2 Lecture 9-28

2.1 Chapter 1 Newton's Laws of Motion (con)

2.1.1 Newton's Third Law

If you are leaning against a wall, it is clear that the wall is exerting a force back onto you. This is often stated as, *for every action there is an equal and opposite reaction.* To be more precise Newton's third law is stated as, "if object 1 exerts a force $\overrightarrow{F}_{21}$ on object 2, then object 2 always exerts an equal and opposite reaction force, $\overrightarrow{F}_{12}$, on object 1", or

$$\boxed{\overrightarrow{F}_{21} = -\overrightarrow{F}_{12}} \tag{1}$$

Think of the gravitational force between the Earth and the Moon.

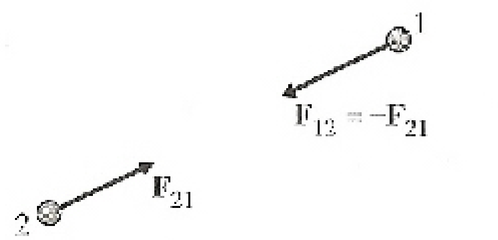

Figure 1-2. Newton's third law states that the reaction force exerted on object 1 by object 2 is equal and opposite to the force exerted by 2 on 1, i.e. $F_{12} = -F_{21}$

As an example we will consider two particles. Assume that an external force is present, and they interact with each other as well. The net force, $\overrightarrow{F}_1$ on particle 1 is

$$\overrightarrow{F}_1 = \overrightarrow{F}_{12} + \overrightarrow{F}_1^{\text{ext}} = \dot{\overrightarrow{p}}_1, \tag{2}$$

where $\dot{\overrightarrow{p}}_1$ is the rate of change in the momentum of particle 1 and similarly

$$\overrightarrow{F}_2 = \overrightarrow{F}_{21} + \overrightarrow{F}_2^{\text{ext}} = \dot{\overrightarrow{p}}_2. \tag{3}$$

Defining the total momentum of the system as $\overrightarrow{P} = \overrightarrow{p}_1 + \overrightarrow{p}_2$, then the rate of change of the total momentum is

$$\dot{\overrightarrow{P}} = \dot{\overrightarrow{p}}_1 + \dot{\overrightarrow{p}}_2 = \overrightarrow{F}_{12} + \overrightarrow{F}_1^{\text{ext}} + \overrightarrow{F}_{21} + \overrightarrow{F}_2^{\text{ext}}. \tag{4}$$

Because of Newton's third law the internal forces cancel and

$$\dot{\overrightarrow{P}} = \overrightarrow{F}_1^{\text{ext}} + \overrightarrow{F}_2^{\text{ext}} = \overrightarrow{F}^{\text{ext}}, \tag{5}$$

where we have defined

$$\vec{F}^{\text{ext}} = \vec{F}_1^{\text{ext}} + \vec{F}_2^{\text{ext}}. \tag{6}$$

This is an important result as it asserts that if there are no external forces, $\vec{F}^{\text{ext}} = 0$, then $\dot{\vec{P}} = 0$, and the total momentum for the pair of particles is conserved. Additionally, the rate of change for the total momentum of the system is determined only by the external force acting on the pair of particles.

The analysis for a system of N particles is a straightforward extension of that used for a two particle system. Consider a particle designated by α. The net force on this particle given by

$$\vec{F}_\alpha = \sum_{\beta \neq \alpha}^{N} \vec{F}_{\alpha\beta} + \vec{F}_\alpha^{\text{ext}} = \dot{\vec{p}}_\alpha. \tag{7}$$

Here the sum over β includes all of the particles other than the α particle as it does not exert a force on itself. This sum is true for any of the N particles in the multiparticle system. The total momentum for this system is given by the sum

$$\vec{P} = \sum_{\alpha=1}^{N} \vec{p}_\alpha. \tag{8}$$

The sum here covers all N particles. Differentiating this expression with respect to time we find

$$\dot{\vec{P}} = \sum_{\alpha=1}^{N} \dot{\vec{p}}_\alpha = \sum_{\alpha=1}^{N} \vec{F}_\alpha \tag{9}$$

From equation (7) this sum is given by

$$\sum_{\alpha=1}^{N} \vec{F}_\alpha = \sum_{\alpha=1}^{N} \sum_{\beta \neq \alpha}^{N} \vec{F}_{\alpha\beta} + \sum_{\alpha=1}^{N} \vec{F}_\alpha^{\text{ext}}. \tag{10}$$

