  and  $\delta y$ ? You could then derive the Euler-Lagrange equations for the action integral with this modification to the Lagrangian. In fact you can do this, and if you did, you would be led to the same modified Lagrange equations. Additionally, since in the derivation of the Euler-Lagrange equations we are interested in small displacements from the correct path, the constant makes no contribution and you only need to add  $\lambda(t)f(x, y)$ . A few simple steps by the student verifies that the same modified Lagrange equations are an immediate consequence of such an approach. From this point forward when using Lagrange multipliers will always consider the modified Lagrangian

$$\mathcal{L}' = \mathcal{L} + \lambda f(x, y)$$

to obtain our modified Lagrangian equations of motion.

Now the modified Lagrange equations, (10) and (11), are still in the form

$$F_i \text{ (generalized force)} = \frac{d}{dt} p_i \text{ (generalized momentum),}$$

hence the components generalized force now takes the form

$$F_x = \frac{\partial \mathcal{L}}{\partial x} + \lambda(t) \frac{\partial f}{\partial x}, \quad \text{and} \quad F_y = \frac{\partial \mathcal{L}}{\partial y} + \lambda(t) \frac{\partial f}{\partial y}. \quad (12)$$

Since  $\partial \mathcal{L} / \partial x = -\partial U / \partial x$  is the  $x$  component of the nonconstraint force and  $dp_x / dt$  is the sum of all the forces on the particle including both the nonconstraint and constraint forces, we must have

$$F_x^{cstr} = \lambda(t) \frac{\partial f}{\partial x}, \quad (13)$$

with a corresponding result for the  $y$  component.

**Atwood Machine** As an example of how these ideas work we will use the concepts of Lagrange multipliers applied to the Atwood machine in figure 7.7.

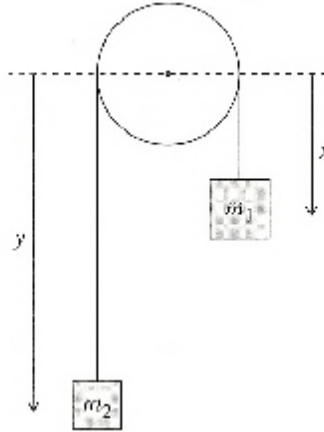


Figure 7.7 Atwood machine (again).

In terms of unconstrained coordinates the Lagrangian is

$$\mathcal{L} = T - U = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 + m_1gx + m_2gy. \quad (14)$$

The constraint equation is simply

$$f(x, y) = x + y = \text{const.} \quad (15)$$

The modified Lagrange equation for  $x$  is

$$\frac{\partial \mathcal{L}}{\partial x} + \lambda(t) \frac{\partial f}{\partial x} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} \rightarrow m_1g + \lambda = m_1\ddot{x}, \quad (16)$$

and similarly the modified Lagrange equation for  $y$  is

$$\frac{\partial \mathcal{L}}{\partial y} + \lambda(t) \frac{\partial f}{\partial y} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}} \rightarrow m_2g + \lambda = m_2\ddot{y}, \quad (17)$$

From the constraint equation we know that  $\ddot{y} = -\ddot{x}$ . Then subtracting the equation for the  $y$  coordinate from that for the  $x$  coordinate eliminates the Lagrange multiplier with the result

$$(m_1 - m_2)g = (m_1 + m_2)\ddot{x} \rightarrow \ddot{x} = (m_1 - m_2)g / (m_1 + m_2). \quad (18)$$

To better understand the two modified Lagrange equations, (16) and (17), it is useful to compare them with the two Newtonian equations. Newton's second law for  $m_1$  is

$$m_1g - T = m_1\ddot{x}, \quad (19)$$

where  $T$  is the tension in the string. Similarly for  $m_2$  Newton's law reads

$$m_2 g - T = m_2 \ddot{y}, \quad (20)$$

where again  $T$  is the tension in the string. These are precisely the two modified Lagrange equations, (16) and (17), with the Lagrange multiplier identified as the negative of the constraint force,

$$F_x^{cstr} = \lambda(t) \frac{\partial f}{\partial x} = -T. \quad (21)$$

Note that minus sign occurred because we measured both  $x$  and  $y$  to be positive in the downward direction while the tension on the string was directed upward. In general the constraint force is given by  $F^{cstr} = \lambda(t) \partial f / \partial x$ , but in this simple example  $\partial f / \partial x = 1$ .

