Solutions Assignment 10   11.17*, 11.19, 11.27**, 11.32***

*For 11.17 add a part (c) in which you find the normal coordinates. Show that by using these coordinates the Lagrangian separates into two independent Lagrangians. **For problem 11.27 assume that $m_1 \neq m_2$. Additionally add a part (d) in which you verify that the zero frequency normal coordinate is (or at least proportional to) the CM coordinate and that the finite frequency normal coordinate is (or at least proportional to) the relative coordinate. *** In 11.32 assume that all masses are equal, $M = m$.

11.17 (a)   The Lagrangian for the double pendulum (small amplitude oscillations) is (see equations 11.39 and 11.40)

$$\mathcal{L} = \frac{1}{2}\left(m_1 + m_2\right)L_1^2\dot{\phi}_1^2 + m_2 L_1 L_2 \dot{\phi}_1\dot{\phi}_2 + \frac{1}{2}m_2 L_2^2\dot{\phi}_2^2 - \frac{1}{2}\left(m_1 + m_2\right)gL_1\phi_1^2 - \frac{1}{2}m_2 gL_2\phi_2^2.$$

For this problem $L_1 = L_2 = L$ and $m_1 = 8m_2 = 8m$, hence the Lagrangian reduces to

$$\mathcal{L} = \frac{9}{2}mL^2\dot{\phi}_1^2 + mL^2\dot{\phi}_1\dot{\phi}_2 + \frac{1}{2}mL^2\dot{\phi}_2^2 - \frac{9}{2}mgL\phi_1^2 - \frac{1}{2}mgL\phi_2^2.$$

The two Lagrange equations of motion are

$$\phi_1 \quad : \quad 9\ddot{\phi}_1 + \ddot{\phi}_2 = -9\frac{g}{L}\phi_1 = -9\omega_o^2\phi_1$$

$$\phi_2 \quad : \quad \ddot{\phi}_1 + \ddot{\phi}_2 = -\frac{g}{L}\phi_2 = -\omega_o^2\phi_2,$$

where $\omega_o^2 = g/L$. The matrix equation that we must solve is

$$\begin{bmatrix} 9\omega_o^2 - 9\omega^2 & -\omega^2 \\ -\omega^2 & \omega_o^2 - \omega^2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The secular or characteristic equation is

$$\det\begin{bmatrix} 9\omega_o^2 - 9\omega^2 & -\omega^2 \\ -\omega^2 & \omega_o^2 - \omega^2 \end{bmatrix} = 8\omega^4 - 18\omega_o^2\omega^2 + 9\omega_o^4 = 0.$$

The solutions for the two normal mode frequencies are

$$\omega_1^2 = \frac{3}{4}\omega_o^2, \text{ and } \omega_2^2 = \frac{3}{2}\omega_o^2.$$

The first normal mode is found from

$$\begin{bmatrix} 9\left(1 - 3/4\right)\omega_o^2 & -3\omega_o^2/4 \\ -3\omega_o^2/4 & \left(1 - 3/4\right)\omega_o^2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

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The solution is $a_1 = a_2/3$. In this mode both pendulums oscillate in phase with the upper pendulum having an amplitude that is $1/3$ that of the lower pendulum.

The second normal mode is found from

$$\begin{bmatrix} 9\left(1-3/2\right)\omega_o^2 & -3\omega_o^2/2 \\ -3\omega_o^2/2 & \left(1-3/2\right)\omega_o^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} -9/2 & -3/2 \\ -3/2 & -1/2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The solution is $a_1 = -a_2/3$. In this mode both pendulums oscillate exactly out of phase with the upper pendulum again having an amplitude that is $1/3$ that of the lower pendulum.