The double sum is a sum over α and β such that all terms in which $\alpha = \beta$ are omitted. Imagine a matrix in which you sum over all of the terms except for those on the diagonal. Since α and β are dummy summation indices we can exchange them and write

$$\sum_{\alpha=1}^{N} \sum_{\beta \neq \alpha}^{N} \vec{F}_{\alpha\beta} = \sum_{\alpha,\beta\ (\alpha \neq \beta)}^{N} \vec{F}_{\alpha\beta} = \sum_{\beta,\alpha\ (\alpha \neq \beta)}^{N} \vec{F}_{\beta\alpha}. \tag{11}$$

In this last step, interchanging the dummy summation indices amounts to simply performing the sum in a different order, but it results in the same total sum. From Newton's third law we know that $\vec{F}_{\alpha\beta} = -\vec{F}_{\beta\alpha}$. Hence this sum must vanish. It might add some insight to note that the matrix given by the components $\vec{F}_{\alpha\beta}$ is an antisymmetric matrix in which $\vec{F}_{\alpha\beta} = -\vec{F}_{\beta\alpha}$ and as in any antisymmetric matrix the diagonal term, $\vec{F}_{\alpha\alpha}$, vanishes. Summing all of the terms in this matrix also vanishes. Since this term vanishes equations (9) and (10) become

$$\dot{\overrightarrow{P}} = \sum_{\alpha=1}^{N} \overrightarrow{F}_{\alpha} = \sum_{\alpha=1}^{N} \overrightarrow{F}_{\alpha}^{\text{ext}}. \tag{12}$$

This is analogous to the result for the two particle system in that the rate of change for the total momentum of all of the particles is given by the sum of the external forces. Clearly in the absence of any external force the total momentum of the N particle system is conserved.

2.1.2 Validity of Newton's Third Law

Within the domain of classical physics, the third law, like the second, is valid to such accuracy that it is taken to be exact. However, it is implicitly assumed that $\overrightarrow{F}_{\alpha\beta}(t) = -\overrightarrow{F}_{\beta\alpha}(t)$. That is the action and reaction forces are taken at the same time. However once relativity becomes important, relative speeds between two inertial frames becomes a reasonable fraction of the speed of light, then events that are simultaneous for one observer are not simultaneous for a different observer. Therefore, the third law cannot be valid once relativity becomes important.

Surprisingly, there is a simple example of a well-known force for which the third law is not true even at nonrelativistic speeds. Consider the two positive charges moving perpendicular to each other as shown in Figure 1-3. Using the

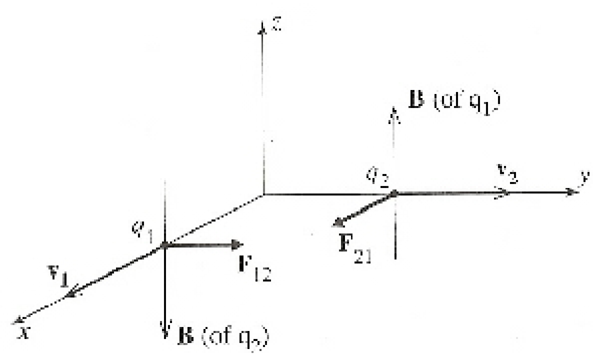

Figure 1-3. Each of the positive charges q_1 and q_2 produce magnetic fields. These fields induce forces F_{12} and F_{21} that do not obey Newton's Third Law.

right hand rule the magnetic field for charged particle 1 traveling in the positive x direction produces a magnetic field pointing in the positive z direction for $y > 0$. Again using the right hand rule this magnetic field produces a force on particle 2 pointing in the positive x direction. A similar analysis shows that the magnetic force on particle 1 is in the positive y direction. Apparently the total momentum for the pair of particles is not conserved. What's up? As it turns out electromagnetic fields also carry momentum and the total momentum of the particles plus that of the electromagnetic fields is conserved. Note that the electrostatic Coulomb force is a central force and does obey the third law. Additionally, as an exercise for the student, it can be shown that the magnetic force when compared to the Coulomb force is reduced by a factor of v^2/c^2. Thus for nonrelativistic velocities, the Coulomb force will dominate and again the third law in the nonrelativistic limit is essentially satisfied.

2.1.3 Newton's Second Law in Cartesian Coordinates

Newton's second law, the equation of motion, is expressed as

$$\overrightarrow{F} = m\ddot{\overrightarrow{r}} \tag{13}$$

Since the basis vectors have no time (or spatial dependence) this expression reduces to three separate equations

$$F_x = m\ddot{x}, \ F_y = m\ddot{y}, \text{ and } F_z = m\ddot{z}. \tag{14}$$

This is a rather elegant result as it shows that Newton's second law in three dimensions is equivalent to three one dimensional equations of the same law.

