3 Lecture 9-30

3.1 Chapter 1 Newton's Laws of Motion (con)

As final example of the concepts from chapter 1, consider a cannon that can fire shells in any direction with the same speed v_0. We wish to show that the cannon can hit any object inside a surface defined by

$$z = \frac{v_0^2}{2g} - \frac{g}{2v_0^2} r^2, \tag{1}$$

where z is the vertical coordinate and r is the cylindrical radial coordinate. The EOM in cylindrical coordinates are $\left(\dot{\phi} = \ddot{\phi} = 0\right)$

$$m\frac{d^2z}{dt^2} = -mg \text{ and } m\frac{d^2r}{dt^2} = 0. \tag{2}$$

Depending on the angle of elevation, θ, $v_{z0} = v_0 \sin\theta$ and $v_{r0} = v_o \cos\theta$. Integrating the EOM yields

$$\frac{dz}{dt} = v_0 \sin\theta - gt \quad \text{and} \quad \frac{dr}{dt} = v_0 \cos\theta. \tag{3}$$

One more integration yields the time dependence for the coordinates of the projectile,

$$z = (v_0 \sin\theta)\, t - \frac{1}{2} g t^2 \quad \text{and} \quad r = (v_0 \cos\theta)\, t. \tag{4}$$

From these two expressions, z as a function of r and θ is

$$z = r\tan\theta - \frac{g}{2v_o^2}\frac{r^2}{\cos^2\theta} = \frac{r}{2\cos^2\theta}\left(2\sin\theta\cos\theta - \frac{g}{v_o^2} r\right). \tag{5}$$

We we see from this solution that $z = 0$ corresponds to $r = 0$, the launch point, and $r = \left(v_o^2 \sin 2\theta\right)/g$, the range of the projectile. The angle that gives the maximum value of z (for a given r) is found from

$$\begin{aligned} \frac{dz}{d\theta} &= \frac{r}{\cos^2\theta} - \frac{g}{v_o^2}\frac{r^2\sin\theta}{\cos^3\theta} = \frac{r}{\cos^2\theta}\left(1 - \frac{gr}{v_o^2}\tan\theta\right) = 0, \\ \tan\theta_{\max} &= \frac{v_o^2}{gr}. \end{aligned} \tag{6}$$

Remembering the trigonometric identity,

$$\frac{1}{\cos^2\theta} = 1 + \tan^2\theta, \tag{7}$$

we find that $z_{\max}$ (for a given r) is given by

$$\begin{aligned} z_{\max} &= \frac{v_o^2}{g} - \frac{gr^2}{2v_o^2}\left(1 + \frac{v_o^4}{r^2 g^2}\right) \\ z_{\max} &= \frac{v_o^2}{2g} - \frac{g}{2v_o^2} r^2. \end{aligned} \tag{8}$$

This is the highest value of z for a given radial distance r. Clearly the projectile can reach any lower value of z. Note that for $r = 0 \rightarrow z_{\max} = v_o^2/2g$, the correct answer for a vertical projectile. For the case $z_{\max} = 0 \rightarrow r = v_o^2/g$ which is the maximum range of a projectile launched at an angle $\pi/4$ above horizontal.

3.2 Chapter 2 Projectiles and Charged Particles

We will now consider projectile motion subject to both gravitational forces and air resistance. Not only will we learn about the effects of air resistance, but also learn some valuable mathematical techniques for solving Newton's equation of motion. We will assume that the frictional drag due to wind resistance, $\overrightarrow{f}$, is always opposite to the velocity, $\overrightarrow{v}$. In doing so we will ignore any lateral component

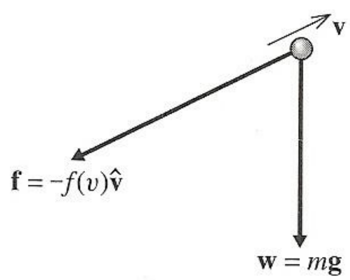

Figure 2-1. A projectile is subject to two forces, the gravitational force $\overrightarrow{w} = m\overrightarrow{g}$, and the force of drag due to air resistance $\overrightarrow{f} = -f\,(v)\,\widehat{v}$.