(b)

Given the initial conditions for this pendulum are $\phi_1 = \phi_2 = \alpha$ and $\dot{\phi}_1 = \dot{\phi}_2 = 0$, then the solutions are in the form

$$\begin{bmatrix} \phi_1\left(t\right) \\ \phi_2\left(t\right) \end{bmatrix} = A_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \cos\omega_1 t + A_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} \cos\omega_2 t.$$

This results in $\dot{\phi}_1\left(0\right) = \dot{\phi}_2\left(0\right) = 0$. The additional initial condition results in

$$\begin{bmatrix} \phi_1\left(0\right) \\ \phi_2\left(0\right) \end{bmatrix} = A_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + A_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \alpha.$$

Solving for A_1 and A_2 we find

$$\begin{aligned} A_1 + A_2 &= \alpha \\ A_1 - A_2 &= \alpha/3 \end{aligned}$$

or

$$A_1 = 2\alpha/3 \quad \text{and} \quad A_2 = \alpha/3.$$

The expression for the solution is

$$\begin{bmatrix} \phi_1\left(t\right) \\ \phi_2\left(t\right) \end{bmatrix} = \frac{2\alpha}{3} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \cos\sqrt{3g/4L}t + \frac{\alpha}{3} \begin{bmatrix} 1 \\ -3 \end{bmatrix} \cos\sqrt{3g/2L}t.$$

Since the ratio of the normal mode frequencies is $\sqrt{2}$ which is irrational, the motion is NOT periodic.

(c) The normal mode coordinates satisfy

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \xi_1 \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} + \xi_2 \begin{bmatrix} 1/3 \\ -1 \end{bmatrix}$$

so that

$$\phi_1 = \left(\xi_1 + \xi_2\right)/3 \quad \text{and} \quad \phi_2 = \xi_1 - \xi_2.$$

In these coordinates the Lagrangian is given by

$$
\mathcal{L} = \frac{9}{2}mL^2\frac{\left(\dot{\xi}_1+\dot{\xi}_2\right)^2}{9} + mL^2\frac{\dot{\xi}_1+\dot{\xi}_2}{3}\left(\dot{\xi}_1-\dot{\xi}_2\right) + \frac{1}{2}mL^2\left(\dot{\xi}_1-\dot{\xi}_2\right)^2
$$
$$
-\frac{9}{2}mgL\frac{\left(\xi_1+\xi_2\right)^2}{9} - \frac{1}{2}mgL\left(\xi_1-\xi_2\right)^2 ,
$$
$$
\mathcal{L} = \frac{4}{3}mL^2\dot{\xi}_1^2 - mgL\xi_1^2 + \frac{2}{3}mL^2\dot{\xi}_2^2 - mgL\xi_2^2 = \mathcal{L}_1 + \mathcal{L}_2.
$$

This separates immediately and for a general solution we find

$$
\xi_1 = A_1\cos\left(\sqrt{3g/4L}\,t - \delta_1\right), \quad \xi_2 = A_2\cos\left(\sqrt{3g/2L}\,t - \delta_2\right).
$$

11.19(a) The x, y coordinates of the pendulum bob are

$$
x_{bob} = x + L\sin\phi, \quad y_{bob} = L\cos\phi.
$$

The velocities are

$$
\dot{x}_{bob} = \dot{x} + L\cos\phi\,\dot{\phi}, \quad \dot{y}_{bob} = -L\sin\phi\,\dot{\phi}.
$$

Squaring the velocities and adding yields

$$
v_{bob}^2 = \dot{x}^2 + 2L\cos\phi\,\dot{\phi}\dot{x} + L^2\dot{\phi}^2 .
$$

For small amplitude oscillations this becomes

$$
v_{bob}^2 = \dot{x}^2 + 2L\dot{\phi}\dot{x} + L^2\dot{\phi}^2
$$

The Lagrangian for this system is

$$
\mathcal{L} = \frac{1}{2}\left((M+m)\dot{x}^2 + 2ML\dot{\phi}\dot{x} + ML^2\dot{\phi}^2\right) - \frac{1}{2}kx^2 + MgL\cos\phi
$$
$$
\mathcal{L} = \frac{1}{2}\left((M+m)\dot{x}^2 + 2ML\dot{\phi}\dot{x} + ML^2\dot{\phi}^2\right) - \frac{1}{2}kx^2 - \frac{1}{2}MgL\phi^2.
$$