As an example consider again (as I am sure that you have seen this example previously) the problem of a block sliding down a plane in the presence of

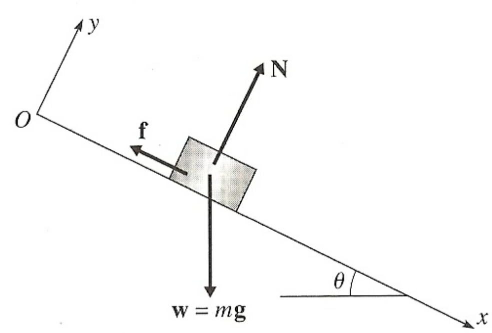

Figure 1-4. A block slides down a slope with incline θ. The forces on the block are the gravitational force, $\overrightarrow{w} = m\overrightarrow{g}$, the normal force of the incline, $\overrightarrow{N}$, and the frictional force of magnitude $f = \mu N$.

friction. With the freedom to choose our reference frame, we find it convenient to choose the x axis to point down the slope and the y axis to be perpendicular to the slope. Note that in this coordinate system the components of the vector $\overrightarrow{g}$ have changed, but the vector itself is unchanged. We will assume that the block starts from rest at both the spatial origin, $x = 0$, and the temporal origin, $t = 0$. Now there is no acceleration in the y direction, which implies that

$$F_y = N - mg\cos\theta = 0. \tag{15}$$

However the block is allowed to accelerate down the plane so that the equation of motion in the x direction is

$$F_x = mg\sin\theta - f = m\ddot{x}. \tag{16}$$

Assuming that the frictional force is proportional to the normal force, $f = \mu N = \mu mg\cos\theta$, then the equation of motion in the x direction reduces to

$$\ddot{x} = g\left(\sin\theta - \mu\cos\theta\right). \tag{17}$$

Since $\ddot{x}$ is independent of time, the solution for a block starting from rest at the origin at $t = 0$ is

$$\dot{x} = g\left(\sin\theta - \mu\cos\theta\right)t, \tag{18}$$

and

$$x = \frac{1}{2}g\left(\sin\theta - \mu\cos\theta\right)t^2. \tag{19}$$

Before we proceed we should stop to examine this solution as a function of θ, the slope of the incline. For angles greater than $\theta = \tan^{-1}\mu$, the block accelerates at ever increasing rates reaching a maximum with an acceleration of g when $\theta = \pi/2$. When $\tan\theta = \mu$ then the frictional force exactly balances the force due to gravity and the block is in equilibrium and remains stationary. What happens when the angle is further reduced so that $\tan\theta < \mu$? Since the role of the frictional force is to retard the motion, it cannot force the block to accelerate back up the incline. This means that the block will remain in equilibrium as the slope is reduced below $\tan^{-1}\mu$ and the frictional force in this regime is given by $f = mg\sin\theta < \mu N$. Another way of stating this, is that the frictional force ranges from zero (when the incline plane is level, perpendicular to the gravitational force) to a maximum given by $f = mg\sin\theta < \mu N$, when $\theta < \tan^{-1}\mu$. Thus our solution is only valid for $\theta \geq \tan^{-1}\mu$.

2.1.4 Newton's Second Law in 2D Polar Coordinates

For circular motion it is often preferable to use polar coordinates which satisfy the coordinate transformation

$$\begin{array}{cc} x = r\cos\phi, & y = r\sin\phi \\ r = \sqrt{x^2+y^2} & \phi = \tan^{-1} y/x \end{array}, \tag{20}$$

where we have to be a little careful with the $\tan^{-1}$ function as it is not single-valued. Just as with Cartesian coordinates it is convenient to make use of a orthonormal basis set. These are defined as

$$\widehat{r} = \overrightarrow{r}/r = \cos\phi\widehat{x} + \sin\phi\widehat{y} = \left(x\widehat{x} + y\widehat{y}\right)/r, \tag{21a}$$

$$\widehat{\phi} = -\sin\phi\widehat{x} + \cos\phi\widehat{y}. \tag{21b}$$

From these definitions, it is clear that our usual position vector $\overrightarrow{r}$ is unchanged, and $\widehat{r}$ is a unit vector pointing in the same direction as $\overrightarrow{r}$. The unit basis vector

$\widehat{\phi}$ is normal to $\widehat{r}$ and points in a direction of increasing ϕ tangent to a circle of radius r centered at the origin. From these definitions it is clear that this basis set depends on ϕ, which implies that if ϕ is time dependent then the basis set $\widehat{r}$ and $\widehat{\phi}$ are time dependent as well.