**Catenary** Before we leave this section on Lagrange multipliers we should note that this technique can be applied to a variety of variational problems. As an example we might wish to determine the shape of a hanging rope of length  $L$ , see figure 7.8. The approach would be to minimize the potential energy of

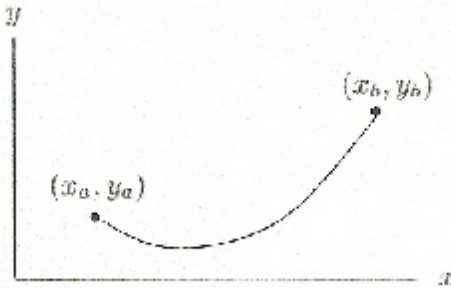


Figure 7.8 A rope of length  $L$  hanging between  $(x_a, y_a)$  and  $(x_b, y_b)$ .

the rope subject to the constraint that its length is fixed. If we define  $\mu$  to be the mass per unit length of the rope,  $\mu = m/L$ , then the potential energy of the rope (relative to the  $x$  axis) is

$$U = \mu g \int_a^b y ds = \mu g \int_a^b y \sqrt{1 + y'^2} dx, \quad (22)$$

subject to the constraint that

$$L = \int_a^b ds = \int_a^b \sqrt{1 + y'^2} dx. \quad (23)$$

Simply adding the function  $f$ ,

$$f = \lambda \left( \int_a^b \sqrt{1 + y'^2} dx - L \right), \quad (24)$$

where  $\lambda$  is a Lagrange multiplier to the potential energy results in

$$U = \mu g \int_a^b y \sqrt{1 + y'^2} dx + \lambda \int_a^b \sqrt{1 + y'^2} dx, \quad (25)$$

where we have dropped the constant  $L$  since it will not effect the resulting Euler Lagrange equations. Redefining the Lagrange multiplier via  $\lambda = -\mu g y_0$  results in the functional  $U$  being expressed as

$$U = \mu g \int_a^b (y - y_0) \sqrt{1 + y'^2} dx. \quad (26)$$

Changing variables to  $z = y - y_0$ , we find  $z' = y'$  and

$$U = \mu g \int_a^b z \sqrt{1 + z'^2} dx. \quad (27)$$

This is exactly the integrand that we analyzed while discussing the soap film problem in which we minimized the surface area of revolution. As the great physicist Feynman proclaimed, "The same equations have the same solutions". So the solution to this problem is the catenary,  $z = c \cosh [(x - x_0) / c]$  or

$$y = y_0 + c \cosh \left( \frac{x - x_0}{c} \right),$$

where  $c$ ,  $x_0$ , and  $y_0$  are determined by the requirements that the rope passes through  $(x_a, y_a)$  and  $(x_b, y_b)$  while holding the length of the rope fixed.

Another interesting problem that can be solved by the method of Lagrange multipliers is that of finding the curve that provides maximum area under a rope of length  $L$ . This is left as an exercise for the student.

### 19.1.2 Conclusion

As we have noted the Lagrangian version of classical mechanics has three main advantages over the Newtonian version. In particular when there are constraints one can greatly simplify the problem with the appropriate choice of independent generalized coordinates. The next task is to determine the kinetic and potential energies in an inertial frame in terms of these coordinates and write down the Lagrangian. The equations of motion then follow automatically as the Lagrange equations which are

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad [i = 1, \dots, n]. \quad (28)$$

These equations take the same form independent of the generalized coordinates. However, there is no guarantee that these equations can be solved, and in most real problems they are not, requiring numerical solution or at least some approximations before they can be solved analytically.

Even in the examples that we have considered which were only moderately complicated, finding the equations of motion by Lagrange's method is remarkably easier than by using Newton's second law. Indeed, some purists object that the Lagrangian approach makes life too easy, removing the need to think about the physics behind the problem.

Often the physics can be formulated from a Hamiltonian point of view where the Hamiltonian is defined as

$$\mathcal{H} = \sum_{i=1}^n p_i \dot{q}_i - \mathcal{L}. \quad (29)$$

This will be considered in more detail in Chapter 13. For now it is useful to know that the Hamiltonian is conserved as long as the Lagrangian has no explicit time dependence. Additionally if the relation between the Cartesian coordinates and the generalized coordinates is time independent then the Hamiltonian and the total energy are equal. But if they are not, then the Hamiltonian will still be a constant of the motion while the energy may not be!

Finally, although the Lagrange equations eliminate the need to determine constraint forces, it is possible in those special cases to determine them using the method of Lagrange multipliers