or lift as that is not the subject of this discussion. Now as we see in Figure 2-1 the frictional force is of the form

$$\overrightarrow{f} = -f(v)\widehat{v}. \tag{9}$$

At lower speeds we will use the approximation (a good one) that the function $f(v)$ is of the form

$$f(v) = bv + cv^2 = f_{\text{lin}} + f_{\text{quad}}, \tag{10}$$

where f_{lin} and f_{quad} are the linear and quadratic terms respectively. As a question, why would you expect there to be no constant term? Now, folks who work in fluid dynamics find it useful to define a dimensionless parameter called the Reynolds number, R where $R = D\rho v/\eta$. Here D is a length measurement that defines the size of the object, e.g. for a sphere D is the diameter, ρ is the density of the fluid that the object is traveling through, v is the velocity of the object relative to the fluid, and η is the viscosity of the fluid. As it turns out, for $R << 1$ the linear term dominates, while for $R >> 1$ the quadratic term dominates.

3.2.1 Linear Air Resistance

In the presence a gravitational field Newton's equation of motion for a particle traveling through a fluid when $R << 1$ becomes

$$m\ddot{\overrightarrow{r}} = m\overrightarrow{g} - b\overrightarrow{v}, \tag{11}$$

where we are measuring y positive in the downward direction. Since no terms depend on the location of the particle, $\overrightarrow{r}$, we can write the equation of motion in terms of $\overrightarrow{v}$ and it becomes

$$m\dot{\overrightarrow{v}} = m\overrightarrow{g} - b\overrightarrow{v}. \tag{12}$$

This is a first order differential equation for $\overrightarrow{v}$ and being linear in $\overrightarrow{v}$ it separates into component equations as

$$m\dot{v}_x = -bv_x \text{ and } m\dot{v}_y = mg - bv_y. \tag{13}$$

The equation for the x component can be easily separated and integrated via

$$\begin{aligned} \int_{v_{x0}}^{v_x} \frac{dv_x}{v_x} &= -\frac{b}{m}\int_0^t dt = -\frac{b}{m}t \\ \ln(v_x/v_{x0}) &= -bt/m \\ v_x(t) &= v_{x0}e^{-t/\tau}, \end{aligned} \tag{14}$$

where $\tau = m/b$. The physics is clear here. The particle starts out at with a horizontal velocity of v_{x0} and then exponentially approaches $v_x = 0$ as $t \to \infty$. To find how the horizontal coordinate depends on time we merely have to integrate this equation and

$$x(t) = v_{x0}\int_0^t e^{-t/\tau}dt = v_{x0}\tau\left(1 - e^{-t/\tau}\right). \tag{15}$$

Here we have assumed that $x(t=0) = 0$. Note that as $t \to \infty$ the position of the particle approaches $v_{x0}\tau$.

The vertical motion satisfies

$$m\dot{v}_y = mg - bv_y, \tag{16}$$

where the positive vertical velocity is down. Eventually the drag will balance out the gravitational pull and $\dot{v}_y \to 0$. When this happens

$$\lim_{t\to\infty} v_y = mg/b = g\tau = v_{\text{ter}}, \tag{17}$$

where we see that the terminal vertical velocity is v_{ter}. We can now rewrite the equation of motion as

$$\begin{aligned} m\dot{v}_y &= -b(v_y - v_{\text{ter}}) \\ \dot{v}_y &= -\frac{b}{m}(v_y - v_{\text{ter}}) = -\frac{1}{\tau}(v_y - v_{\text{ter}}). \end{aligned} \tag{18}$$