Now Newton's second law is still expressed as

$$\overrightarrow{F} = m\ddot{\overrightarrow{r}}. \tag{22}$$

To see what form Newton's equation takes in these coordinates we must first find $\dot{\overrightarrow{r}}$. Since this involves the time derivative of both r and $\widehat{r}$, we first note from the definitions of the basis vectors that

$$\frac{d}{dt}\widehat{r} = -\sin\phi\widehat{x}\frac{d\phi}{dt} + \cos\phi\widehat{y}\frac{d\phi}{dt} = \dot{\phi}\widehat{\phi}. \tag{23}$$

Thus $\dot{\overrightarrow{r}}$ is given by

$$\dot{\overrightarrow{r}} = \dot{r}\widehat{r} + r\dot{\phi}\widehat{\phi}. \tag{24}$$

Now $\ddot{\overrightarrow{r}}$ will involve the time derivative of both basis vectors. In a similar manner from the definition of the basis vectors we can determine the time derivative of $\widehat{\phi}$ as

$$\frac{d}{dt}\widehat{\phi} = -\cos\phi\frac{d\phi}{dt}\widehat{x} - \sin\phi\frac{d\phi}{dt}\widehat{y} = -\dot{\phi}\widehat{r}. \tag{25}$$

The acceleration in terms of a polar coordinate basis is then

$$\overrightarrow{a} = \ddot{\overrightarrow{r}} = \ddot{r}\widehat{r} + 2\dot{r}\dot{\phi}\widehat{\phi} + r\ddot{\phi}\widehat{\phi} - r\dot{\phi}^2\widehat{r}. \tag{26}$$

The component form of Newton's second law then takes the form

$$F_r = ma_r = m\left(\ddot{r} - r\dot{\phi}^2\right), \tag{27}$$

and

$$F_\phi = ma_\phi = m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right). \tag{28}$$

These equations in polar coordinates are much more complicated (messy) than those in Cartesian coordinates. You may be thinking that there might not be any reason to use Newton's law in these coordinates. However let's consider the example of a skateboard (or a ball) in semicircular trough, Figure 1-5. In this geometry $r = R$ the radius of the trough and $\dot{r} = \ddot{r} = 0$. Newton's equations then reduce to

$$F_r = ma_r = -mR\dot{\phi}^2, \text{ and } F_\phi = ma_\phi = mR\ddot{\phi}. \tag{29}$$

The equation of interest for this geometrical configuration is in the ϕ direction (much as with the inclined plane where we were interested in the x direction). The force in the ϕ direction comes from the weight of the skateboard and is

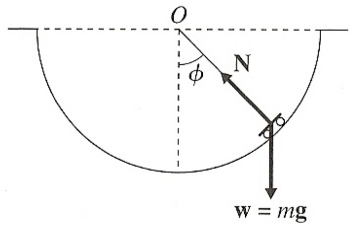

Figure 1-5. A skateboard in a semicircular tourgh of radius R whose position is determined by the angle ϕ as measured from the bottom. The forces on the skate board are its weight, $\overrightarrow{w} = m\overrightarrow{g}$, and the normal force $\overrightarrow{N}$.

given by $F_\phi = -mg\sin\phi$. Newton's equation then takes the form

$$R\ddot{\phi} + g\sin\phi = 0. \tag{30}$$

The solution to this differential equation cannot be expressed in terms of elementary functions (trigonometric functions, exponential functions, etc.). However in the limit of small angular displacement, $\phi << 1$, $\sin\phi \simeq \phi$. In this limit our differential equation describing the motion of the skateboard is

$$\ddot{\phi} + \omega^2\phi = 0, \tag{31}$$

where $\omega^2 = g/R$. By inspection it can be seen that the solution to this equation is given by

$$\phi = A\cos\omega t + B\sin\omega t. \tag{32}$$

The two unknown constants are determined by the initial conditions $\phi\,(t=0) = \phi_o$ and $\dot{\phi}\,(t=0) = 0$. They are $A = \phi_o$ and $B = 0$ so that our final solution is

$$\phi = \phi_o\cos\omega t. \tag{33}$$

Using these initial conditions we could have also directly integrated this equation by using the chain rule

$$\ddot{\phi} = \left(\frac{d}{d\phi}\dot{\phi}\right)\frac{d\phi}{dt} = \dot{\phi}\frac{d}{d\phi}\dot{\phi} = \frac{1}{2}\frac{d}{d\phi}\dot{\phi}^2. \tag{34}$$

Now integrating equation (31), yields

$$\dot{\phi}^2 = -2\omega^2\int_{\phi_o}^{\phi}\phi d\phi = \omega^2\left(\phi_o^2 - \phi^2\right). \tag{35}$$