The two equations of motion are

$$
x \quad : \quad (M+m)\ddot{x} + ML\ddot{\phi} = -kx.
$$
$$
\phi \quad : \quad M\ddot{x} + ML\ddot{\phi} = -Mg\phi.
$$

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(b) Assuming units in which that $m = M = L = g = 1$ and $k = 2$, the matrix equation we must solve is

$$\begin{bmatrix} 2 - 2\omega^2 & -\omega^2 \\ -\omega^2 & 1 - \omega^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The secular or characteristic equation that results is

$$\det \begin{bmatrix} 2 - 2\omega^2 & -\omega^2 \\ -\omega^2 & 1 - \omega^2 \end{bmatrix} = \omega^4 - 4\omega^2 + 2 = 0,$$

with solutions

$$\omega_1^2 = 2 - \sqrt{2} \ \text{ and } \ \omega_2^2 = 2 + \sqrt{2}.$$

The solution for ω_1 is found from

$$\begin{bmatrix} 2 - 2\left(2 - \sqrt{2}\right) & -\left(2 - \sqrt{2}\right) \\ -\left(2 - \sqrt{2}\right) & 1 - \left(2 - \sqrt{2}\right) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -2 + 2\sqrt{2} & -2 + \sqrt{2} \\ -2 + \sqrt{2} & -1 + \sqrt{2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{2}\left(2 - \sqrt{2}\right) & -2 + \sqrt{2} \\ \sqrt{2}\left(1 - \sqrt{2}\right) & -1 + \sqrt{2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \sqrt{2} a_1 = a_2$$

In this mode the cart and pendulum are in phase with the amplitude of the cart smaller than the amplitude of the pendulum bob by a factor of $\sqrt{2}$.

The solution for ω_2 is found from

$$\begin{bmatrix} 2 - 2\left(2 + \sqrt{2}\right) & -\left(2 + \sqrt{2}\right) \\ -\left(2 + \sqrt{2}\right) & 1 - \left(2 + \sqrt{2}\right) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -2 - 2\sqrt{2} & -2 - \sqrt{2} \\ -2 - \sqrt{2} & -1 - \sqrt{2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\sqrt{2}\left(2 + \sqrt{2}\right) & -2 - \sqrt{2} \\ -\sqrt{2}\left(1 + \sqrt{2}\right) & -1 - \sqrt{2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow -\sqrt{2} a_1 = a_2$$

In this mode the cart and pendulum are exactly out of phase with the amplitude of the cart smaller than the amplitude of the pendulum bob by a factor of $\sqrt{2}$.

11.27 (a) The Lagrangian for this system is

$$\mathcal{L} = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k \left(x_1 - x_2\right)^2.$$

The equations of motion are

$$\begin{aligned} m_1 \ddot{x}_1 &= -k x_1 + k x_2 \\ m_2 \ddot{x}_2 &= k x_1 - k x_2, \end{aligned}$$

The matrix equation that we must solve is

$$\begin{bmatrix} k - \omega^2 m_1 & -k \\ -k & k - \omega^2 m_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

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The resulting secular equation is

$$\det \begin{bmatrix} k - \omega^2 m_1 & -k \\ -k & k - \omega^2 m_2 \end{bmatrix} = m_1 m_2 \omega^4 - (m_1 + m_2) k \omega^2 = 0,$$

with solutions

$$\omega_1^2 = 0 \ \text{ and } \ \omega_2^2 = \frac{(m_1 + m_2) k}{m_1 m_2} = k/\mu,$$

where μ is the reduced mass of this system.

(b)   For the nonzero frequency the matrix equation becomes

$$\begin{bmatrix} k - kM/m_2 & -k \\ -k & k - kM/m_1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -m_1 k/m_2 & -k \\ -k & -m_2 k/m_1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

with solution

$$m_1 a_1 = -m_2 a_2 \rightarrow m_1 a_1 + m_2 a_2 = 0.$$

In this mode as one mass moves relative to the center of mass the other moves in the opposite direction with an amplitude that keeps the center of mass fixed (or at least moving uniformly)

(c)   At zero frequency the equation of motion reduces to no restoring force which implies uniform translation. For the zero frequency we find

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

with solution

$$a_1 = a_2.$$

Both masses travel with equal amplitude (this keeps the center of mass moving at a uniform velocity).