Again we can easily integrate this equation and find

$$\begin{aligned}
\int_{v_{y0}}^{v_y} \frac{dv_y}{v_y - v_{\mathrm{ter}}} &= \ln \frac{v_y - v_{\mathrm{ter}}}{v_{y0} - v_{\mathrm{ter}}} = -\frac{t}{\tau} \\
v_y - v_{\mathrm{ter}} &= \left(v_{y0} - v_{\mathrm{ter}}\right) e^{-t/\tau}, \qquad (19)
\end{aligned}$$

or

$$v_y = v_{y0} e^{-t/\tau} + v_{\mathrm{ter}} \left(1 - e^{-t/\tau}\right). \qquad (20)$$

We see from this solution that v_y starts off at v_{y0} and approaches v_{ter} as $t \to \infty$. Both of these limits were what we anticipated. Of some interest is the solution where we drop the particle from rest, i.e. $v_{y0} = 0$. In that case for small times, $t << \tau$ the velocity behaves as $v_y = v_{\mathrm{ter}} t/\tau = (g\tau)\, t/\tau = gt$, just as we would suspect. However for larger times the velocity asymptotically approaches v_{ter}.

In general, assuming that the particles starts at $x = y = 0$, the particle's vertical position is given by

$$\begin{aligned}
y(t) &= v_{\mathrm{ter}} t - \tau \left(v_{y0} - v_{\mathrm{ter}}\right) \left[e^{-t/\tau}\right]_0^t \\
y(t) &= v_{\mathrm{ter}} t + \left(v_{y0} - v_{\mathrm{ter}}\right) \tau \left(1 - e^{-t/\tau}\right). \qquad (21)
\end{aligned}$$

To determine the trajectory of the particle, we need to invert the expression we found for $x(t)$ to find $t(x)$ and then substitute that result into $y(t)$ to find $y(x)$. Before we do that it is a bit more convenient to define y positive vertically upward. The original differential equation remains unchanged except for the term that results from the gravitational acceleration which changes sign. Since $v_{\mathrm{ter}} = mg/b$, this is equivalent to letting $v_{\mathrm{ter}} \to -v_{\mathrm{ter}}$. Using the outlined procedure we find

$$\begin{aligned}
&x/v_{x0}\tau = 1 - e^{-t/\tau} \\
&t = -\tau \ln\left(1 - x/v_{x0}\tau\right). \qquad (22)
\end{aligned}$$

Substituting this result into the expression for $y(t)$ yields

$$y(x) = v_{\mathrm{ter}} \tau \ln\left(1 - x/v_{x0}\tau\right) + \left(v_{y0} + v_{\mathrm{ter}}\right) x/v_{x0} \qquad (23)$$

This equation is not particularly enlightening and hence is plotted in Figure 2-2. Note that y has an asymptote (as shown in the figure) at $x = v_{x0}\tau$.

To solve for the range R, we use the condition $y(R) = 0$, or

$$v_{\mathrm{ter}} \tau \ln\left(1 - R/v_{x0}\tau\right) + \left(v_{y0} + v_{\mathrm{ter}}\right) R/v_{x0} = 0. \qquad (24)$$

We see that one solution occurs at $R = 0$. Clearly this corresponds to the initial starting point and is not of interest here. Since this is a transcendental equation it cannot be solved analytically for $R \neq 0$, and in general it must be solved numerically. That being said physical insight can often be gained by finding approximate analytical solutions. We can do that for the case when