Using the technique of separation of variables we can rewrite this equation as

$$\begin{aligned} \frac{d\phi}{\sqrt{\phi_o^2 - \phi^2}} &= -\omega dt, \\ \int_{\phi_o}^{\phi}\frac{d\phi}{\sqrt{\phi_o^2 - \phi^2}} &= -\omega\int_0^t dt = -\omega t. \end{aligned} \tag{36}$$

Note that we chose the minus sign when we took the square root as ϕ is initially decreasing from ϕ_o. Now this integral is easily performed with the substitution $\phi = \phi_o \cos\theta$ which leads to $d\phi = -\phi_o \sin\theta d\theta$, and the integral becomes

$$-\int_0^\theta \frac{-\phi_o \sin\theta d\theta}{\phi_o \sin\theta} = -\theta = -\omega t. \tag{37}$$

Since $\phi = \phi_o \cos\theta$, we find the same solution

$$\phi = \phi_o \cos\omega t. \tag{38}$$

There are several points to be made in this approach. The first is the useful trick with the chain rule shown in equation (34). A similar trick, see problems 2.12 and 2.13, is often used to integrate Newton's equation of motion in one dimension whenever the forcing term in independent of both the time and velocity. The second point is that the initial conditions are employed in a natural way as limits in the integration. Additionally, since the solution required two integrations, there were two constants of integration to be determined. In general, the solutions to a second order differential equations (at least all of the ones that we shall encounter in this course) contain precisely two independent constants to be determined by the initial conditions or other constraints determined by the relevant physics of the problem.

As final example of the concepts from chapter 1, consider a cannon that can fire shells in any direction with the same speed v_0. We wish to show that the cannon can hit any object inside a surface defined by

$$z = \frac{v_0^2}{2g} - \frac{g}{2v_0^2} r^2, \tag{39}$$

where z is the vertical coordinate and r is the cylindrical radial coordinate. The EOM in cylindrical coordinates are $\left(\dot{\phi} = \ddot{\phi} = 0\right)$

$$m\frac{d^2 z}{dt^2} = -mg \;\text{ and }\; m\frac{d^2 r}{dt^2} = 0. \tag{40}$$

Depending on the angle of elevation, θ, $v_{z0} = v_0 \sin\theta$ and $v_{r0} = v_o \cos\theta$. Integrating the EOM yields

$$\frac{dz}{dt} = v_0 \sin\theta - gt \;\text{ and }\; \frac{dr}{dt} = v_0 \cos\theta. \tag{41}$$

One more integration yields the time dependence for the coordinates of the projectile,

$$z = (v_0 \sin\theta)\, t - \frac{1}{2} g t^2 \;\text{ and }\; r = (v_0 \cos\theta)\, t. \tag{42}$$

From these two expressions, z as a function of r and θ is

$$z = r\tan\theta - \frac{g}{2v_o^2}\frac{r^2}{\cos^2\theta} = \frac{r}{2\cos^2\theta}\left(2\sin\theta\cos\theta - \frac{g}{v_o^2} r\right). \tag{43}$$

We we see from this solution that $z = 0$ corresponds to $r = 0$, the launch point, and $r = \left(v_o^2 \sin 2\theta\right)/g$, the range of the projectile. The angle that gives the maximum value of z (for a given r) is found from

$$\begin{aligned} \frac{dz}{d\theta} &= \frac{r}{\cos^2\theta} - \frac{g}{v_o^2}\frac{r^2 \sin\theta}{\cos^3\theta} = \frac{r}{\cos^2\theta}\left(1 - \frac{gr}{v_o^2}\tan\theta\right) = 0, \\ \tan\theta_{\max} &= \frac{v_o^2}{gr}. \end{aligned} \tag{44}$$

Remembering the trigonometric identity,

$$\frac{1}{\cos^2\theta} = 1 + \tan^2\theta, \tag{45}$$

we find that $z_{\max}$ (for a given r) is given by

$$\begin{aligned} z_{\max} &= \frac{v_o^2}{g} - \frac{gr^2}{2v_o^2}\left(1 + \frac{v_o^4}{r^2 g^2}\right) \\ z_{\max} &= \frac{v_o^2}{2g} - \frac{g}{2v_o^2} r^2. \end{aligned} \tag{46}$$

This is the highest value of z for a given radial distance r. Clearly the projectile can reach any lower value of z. Note that for $r = 0 \to z_{\max} = v_o^2/2g$, the correct answer for a vertical projectile. For the case $z_{\max} = 0 \to r = v_o^2/g$ which is the maximum range of a projectile launched at an angle $\pi/4$ above horizontal.