(d)   The general solution is

$$\mathbf{x} = A_1 \begin{bmatrix} m_2 \\ -m_1 \end{bmatrix} \cos(\omega_1 t - \delta_1) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} (vt + x_o)$$

and the normal coordinates are

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \xi_2 \begin{bmatrix} m_2 \\ -m_1 \end{bmatrix} + \xi_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

To solve for the normal coordinates we invert the algebraic relations

$$\begin{aligned} x_1 &= m_2 \xi_2 + \xi_1 \\ x_2 &= -m_1 \xi_2 + \xi_1. \end{aligned}$$

The normal coordinate for $\omega_1^2 = 0$ is

$$\xi_1 = \frac{\det \begin{bmatrix} m_2 & x_1 \\ -m_1 & x_2 \end{bmatrix}}{m_2 + m_1} = \frac{m_2 x_2 + m_1 x_1}{m_2 + m_1} = X$$

where X is the center of mass of the two masses. The normal coordinate for $\omega_2^2 = k/\mu$ is

$$\xi_2 = \frac{\det \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix}}{m_2 + m_1} = \frac{x_1 - x_2}{m_2 + m_1} = x/M$$

where x is the relative coordinate.

11.32 (a)   The Lagrangian for this system is

$$\mathcal{L} = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 \right) - \frac{1}{2} k \left(x_2 - x_1 \right)^2 - \frac{1}{2} k \left(x_3 - x_2 \right)^2$$

The three equations of motion are

$$x_1 \quad : \quad m\ddot{x}_1 = -k \left(x_1 - x_2 \right) = -k x_1 + k x_2$$

$$x_2 \quad : \quad m\ddot{x}_2 = -k \left(x_2 - x_1 \right) - k \left(x_2 - x_3 \right) = k x_1 - 2k x_2 + k x_3$$

$$x_3 \quad : \quad m\ddot{x}_3 = -k \left(x_3 - x_2 \right) = -k x_3 + k x_2.$$

The resulting matrix equation is

$$\begin{bmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - m\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The secular equation is

$$\det \begin{bmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - m\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{bmatrix} = -m^3\omega^6 + 4km^2\omega^4 - 3k^2 m\omega^2 = 0,$$

$$\omega^2 \left(\omega^4 - 4 \left(k/m \right) \omega^2 + 3 \left(k/m \right)^2 \right) = \omega^2 \left(\omega^2 - k/m \right) \left(\omega^2 - 3k/m \right) = 0.$$

Clearly one of the solutions occurs at $\omega^2 = 0$.

(b)   The normal mode solution for $\omega^2 = k/m$ is found from

$$\begin{bmatrix} 0 & -k & 0 \\ -k & k & -k \\ 0 & -k & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The solutions are $a_2 = 0$, and $a_1 = -a_3$. In this mode the center mass remains stationary while the outside masses oscillate exactly out of phase with equal amplitudes. Clearly this explains why $\omega^2 = k/m$.

The normal mode solution for $\omega^2 = 3k/m$ is found from

$$\begin{bmatrix} -2k & -k & 0 \\ -k & -k & -k \\ 0 & -k & -2k \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The solutions are $a_2 = 0$, and $a_1 = -a_3$. In this mode the center mass remains stationary while the outside mass oscillate exactly out of phase with equal amplitudes. Clearly this explains why $\omega^2 = k/m$. The solution is $a_2 = -2a_1 = -2a_3$. The two outer mass oscillate in phase with equal amplitudes while the center mass oscillates exactly out of phase with twice the amplitude of the outer masses. This is consistent with $\omega^2 = 3k/m$.

(c) The normal mode solution for $\omega^2 = 0$ is found from

$$\begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The solution is simply $a_1 = a_2 = a_3$. This is a uniform translation of the three masses which explains why $\omega^2 = 0$.