Small Amplitude Fluctuations About Equilibrium

Problem 7.47 (b,c).

(b) The equation of motion for a system with a single generalized coordinate was shown to be of the form

$$A(q)\ddot{q} = -\frac{dU}{dq} - \frac{1}{2}\frac{dA}{dq}\dot{q}^2 . \tag{1}$$

Given a potential energy function (1-D) we have previously shown that the points of equilibrium occur whenever

$$\frac{dU}{dq} = 0.$$

This may occur for a set of q_o's for which $dU/dq = 0$, or whenever U is at a maximum or a minimum. Given an equation of motion such as that in equation (1), equilibrium implies that $\dot{q} = \ddot{q} = 0$. Hence, consistent with our earlier conclusions, equilibrium still requires $dU/dq = 0$.

(c) Now an important aspect of being at equilibrium is whether it is a position of stable or unstable equilibrium. Earlier in the class we demonstrated that a stable equilibrium occurs at a minimum in the potential energy function. From basic calculus this requires

$$\frac{d^2U(q_o)}{dq^2} > 0,$$

where the second derivative is evaluated at a position of equilibrium, $q = q_o$. For small amplitude fluctuations about equilibrium the potential energy can be expanded in a Taylor's series via

$$U(q) \simeq U(q_o) + \frac{dU(q_o)}{dq}(q - q_o) + \frac{1}{2}\frac{d^2U(q_o)}{dq^2}(q - q_o)^2 . \tag{2}$$

Since the first derivative vanishes this expansion reduces to

$$U(q) \simeq U(q_o) + \frac{1}{2}\frac{d^2U(q_o)}{dq^2}(q - q_o)^2 .$$

This expression is analogous to that for a spring stretched (or compressed) from it equilibrium position q_o with an effective spring constant, k_{eff}, being given by

$$k_{eff} = \frac{d^2U(q_o)}{dq^2}.$$

Clearly if the curvature of the potential energy function is positive then $k_{eff} > 0$ and the equilibrium position at q_o is stable.

Now consider the equation of motion, equation (1), for small amplitude fluctuations from equilibrium. In that case $q = q_o + \epsilon$, where ϵ is small. Since q_o is a constant the equation of motion is linearized and reduces to

$$A(q_o)\ddot{\epsilon} = -\left.\frac{dU(q)}{dq}\right|_{q=q_o+\epsilon}. \tag{3}$$

To determine the quantity

$$\left.\frac{dU(q)}{dq}\right|_{q=q_o+\epsilon}$$

for small ϵ we could simplify differentiate the Taylor's expansion, equation (2), above. However it is straightforward to note that a first order Taylor's expansion of dU/dq about q_o yields

$$\left.\frac{dU(q)}{dq}\right|_{q=q_o+\epsilon} = \frac{dU(q_o)}{dq} + \frac{d^2U(q_o)}{dq^2}(q - q_o) = \frac{d^2U(q_o)}{dq^2}\epsilon.$$

Substituting this result into the equation of motion for small amplitude fluctuations, equation (3), yields

$$A(q_o)\ddot{\epsilon} = -\frac{d^2U(q_o)}{dq^2}\epsilon. \tag{4}$$

This equation is analogous to Hook's law and harmonic oscillations occur as long as the curvature of the potential energy function at the equilibrium position q_o is greater than zero, $k_{eff} > 0$. That is the forcing function on the right hand side is negative corresponding to a restoring force.

The equation of motion has an additional property in that we can obtain the frequency of small amplitude oscillations. From equation (4) these are given by

$$\omega^2 = \frac{1}{A(q_o)}\frac{d^2U(q_o)}{dq^2} = \frac{k_{eff}}{m_{eff}},$$

where the effective mass is $m_{eff} = A(q_o)$. So the analogy with Hook's law is complete. For small amplitude oscillations about equilibrium the angular frequency is found from the effective spring constant divided by the effective mass.

Problem 8.13(b,c)

(b) Consider two orbiting particles of reduced mass μ which interact via the potential energy function

$$U(r) = \frac{1}{2}kr^2,$$

where r is the relative distance between them. The effective interaction for two orbiting particles is

$$U_{eff}(r) = U(r) + \frac{1}{2}\frac{L^2}{\mu r^2}.$$

Their equilibrium position for a circular orbit is found from

$$\frac{dU_{eff}(r_o)}{dr} = kr_o - \frac{L^2}{\mu r_o^3} = 0, \tag{5}$$

which has a solution

$$r_o^4 = \frac{L^2}{\mu k}.$$

(c) To determine frequency of small amplitude oscillations about r_o, we consider the equation of motion for the relative coordinate

$$\mu \ddot{r} = -\frac{dU_{eff}(r)}{dr}.$$

As we saw in the solution for 7.47, for small amplitude oscillations, $r = r_o + \epsilon$, this becomes

$$\mu \ddot{\epsilon} = -\frac{dU_{eff}(r_o + \epsilon)}{dr} = -\frac{dU_{eff}^2(r_o)}{dr^2}\epsilon. \tag{6}$$

The curvature or second derivative of the potential energy function at equilibrium is

$$\frac{dU_{eff}^2(r_o)}{dr^2} = k + 3\frac{L^2}{\mu r_o^4}.$$

From equation (5) we can substitute for k with the result.

$$\frac{dU_{eff}^2(r_o)}{dr^2} = \frac{L^2}{\mu r_o^4} + 3\frac{L^2}{\mu r_o^4} = 4\frac{L^2}{\mu r_o^4}.$$

First we note that this expression is positive definite, hence we must have a stable equilibrium with harmonic oscillations about equilibrium. Dividing equation (6) by the reduced mass yields

$$\ddot{\epsilon} = -\frac{1}{\mu}\frac{dU_{eff}^2(r_o)}{dr^2}\epsilon = -4\frac{L^2}{\mu^2 r_o^4}\epsilon. \tag{7}$$

Hence the frequency of radial oscillations is

$$\omega_r = 2\frac{L}{\mu r_o^2}$$

Since the angular momentum for a circular orbit is $L = \mu r_o^2 \dot{\phi}$, we see that

$$\omega_r = 2\dot{\phi}.$$

This means that the period of radial oscillations and the orbital period are related via

$$\tau_{ang} = 2\tau_{rad}.$$

Hence the orbit is closed.