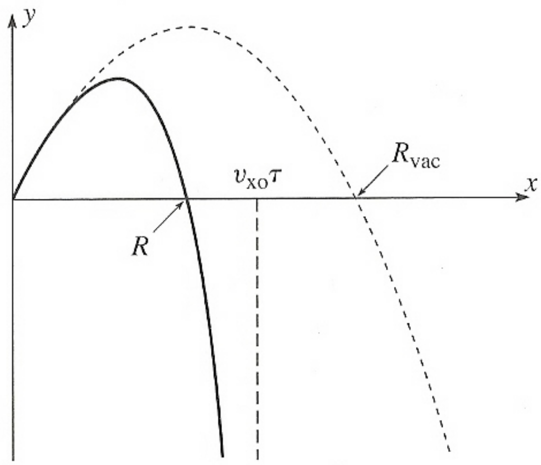

Figure 2-2. The solid line is the trajectory of a projectile subject to linear drag and the dashed line is the trajectory in a vacuum. Initially the curves are similar, but they soon diverge as air resistance slows the projectile downwith a vertical asymptote at $x = v_{xo}t$. The respective ranges are R and $R_{\rm vac}$.

$R << v_{x0}\tau$ which is the case when the drag is small. In this limit we can expand the natural log function. In case you don't remember this expansion and don't feel like taking the necessary derivatives to form the Taylor's expansion then consider the expansion

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + \cdots \\ \int \frac{dx}{1-x} &= -\ln(1-x) = const + x + x^2/2 + x^3/3 + \cdots. \end{aligned}$$

Since the $\ln(1-x)$ vanishes at $x = 0$, the unknown constant must also vanish. Thus the expansion for $\ln(1-x)$ is

$$\ln(1-x) = -\left(x + x^2/2 + x^3/3 + \cdots\right) \tag{25}$$

Expanding the log function to third order in $R/v_{x0}\tau$ and simplifying yields

$$\begin{aligned} -v_{\rm ter}\tau\left[\frac{R}{v_{x0}\tau} + \frac{1}{2}\left(\frac{R}{v_{x0}\tau}\right)^2 + \frac{1}{3}\left(\frac{R}{v_{x0}\tau}\right)^3\right] + (v_{y0} + v_{\rm ter})\frac{R}{v_{x0}} &\simeq 0 \\ -v_{\rm ter}\left[\frac{R}{v_{x0}^2\tau} + \frac{2}{3}\frac{R^2}{v_{x0}^3\tau^2}\right] + 2\frac{v_{y0}}{v_{x0}} &\simeq 0. \end{aligned} \tag{26}$$

Replacing $v_{\rm ter}/\tau$ with g further reduces this to

$$R \simeq 2\frac{v_{y0}v_{x0}}{g} - \frac{2}{3}\frac{R^2}{v_{x0}\tau} \tag{27}$$

The leading term here is the result that you obtain for the range of a projectile in a vacuum. Since, in our approximation, the next term is a small correction we can replace R in that term with the vacuum solution, $R_{\rm vac}$, and find

$$R \simeq R_{\rm vac}\left(1 - \frac{4}{3}\frac{v_{y0}}{g\tau}\right) = R_{\rm vac}\left(1 - \frac{4}{3}\frac{v_{y0}}{v_{\rm ter}}\right). \tag{28}$$

This expression is simple enough that we can make some physical observations. First the drag always reduces the range when compared to the vacuum result. This is true even if we consider higher order terms in the expansion for the log function. The correction depends only on the ratio $v_{y0}/v_{\rm ter}$. In general if $v/v_{\rm ter} << 1$ throughout the trajectory then the effect of linear drag is also small. If however $v/v_{\rm ter}$ is on the order of 1 or even larger then our approximation is no longer valid.

3.2.2 Quadratic Air Resistance

While we can find examples for which the drag of an object is linear with respect to its velocity, notably very small objects or for very small velocities, e.g. the Millikan oil drops, more obvious examples such as baseballs etc. are subject to quadratic drag. For this case the x and y components of the equation of motion are not in general separable. Additionally the equations are nonlinear which are often significantly more complicated than linear differential equations. For these reasons we shall consider purely horizontal or vertical motion.

In the case of purely horizontal motion, Newton's equation of motion is given by

$$m\frac{dv_x}{dt} = -cv_x^2. \tag{29}$$

This equation is easily separated which allows us to obtain the integrals

$$m\int_{v_{x0}}^{v_x} \frac{dv_x}{v_x^2} = -c\int_0^t dt. \tag{30}$$

These integrals are well known and we find

$$m\left(\frac{1}{v_{x0}} - \frac{1}{v_x}\right) = -ct.$$

Solving for v_x yields

$$\begin{aligned} \frac{1}{v_x} &= \frac{1}{m}ct + \frac{1}{v_{x0}} = \frac{1}{v_{x0}}\left(1 + cv_{x0}t/m\right) \\ v_x &= \frac{v_{x0}}{1 + t/\tau}, \end{aligned} \tag{31}$$

where

$$\tau = m/cv_{x0}. \tag{32}$$

This is a different time constant that the one we obtained for the case of linear drag. Here when $t = \tau$ the velocity is reduced by a factor of 2 versus e^{-1}. NTL, both time constants give us a measure of the time required for wind resistance to slow the motion of the object appreciably.

To find the position of the object as a function of time we merely integrate this solution via

$$\begin{aligned} x &= x_0 + \int_0^t \frac{v_{x0}}{1 + t/\tau} dt \\ x &= x_0 + v_{x0}\tau \ln\left(1 + t/\tau\right). \end{aligned} \tag{33}$$

The velocity still goes to zero as $t \to \infty$, but in this case it does so much more slowly. So slow in fact that x increases without limit. Remember however that when the velocity of the particle becomes small enough the drag becomes linear and the velocity will begin to fall off exponentially. Thus no real body can coast to infinity.

For vertical motion Newton's equation of motion is

$$m\frac{dv_y}{dt} = mg - cv_y^2, \tag{34}$$

where we are measuring positive y to be vertically down. Again as in the linear case it is useful to find the terminal velocity. In this situation $dv_y/dt = 0$ (the same as in the linear case) and

$$v_{\text{ter}} = \sqrt{mg/c}. \tag{35}$$

Rewriting the equation of motion in terms of the terminal velocity yields

$$\frac{dv_y}{dt} = \frac{g}{v_{\text{ter}}^2}\left(v_{\text{ter}}^2 - v_y^2\right) \tag{36}$$

We will assume that the object (ball) is dropped from rest. Then using the technique of separation of variables we find

$$\int_0^{v_y} \frac{dv_y}{v_{\text{ter}}^2 - v_y^2} = \frac{g}{v_{\text{ter}}^2}\int_0^t dt. \tag{37}$$

This integrand can be expanded into partial fractions,

$$\frac{1}{v_{\text{ter}}^2 - v_y^2} = \frac{1}{2v_{\text{ter}}}\left(\frac{1}{v_{\text{ter}} - v_y} + \frac{1}{v_{\text{ter}} + v_y}\right), \tag{38}$$

which enables us to perform the integral in a straightforward fashion. The result is

$$\ln\frac{v_{\text{ter}} + v_y}{v_{\text{ter}} - v_y} = \frac{2gt}{v_{\text{ter}}}. \tag{39}$$

Solving for v_y leads to

$$\begin{aligned} v_{\text{ter}} + v_y &= (v_{\text{ter}} - v_y) \exp 2gt/v_{\text{ter}} \\ v_y &= v_{\text{ter}} \frac{e^{2gt/v_{\text{ter}}} - 1}{e^{2gt/v_{\text{ter}}} + 1} = v_{\text{ter}} \tanh gt/v_{\text{ter}}. \end{aligned} \tag{40}$$

For $gt/v_{\text{ter}} << 1$ this expression reduces to

$$v_y = gt, \tag{41}$$

which is what you would expect for a falling object. However, the hyperbolic tangent rapidly approaches 1 as gt/v_{ter} increases beyond 1, so that the velocity of the object quickly reaches its terminal velocity. To find the distance the object has fallen we simply integrate the vertical velocity to find

$$y = \frac{v_{\text{ter}}^2}{g} \ln \cosh gt/v_{\text{ter}} \tag{42}$$